Formal Reasoning 2017
Solutions Test Block 1: Propositional and Predicate Logic
(25/09/17)

1.
$$(\neg a \lor b) \land ((\neg b) \lor a) \leftrightarrow a \leftrightarrow b$$

(a) Write this formula according to the official grammar from the course notes. (10 points)

$$\underbrace{\left(\underbrace{\left(\underbrace{(\neg a \lor b)}_{\lor} \land \underbrace{(\neg b \lor a)}_{\lor}\right)}_{\land} \leftrightarrow \underbrace{(a \leftrightarrow b)}_{\leftrightarrow}\right)}_{\leftrightarrow}$$

(b) Give the full truth table of this formula. (10 points)

a	b	$\neg a$	$\neg b$	$(\neg a \lor b)$	$(\neg b \lor a)$	$((\neg a \lor b) \land (\neg b \lor a))$	$(a \leftrightarrow b)$	$(((\neg a \lor b) \land (\neg b \lor a)) \leftrightarrow (a \leftrightarrow b))$
0	0	1	1	1	1	1	1	1
0	1	1	0	1	0	0	0	1
1	0	0	1	0	1	0	0	1
1	1	0	0	1	1	1	1	1

2. *In autumn the sun is over the southern hemisphere, and therefore it is cold here now.*

Formalize this English sentence as a formula of propositional logic using the following dictionary as well as possible: (20 points)

A	it is autumn
C	it is cold
S	the sun is over the southern hemisphere

$$(A \to S) \land A \land (S \to C) \equiv A \land S \land C$$

3. The following statement holds:

$$\neg(f \leftrightarrow g) \equiv (f \lor g) \land \neg(f \land g)$$

Explain what this statement *says* in terms of truth tables or models. Note that you do not have to *show* that this statement holds. (10 points)

This statement says that the formula $\neg(f \leftrightarrow g)$ is logically equivalent to the formula $(f \lor g) \land \neg(f \land g)$. In terms of truth tables this means that for each row in the truth table the two formulas have the same value. And in terms of models this means that for each model v it holds that $v(\neg(f \leftrightarrow g)) = v((f \lor g) \land \neg(f \land g))$.

4. *There is a tall woman that all nice men like.*

Formalize this English sentence as a formula of predicate logic using the following dictionary as well as possible: (20 points)

M	domain of men
W	domain of women
$N(x)$	x is nice
$T(x)$	x is tall
$L(x, y)$	x likes y

$$\underbrace{\left(\exists w \in W \underbrace{\left(T(w) \land \underbrace{\left(\forall m \in M \underbrace{(N(m) \to L(m,w))}_{\to}\right)}_{\forall}\right)}_{\land}\right)}_{\exists}$$

5.

$$\exists x \in W \neg T(x) \vDash \exists x \in W\, (T(x) \to N(x))$$

Does this statement hold? Explain your answer. (10 points)

This statement expresses that the formula $\exists x \in W\, (T(x) \to N(x))$ follows from the formula $\exists x \in W \neg T(x)$.

Or in other words,

$$\vDash (\exists x \in W \neg T(x)) \to (\exists x \in W\, (T(x) \to N(x)))$$

Or in again other words, the formula

$$(\exists x \in W \neg T(x)) \to (\exists x \in W\, (T(x) \to N(x)))$$

is logically true.

Stating that the formula

$$(\exists x \in W \neg T(x)) \to (\exists x \in W\, (T(x) \to N(x)))$$

is logically true, implies that the formula must hold in any model under any interpretation. So let M_5 and I_5 be an arbitrary model and interpretation. We make a case distinction.

- Assume that $(M_5, I_5) \vDash \exists x \in W \neg T(x)$. Then we know that there must be some $y \in W$, such that $\neg T(y)$ holds. But then $T(y)$ does not hold, which implies that $T(y) \to N(y)$ holds, because of the false premise of the implication. And in particular this means that $(M_5, I_5) \vDash \exists x \in W\, (T(x) \to N(x))$. So in this case $(M_5, I_5) \vDash (\exists x \in W \neg T(x)) \to (\exists x \in W\, (T(x) \to N(x)))$.

- Assume that $(M_5, I_5) \not\models \exists x \in W\, \neg T(x)$. Then automatically $(M_5, I_5) \models (\exists x \in W\, \neg T(x)) \rightarrow (\exists x \in W\, (T(x) \rightarrow N(x)))$, because of the false premise of the implication. So also in this case $(M_5, I_5) \models (\exists x \in W\, \neg T(x)) \rightarrow (\exists x \in W\, (T(x) \rightarrow N(x)))$.

So in all cases

$$(M_5, I_5) \models (\exists x \in W\, \neg T(x)) \rightarrow (\exists x \in W\, (T(x) \rightarrow N(x)))$$

But because M_5 and I_5 are arbitrary, we may generalize this to

$$\models (\exists x \in W\, \neg T(x)) \rightarrow (\exists x \in W\, (T(x) \rightarrow N(x)))$$

which implies that the statement

$$\exists x \in W\, \neg T(x) \models \exists x \in W\, (T(x) \rightarrow N(x))$$

holds.

6. Give an interpretation I_6 in the model $M_6 = (\mathbb{N}, +, 0, 1, <, \leq)$ under which the following formula is true: (10 points)

$$\exists x, y \in D\, (P(x) \wedge P(y) \wedge \forall z \in D\, [P(z) \rightarrow \neg(x = z \leftrightarrow y = z)])$$

Note that you do not need to explain your answer.

Note that from question 3 it follows that $\neg(x = z \leftrightarrow y = z)$ is equivalent to an exclusive or, which means that $x = z$ or $y = z$ but not both. So if we define a predicate P that holds for exactly two elements of $\mathbb{N}$ then the formula is true. We can indeed achieve this by defining P as follows

D	$\mathbb{N}$
$P(x)$	$x \leq 1$

Obviously P only holds for the natural numbers 0 and 1. So if we take $x = 0$ and $y = 1$ we get that $P(0)$ and $P(1)$ both hold. In addition, for any natural number z, if $z \leq 1$ then $P(z)$ holds and indeed $z = 0$ or $z = 1$ but not both. And if $z > 1$ the predicate $P(z)$ does not hold, so the implication holds by default. So this formula is indeed true under this interpretation and the given model.