Formal Reasoning 2014
Solutions Test Block 5: Modal logic
(16/12/14)

Within the second and the fourth exercise we use this dictionary:

S	I study during the Christmas break
W	During the Christmas break I go skiing

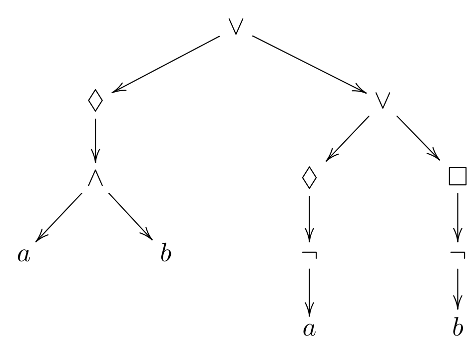
1. Write the following formula of modal logic in such a way that it complies to the official grammar and draw the corresponding parse tree. (15 points)

$$\Diamond(a \wedge b) \vee \Diamond\neg a \vee \Box\neg b$$

The formula is:

$$(\Diamond(a \wedge b) \vee (\Diamond\neg a \vee \Box\neg b))$$

The corresponding parse tree is:



2. Give an English sentence that approximates the meaning of the formula from deontic logic as well as possible:

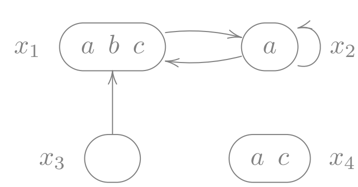
(10 points)

$$\Diamond(S \wedge W) \vee \Diamond\neg S \vee \Box\neg W$$

Within deontic logic $\Box f$ means 'f ought to be' and $\Diamond f$ means 'f is permissible'. A quite literal translation now gives:

It is permissible to study during the Christmas break and to go skiing during the Christmas break, and/or, it is permissible not to study during the Christmas break, and/or I ought to be not going skiing during the Christmas break.

3. Let the Kripke-model $\mathcal{M}_3$ be defined as: (15 points)



Is the following statement true?

$$\mathcal{M}_3 \models \Diamond(a \wedge b) \vee \Diamond\neg a \vee \Box\neg b$$

Explain your answer.

The table represents the $\Vdash$-relation in this model:

$\Vdash$	a	b	$a \wedge b$	$\neg a$	$\neg b$	$\Diamond(a \wedge b)$	$\Diamond\neg a$	$\Box\neg b$	$\Diamond\neg a \vee \Box\neg b$	$\Diamond(a \wedge b) \vee \Diamond\neg a \vee \Box\neg b$
x_1	1	1	1	0	0	0	0	1	1	1
x_2	1	0	0	0	1	1	0	0	0	1
x_3	0	0	0	1	1	1	0	0	0	1
x_4	1	0	0	1	0	0	0	1	1	1

Because we see that in every world x_i it holds that $x_i \Vdash \Diamond(a \wedge b) \vee \Diamond\neg a \vee \Box\neg b$, it also holds that $\mathcal{M}_3 \models \Diamond(a \wedge b) \vee \Diamond\neg a \vee \Box\neg b$.

4. This exercise is about the Dutch sentence:

   During the Christmas break I study until I go skiing.

   (a) Give an LTL formula that approximates the meaning of this sentence as well as possible. You may assume that this statement is given on the first day of the Christmas break. (10 points)

   The most logical solution seems to be $S\,\mathcal{U}\,W$.

   (b) According to the English sentence, is it possible that every now and then you are not studying before you go skiing? (5 points)

   No, apparently you are forced to study continuously. With 'continuously' we mean 'every time unit'. So if the time unit is taken as seconds, you are forced to study each second until the second in which yo go skiing. And if the time unit is taken as days, you have to study each day. But it is not clear whether this should be the whole 24 hours of this day. The formula $S\,\mathcal{U}\,W$ happens to be true in x_i whenever for all moments $x_i, x_{i+1}, \ldots, x_{j-1}$ the formula S holds and in x_j the formula W holds, where $j \geq i$.

   (c) According to your interpretation, does it follow from the sentence that you are no longer studying while gone skiing? Explain why your formula matches your interpretation of the sentence in this respect. (5 points)

   It is suggested, but it is not enforced. This is resembled in the formula $S\,\mathcal{U}\,W$, because it doesn't state anything about this. The only requirement for moment x_j (see previous answer) is that formula W holds. Nothing is stated or enforced about the formula S at moment x_j.

5. This exercise is about the LTL formula

$$\mathcal{F}(a \wedge b) \vee \mathcal{F}\neg a \vee \mathcal{G}\neg b$$

   (a) Explain what this formula means. (10 points)

   There will be a moment in which propositions a and b hold (or a and b already hold in the current moment), or there will be a moment in which proposition a does not hold (or a already doesn't hold in the current moment), or from now on (including now) the proposition b never holds.

   (b) Give an LTL Kripke model in which this formula is true. Explain your answer. (10 points)

   Let $\mathcal{M}_5 = \langle W, R, V \rangle$, where

$$\begin{aligned} W &= \{x_i \mid i \in \mathbb{N}\} \\ R(x_i) &= \{x_j \mid j \in \mathbb{N}, j \geq i\} \qquad \text{for } i \in \mathbb{N} \\ V(x_i) &= \emptyset \qquad \text{for } i \in \mathbb{N} \end{aligned}$$

Because for all $i \in \mathbb{N}$ $x_i \not\Vdash b$, we know that $x_i \Vdash \mathcal{G}\neg b$ for each world x_i. But then, due to the $\vee$-operators, we know that formula $x_i \Vdash \mathcal{F}(a \wedge b) \vee \mathcal{F}\neg a \vee \mathcal{G}\neg b$, also hold for each world x_i. But this implies that $\mathcal{M}_5 \vDash \mathcal{F}(a \wedge b) \vee \mathcal{F}\neg a \vee \mathcal{G}\neg b$.

(c) Give a definition of logically true within LTL and explain whether the given formula complies to that definition or not. (10 points)

A formula is logically true within LTL if it holds in each model.

This formula is indeed logically true. We give a proof by contradiction. Assume that there exist a model $\mathcal{M}'_5$ for which the formula doesn't hold.

Then within this model there must be a world x_i for which $x_i \not\Vdash \mathcal{F}(a \wedge b)$, $x_i \not\Vdash \mathcal{F}\neg a$ and $x_i \not\Vdash \mathcal{G}\neg b$. From $x_i \not\Vdash \mathcal{F}\neg a$ it follows that for all j such that $j \geq i$ it holds that $x_j \Vdash a$. From $x_i \not\Vdash \mathcal{G}\neg b$ it follows that there exists k such that $k \geq i$ and $x_k \Vdash b$. But then $x_k \Vdash a \wedge b$ also holds and hence also $x_i \Vdash \mathcal{F}(a \wedge b)$. However, this is a contradiction, so such a model $\mathcal{M}'_5$ doesn't exist.