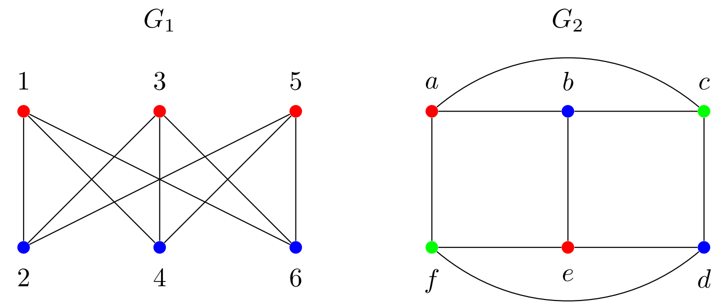
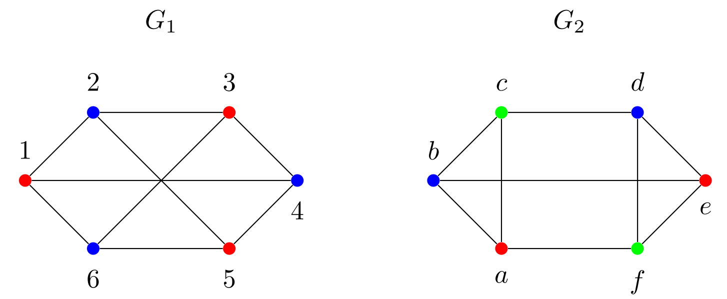
Formal Reasoning 2017
Solutions Test Block 3: Discrete Mathematics and Modal Logic
(20/12/17)

1. (a) Give two non-isomorphic graphs G_1 and G_2 that have six vertices such that each vertex has degree three.

Take for instance:



The same graphs, but in a different representation:



It is clear that G_1 and G_2 have six vertices and that each vertex has degree three. Some reason why the graphs cannot be isomorphic:

- Graph G_1 is not planar but G_2 is. Note that $G_1 = K_{3,3}$ and for that graph we know there is no planar representation. And for G_2 the planar representation is given.
- The chromatic number of G_1 is 2 and the chromatic number of G_2 is 3. See the answer to the next question.
- Graph G_1 is bipartite and G_2 is not. This follows for instance from the chromatic number.
- The shortest cycle in G_1 has length 4 and the shortest cycle in G_2 has length 3. Bipartite graphs don't have cycles of odd length and cycles cannot have length two either. So a shortest cycle in G_1 is $1 \to 2 \to 3 \to 4$. In G_2 a shortest cycle is $1 \to 2 \to 3$.

We didn't prove it but we think these are the only two possible graphs.

(b) Give the chromatic number of your graphs G_1 and G_2. Explain your answer.

The chromatic number of G_1 is two, because it is a bipartite graph. And because of edge $(1, 2)$ it cannot be colored with one color.

Note that graph G_2 contains a triangle a, b and c. And for triangles we know they cannot be colored with less than three colors. And the picture above proves that it is indeed possible to color G_2 with exactly three colors. So its chromatic number is three.

(c) Explain whether your graph G_1 has an Euler path and whether your graph G_2 has a Hamilton path. In both cases, if such a path exists, give it explicitly and if such a path doesn't exist, explain why it doesn't exist.

An Euler path is a path that contains all edges exactly once. A Hamilton path is a path that contains all vertices exactly once.

Since we don't know which graph is your G_1 we give the answers for both our G_1 and G_2.

None of the graphs have an Euler path, because in both graphs there are six vertices with degree three, and for an Euler path to exist there must be at most two vertices of an odd degree.

Both graphs have a Hamilton path. For G_1 take $1 \to 2 \to 3 \to 4 \to 5 \to 6$. For G_2 take $a \to b \to c \to d \to e \to f$.

2. We define the so-called *subfactorial* of n, denoted as $!n$, by this recursive definition:

$$\begin{aligned} !0 &= 1 \\ !(n+1) &= (n+1)\cdot !n + (-1)^{n+1} \qquad \text{for } n \geq 0 \end{aligned}$$

(a) Compute $!2$ and explain how you did this.

We use the recursive definition:

$$\begin{aligned} !0 &= 1 \\ !1 &= !(0+1) \\ &= (0+1)\cdot !0 + (-1)^{0+1} \\ &= 1 \cdot 1 + (-1)^1 \\ &= 1 - 1 \\ &= 0 \\ !2 &= !(1+1) \\ &= (1+1)\cdot !1 + (-1)^{1+1} \\ &= 2 \cdot 0 + (-1)^2 \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

(b) Prove by induction that $!n$ is even whenever n is odd and that $!n$ is odd whenever n is even, for all natural numbers n.

Proposition: 0

$!n$ is even whenever n is odd and $!n$ is odd whenever n is even for all $n \geq 0$.

Proof by induction on n. 1

We first define our predicate P as:
$P(n)$:= $!n$ is even whenever n is odd and $!n$ is odd whenever n is even 2

Base Case. We show that $P(0)$ holds, i.e. we show that 3
$!0$ is even whenever 0 is odd and $!0$ is odd whenever 0 is even.
This indeed holds, because 4
0 is even and $!0 = 1$ which is odd.

