Formal Reasoning 2016
Solutions Test Block 3: Languages & Automata

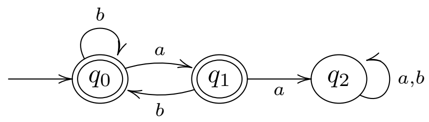
(19/10/16)

1. We define the language

$$L_1 := \{w \in \{a, b\}^* \mid \text{no two } a\text{'s in } w \text{ are next to each other}\}$$

   (a) Give a regular expression for this language.
   Take for instance $r = (b \cup (ab))^*(a \cup \lambda)$.

   (b) Give a finite automaton that recognizes this language.



   (c) Give a context-free grammar for this language which uses only one nonterminal, namely S.
   Note: Grammars that produce language L_1 using more than one nonterminal will also score some points.
   This is a right-linear grammar that corresponds with the automaton above.

$$\begin{aligned} S &\to aB \mid bS \mid \lambda \\ B &\to bS \mid \lambda \end{aligned}$$

   If we substitute the rules for B in the first line we get:

$$S \to abS \mid a \mid bS \mid \lambda$$

   Another solution is

$$S \to SbS \mid a \mid \lambda$$

2. We define the grammar G_2 by the rules:

$$\begin{aligned} S &\to aA \mid aB \\ A &\to aS \\ B &\to BB \mid b \mid \lambda \end{aligned}$$

   (a) Write G_2 as a triple $\langle \Sigma, V, R \rangle$.

$$\langle \{a, b\}, \{S, A, B\}, \{S \to aA, S \to aB, A \to aS, B \to BB, B \to b, B \to \lambda\} \rangle$$

   (b) We want to show that $b \notin \mathcal{L}(G_2)$, and consider for this the property:

$$P(w) := w \text{ does not start with } b$$

   Show that this is not an invariant of G_2.
   Let $v = B$ and $v' = b$. Then $P(v)$ holds, since v starts with a B. It is also clear that $v \to v'$. But $P(v')$ doesn't hold, since v' does start with a b. So $P(w)$ is not an invariant.

(c) Is G_2 right-linear? Explain your answer.

No, it is not. In the rule $B \to BB$ there are two nonterminals in the right hand side. Obviously only one of them can be at the right end.

(d) Is $\mathcal{L}(G_2)$ regular? Explain your answer.

Yes, we have that $\mathcal{L}(G_2) = \mathcal{L}(a(aa)^*b^*)$. Note that this language contains all words that start with an odd number of a's, followed by any number of b's.

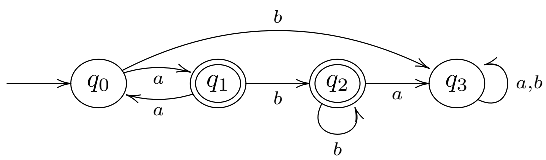
We can also derive that the language is regular from the fact that a right linear grammar G_2' exists:

$$\begin{aligned} S &\to aA \mid aB \\ A &\to aS \\ B &\to bB \mid \lambda \end{aligned}$$

And we could have stated that the language is regular because we give a finite automaton in question 2e.

(e) Give a minimal finite automaton M_2 such that $L(M_2) = \mathcal{L}(G_2)$.

Note: You don't have to prove that your M_2 has indeed a minimal number of states.



It cannot be done with less than four states:

- Obviously we need starting state q_0, which is not a final state because λ is not accepted.
- Since words cannot start with b we need the sink q_3.
- Since a is accepted we need a final state q_1 reachable from q_0 with an a.
- Since aa is not accepted, we need to go from q_1 to a non-final state with an a.
  - This state cannot be sink q_3 because aaa needs to be accepted.
  - It can be q_0 as illustrated in the automaton.
  - It can be a new q_4, but then we have q_0, q_1, q_3 and q_4, which means that we can't do it with three states.
  - So the only way to do it with three states is by going back to q_0.
- Because ab is accepted we need to go from q_1 to a final state with a b.
  - If this final state would be q_1, then the automaton would also accept $abaa$, which is not allowed.
  - So we need a new final state q_2 to make sure that once we encountered b's, we cannot go back accepting a's anymore.

- But then we have already four states. So it cannot be done with less than four.

3. Does there exist a language L with alphabet $\Sigma = \{a, b\}$ such that

$$L^* \cap \overline{L}^* = \Sigma^*$$

holds? Explain your answer.

No, such a language L does not exist. If $L^* \cap \overline{L}^* = \Sigma^*$, then it must be that $\Sigma^* \subseteq L^*$ and $\Sigma^* \subseteq \overline{L}^*$. But $a \in \Sigma^* = \{a, b\}^*$. Hence $a \in L^*$ and $a \in \overline{L}^*$. But if $a \in L^*$ then there must be $k \in \mathbb{N}$ such that $a = w_1 w_2 \cdots w_k$, where $w_i \in L$ for all $i \in \{1, \ldots, k\}$. However, because the length of a equals 1, it follows that there must be some $i \in \{1, \ldots, k\}$ such that $a = w_i$ and either $k = 1$ (and hence $i = 1$) or $w_j = \lambda$ for all $j \in \{1, \ldots, k\} \setminus \{i\}$. But this means that $a \in L$. Using the same kind of reasoning we derive that $a \in \overline{L}$. So $a \in L \cap \overline{L}$, but $L \cap \overline{L} = \emptyset$ by definition. So we have a contradiction and our assumption that such a language L exists cannot be true.