Formal Reasoning 2017
Exam
(22/01/18)

Before you read on, write your name, student number and study on the answer sheet! This exam consists of eighteen exercises (three exercises for each chapter of the course notes) and each of these exercises is worth five points. The mark for this test is the number of points divided by ten. The first ten points are free. Good luck!

1. Give a formula of propositional logic that gives the meaning of the English sentence:

   A rainbow only occurs if it rains and the sun shines at the same time.

   Use the dictionary:

R	it rains
S	the sun shines
RB	there is a rainbow

2. Write the formula of propositional logic

   $$(a \to b \to c) \to (a \to b) \to a \to c$$

   according to the official grammar in the course notes, and give the truth table.

3. Give a formula that is logically equivalent to the formula of propositional logic

   $$a \to (b \to a)$$

   but that does not contain the atoms a and b. This implies that you may use other atoms. Explain your answer.

4. Give a formula of predicate logic with equality that gives the meaning of the English sentence:

   Two is the unique even prime number.

   Use the dictionary:

domains	N	the domain of natural numbers $\mathbb{N}$
constants	d	the number two
predicates	$E(x)$	x is even
	$P(x)$	x is prime

5. Give a model M_5, and an interpretation I_5 in M_5 such that:

   $$(M_5, I_5) \vDash \forall x \in T\, \neg R(x, x) \wedge \exists x_1, x_2, x_3 \in T\, [R(x_1, x_2) \wedge R(x_2, x_3) \wedge R(x_3, x_1)]$$

   Explain your answer. Make sure that your model is unambiguous and not open for discussion.

6. Explain the meaning of the following statement:

$$\forall x \in D\,[A(x) \rightarrow \neg B(x)] \equiv \neg \exists x \in D\,[A(x) \wedge B(x)]$$

Does this statement hold? Explain your answer.

7. Give a non-empty language $L_7 \subseteq \{a,b\}^*$ such that $L_7^R \subseteq \overline{L_7}$, and such that for every word $w \in L_7$ also $aw \in L_7$. Explain your answer.

8. Give a regular expression for the language:

$$L_8 := \{w \in \{a,b,c\}^* \mid \text{in } w \text{ the symbol } a \text{ is never directly followed by } c\}$$

9. Consider the context-free grammar G_9:

$$\begin{aligned} S &\rightarrow aSa \mid B \\ B &\rightarrow bB \mid \lambda \end{aligned}$$

This grammar produces the language:

$$\mathcal{L}(G_9) = \{a^n b^m a^n \mid n, m \in \mathbb{N}\}$$

Someone wants to show that

$$abaa \notin \mathcal{L}(G_9)$$

and proposes the property

$$P(w) := [w \text{ is a palindrome, i.e., } w^R = w]$$

as an invariant to establish this. Does that work? Explain your answer.

10. We define the language:

$$\begin{aligned} L_{10} &:= \{uav \mid u, v \in \{a,b\}^* \text{ and } |v| = 2\} \\ &= \{w \in \{a,b\}^* \mid w \text{ has an } a \text{ two places from the end}\} \end{aligned}$$

For example $\underline{a}bb \in L_{10}$, but $aba\underline{b}ab \notin L_{10}$ and $ab \notin L_{10}$ (too short).

Give a *non*-deterministic finite automaton with at most four states and a single final state that recognizes the language L_{10}.

11. Give a *deterministic* finite automaton that recognizes the language L_{10} from the previous exercise.

12. Let be given a deterministic finite automaton $M = \langle \Sigma, Q, q_0, F, \delta \rangle$. We want to define a deterministic finite automaton $M' = \langle \Sigma', Q', q_0', F', \delta' \rangle$ such that

$$L(M') = \overline{L(M)}$$

Give definitions for Σ', Q', q_0', F' and δ' in terms of Σ, Q, q_0, F and δ and explain why M' has the required property.

13. Give a connected planar graph G_{13} with chromatic number three that does not contain any triangles (subgraphs isomorphic to K_3) and that does not have a Hamilton path. Explain your answer.

14. We define the sequence $a_3, a_4, a_5, \dots$ using the recursive equations:

$$\begin{aligned} a_3 &:= 2 \\ a_{n+1} &:= a_n + 2n + 1 \qquad\qquad \text{for } n \geq 3 \end{aligned}$$

We have for example $a_3 = 2$, $a_4 = 9$, $a_5 = 18$, $\dots$, $a_{45} = 2018$, $\dots$

Use induction to prove that

$$a_n = n^2 - 7$$

for all $n \geq 3$.

15. Someone divides nine distinguishable objects by first taking three objects and putting them aside, and then dividing the other six objects in three non-empty indistinguishable piles. Give the number of different combinations of three piles that can be formed this way. If you use binomial coefficients and/or Stirling numbers in your answer, then give a relevant part of the corresponding triangles as well.

16. Give a formula of modal logic that gives the meaning of the English sentence:

When it rains, I get wet, but now it rains and I don't get wet!

Use the dictionary:

R	it rains
W	I get wet

17. Give a Kripke model $\mathcal{M}_{17}$ such that:

$$\mathcal{M}_{17} \vDash \Box(a \to b) \land a \land \neg b$$

Explain what this statement means, and explain why your Kripke model makes this true.

18. Give an LTL formula in which the connectives $\mathcal{G}$ and $\to$ do not occur, that says that directly after the proposition a is true, the proposition c is not true.

Note that $\Box$ is alternative syntax for $\mathcal{G}$ and therefore is also disallowed. Also, if you only manage to write such a formula using these forbidden connectives, you can do so for partial points.