Induction Step. Let k be any natural number such that $k \geq 0$. 5
Assume that we already know that $P(k)$ holds, i.e. we assume that 6
$!k$ is even whenever k is odd and $!k$ is odd whenever k is even.
(Induction Hypothesis IH)
We now show that $P(k+1)$ also holds, i.e. we show that 7
$!(k+1)$ is even whenever $k+1$ is odd and $!(k+1)$ is odd whenever $k+1$ is even.
This indeed holds, because we can make a case distinction on k being odd or even and get the following two results. 8

- If we assume that k is odd, we have to prove that $!(k+1)$ is odd.
  This holds because
  - $!(k+1) = (k+1)\cdot !k + (-1)^{k+1}$
  - and $k+1$ is even,
  - and by induction $!k$ is even,
  - hence $(k+1)\cdot !k$ is also even,
  - $(-1)^{k+1}$ is always odd, independent of the value of k,
  - hence $(k+1)\cdot !k + (-1)^{k+1}$ is odd,
  - so $!(k+1)$ is odd.
- If we assume that k is even, we have to prove that $!(k+1)$ is even.
  This holds because
  - $!(k+1) = (k+1)\cdot !k + (-1)^{k+1}$
  - and $k+1$ is odd,
  - and by induction $!k$ is odd,
  - hence $(k+1)\cdot !k$ is also odd,
  - $(-1)^{k+1}$ is always odd, independent of the value of k,
  - hence $(k+1)\cdot !k + (-1)^{k+1}$ is even,
  - so $!(k+1)$ is even.

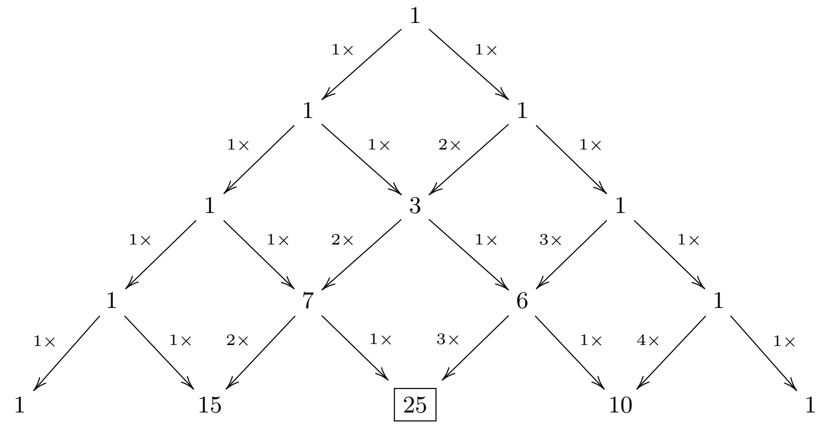
So in both cases we have proven what we needed to prove, so the induction stepholds.

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$. 9

3. In how many ways can we distribute five distinguishable objects into three non-empty indistinguishable groups? Write your answer in terms of Stirling numbers of the second kind and give a sufficiently large part of the

triangle for these Stirling numbers to make sure that all numbers you use are visible and marked.

The answer is $\left\{ {5 \atop 3} \right\} = 25$. Because we have to divide the six objects into three non-empty groups, we don't have to take into account the values for $\left\{ {5 \atop 2} \right\}$ and $\left\{ {5 \atop 1} \right\}$, because these have empty groups.



4. (a) Using the dictionary

S	it snows

give an English rendering of the formula $S \to \neg\Box\neg S$ according to doxastic logic.

In doxastic logic $S \to \neg\Box\neg S$ means:

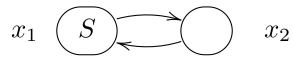
If it snows then I don't believe that it doesn't snow.

Or if we use the valid transformation to $S \to \Diamond S$:

If it snows then the fact that it snows doesn't contradict my beliefs.

(b) A formula f is called *true in the logic D* if f is true in all serial Kripke models. The notation for this is $\vDash_D f$. Show that $\vDash_D S \to \neg\Box\neg S$ does not hold.

So we have to give a serial Kripke model $\mathcal{M}_1$ for which $\mathcal{M}_1 \vDash S \to \neg\Box\neg S$ does not hold. So in other words, each world in $\mathcal{M}_1$ must have at least one outgoing arrow to be a serial Kripke model and there must be at least one world x_1 in $\mathcal{M}_1$ such that $x_1 \Vdash S \to \neg\Box\neg S$ doesn't hold. This implies that S should hold in x_1 but that $\neg\Box\neg S$ does not hold in x_1, which obviously implies that $\Box\neg S$ should hold in x_1. And this means that in all reachable worlds of x_1 formula S should not hold. But this is easy. Take for instance:

$\mathcal{M}_1 :$ 

Now we prove that this model is indeed a correct counterexample.

- Note that $\mathcal{M}_1$ is serial because all worlds have at least one reachable world.
- Clearly $x_1 \Vdash S$.
- And also clearly $x_2 \nVdash S$, hence $x_2 \Vdash \neg S$.

- Since $R(x_1) = \{x_2\}$ we get that $x_1 \Vdash \Box\neg S$.
- But then we get that $x_1 \nVdash \neg\Box\neg S$.
- And using the truth table of the implication we now get we get that $x_1 \nVdash S \to \neg\Box\neg S$.
- And hence $\mathcal{M}_1 \nvDash S \to \neg\Box\neg S$.
- And hence $\nvDash_D S \to \neg\Box\neg S$.

(c) What is the counterpart of the formula $S \to \neg\Box\neg S$ in LTL, if you may only use the operators $\mathcal{F}$, $\mathcal{X}$, $\mathcal{U}$, $\neg$, $\wedge$, $\vee$, $\to$ and $\leftrightarrow$, and the new formula must have the exact same meaning as the original one? *Note:* You don't have to use all these operators, but you are not allowed to use operators that aren't listed.

The counterpart of formula $S \to \neg\Box\neg S$ using only allowed operators in LTL is $S \to \mathcal{F}S$.