Formal Reasoning 2014
Solutions Additional Test
(07/01/15)

1. Give three distinct models in which the proposition $a \to b \leftrightarrow c$ holds. (15 points)

   In order to see in which models a formula holds easily, we create the corresponding truth table. The columns of the table immediately give the order for parsing the formula.

a	b	c	$a \to b$	$(a \to b) \leftrightarrow c$
0	0	0	1	0
0	0	1	1	1
0	1	0	1	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	0
1	1	0	1	0
1	1	1	1	1

   From this table it follows that there are four models in which the formula is true:

   - v_1 where $v_1(a) = 0$, $v_1(b) = 0$ en $v_1(c) = 1$
   - v_3 where $v_3(a) = 0$, $v_3(b) = 1$ en $v_3(c) = 1$
   - v_4 where $v_4(a) = 1$, $v_4(b) = 0$ en $v_4(c) = 0$
   - v_7 where $v_7(a) = 1$, $v_7(b) = 1$ en $v_7(c) = 1$

2. Translate the following English sentence to a formula of predicate logic with equality.

   There is exactly one man who loves exactly one woman.

   Use this dictionary:

M	the domain of men
V	the domain of women
$H(x, y)$	x loves y

   (20 points)

   We start by introducing an abbreviation $HPV(x)$ with the meaning: x loves exactly one woman.

$$HPV(x) := \exists y \in V\, [H(x, y) \land \forall z \in V\, [H(x, z) \to z = y]]$$

   We use this abbreviation twice in the formalization of the whole sentence:

$$\exists x \in M\, [HPV(x) \land \forall y \in M\, [HPV(y) \to y = x]]$$

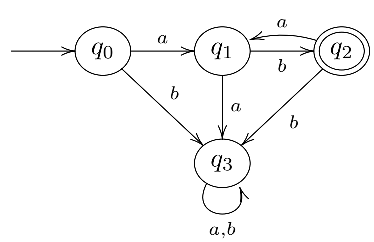
3. Give a finite (deterministic) automaton for the language

$$L_3 := \mathcal{L}((ab)^*) \cap \overline{\{w \in \{a,b\}^* \mid w \text{ does not contain } b\}^*}$$

(20 points)

- The language $\mathcal{L}((ab)^*) = \{\lambda, ab, abab, ababab, \dots\}$.
- The language $\{w \in \{a,b\}^* \mid w \text{ does not contain } b\} = \{\lambda, a, aa, aaa, \dots\}$.
- The language $\{w \in \{a,b\}^* \mid w \text{ does not contain } b\}^* = \{\lambda, a, aa, aaa, \dots\}$.
- The language $\overline{\{w \in \{a,b\}^* \mid w \text{ does not contain } b\}^*} = \{w \in \{a,b\}^* \mid w \text{ contains a } b\}$.
- But then the requested intersection is the language: $\{ab, abab, ababab, \dots\}$.

A finite (deterministic) automaton that accepts this language is:



4. We want to show that the sum of the degrees of all vertices in a graph is always even. Prove this by induction to the number of edges in the graph.

(20 points)

Proposition:

The sum of the degrees of all vertices in a graph with n edges is even, for all $n \geq 0$.

Proof by induction on n. 1

We first define our predicate P as:

$P(n)$:= The sum of the degrees of all vertices in a graph with n edges is even 2

Base Case. We show that $P(0)$ holds, i.e. we show that 3

the sum of the degrees of all vertices in a graph with 0 edges is even.

This indeed holds, because 4

in a graph with zero edges, each vertex has degree 0. Hence the sum of all these degrees is also 0. And 0 is even.

Induction Step. Let k be any natural number such that $k \geq 0$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6
the sum of the degrees of all vertices in a graph with k edges is even.
(Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7
the sum of the degrees of all vertices in a graph with $k+1$ edges is even.

This indeed holds, because of this argument. Let G be a graph 8
with $k+1$ edges and let (p, q) be one of these edges in the graph. If we omit edge (p, q) from the graph, we get a subgraph G' with the same vertices and almost the same edges. But G' has only k edges. By applying the induction hypothesis we know that the sum of the degrees of all vertices in G' is even, say $2r$ for some $r \in N$. However, by construction it follows that the sum of the degrees of all vertices in graph G equals $2r + 1 + 1$, because both the degrees of p and q are exactly one higher. Hence the sum of the degrees in G is $2r + 2 = 2(r + 1)$ and hence in particular also even.

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$. 9

5. Give an LTL formula that formalizes the following property and explain your answer:

   a becomes true before b

   More precisely: a and b are both at least once true (but they don't have to stay true), and the moment on which a becomes true for the first time, is strictly earlier than the moment on which b becomes true for the first time.

   (15 points)

$$(\neg a \wedge \neg b)\ \mathcal{U}\ (a \wedge \neg b \wedge \mathcal{F} b)$$

The second part of this formula says that $a \wedge \neg b \wedge \mathcal{F} b$ will be true at some moment in time. Hence there will be a moment in which a is true and b is false, but b will become true some time later. Although $\mathcal{F} b$ implies that this later moment may be the same moment that a is true, however, this is excluded by $\neg b$. The first part of this formula prevents that there is an earlier moment in which b becomes true. For instance, if we had only written $\mathcal{F}(a \wedge \neg b \wedge \mathcal{F} b)$ this situation was not excluded.