Formal Reasoning 2014
Solutions Test Block 2: Predicate Logic
(30/09/14)

The first three exercises use the following 'dictionary'.

P	the domain of persons
$W(x)$	x is a woman
$H(x)$	x is happy
$L(x, y)$	x loves y

1. Give a formula of predicate logic that approximates the meaning of the following sentence as well as possible:

   (a) *There exists at least one happy person.* (10 points)

   $$\exists x \in P\, H(x)$$

   (b) *There exists exactly one happy person.* (10 points)

   $$\exists x \in P\, (H(x) \wedge \forall y \in P\, (H(y) \rightarrow y = x))$$

   (c) *There exist exactly two happy persons.* (10 points)

   $$\exists x_1, x_2 \in P\, (x_1 \neq x_2 \wedge H(x_1) \wedge H(x_2) \wedge \forall y \in P\, (H(y) \rightarrow y = x_1 \vee y = x_2))$$

2. Give a formula of predicate logic that approximates the meaning of the following Dutch sentence as well as possible: (15 points)

   > *A man is happy if there is a woman who loves him whom he also loves.*

   A 'man is happy if ...' must be interpreted as 'for all men it holds that if they are happy...', because the interpretation 'there exists a man such that if he is happy...' gives a very strange formula (which also holds if there is a man who is happy for another reason even if there are no men that are happy for the reason from the sentence).

   $$\forall x \in P\, (\neg W(x) \wedge [\exists y \in P\, (W(y) \wedge H(y, x) \wedge L(x, y))] \rightarrow H(x))$$

3. Consider the following formula of predicate logic:

   $$\forall x \in P\, [\neg \exists y \in P\, L(y, x) \wedge W(x) \rightarrow \neg H(x)]$$

(a) Write this formula according to the official grammar from the course notes.

(10 points)

$$(\forall x \in P\,((\neg(\exists y \in P\,L(y,x)) \wedge W(x)) \rightarrow \neg H(x)))$$

(b) Give an English sentence that corresponds to this formula as well as possible. (15 points)

Women whom nobody loves are not happy.

4. What does $f \vDash g$ means in predicate logic? Give the definition in terms of interpretations within models. (10 points)

The notation stands for 'g is a logical consequence of f'. This means that for *each* interpretation within a model that makes f true, it must hold that g is also true under the same interpretation within that model.

5. Give an interpretation I_5 within a model M_5 such that the following formula is not true.

$$\Big(\forall x \in D\,\big(P(x) \vee Q(x)\big)\Big) \rightarrow \Big(\big(\forall x \in D\,P(x)\big) \vee \big(\forall x \in D\,Q(x)\big)\Big)$$

Explain your answer. (10 points)

Take for instance $M_5 := (\mathbb{N}, 0)$ with the interpretation I_5:

$$\begin{aligned} D &\longrightarrow \mathbb{N} \\ P(x) &\longrightarrow (x = 0) \\ Q(x) &\longrightarrow (x \neq 0) \end{aligned}$$

The first part of the implication says that each natural number is either equal to zero or not equal to zero, which is true. However, the right part of the implications states that either each natural number is equal to zero, or each natural number is not equal to zero. And this is obviously not true. So the implication does not hold.