Formal Reasoning 2017
Solutions Test Blocks 1, 2 and 3: Additional Test
(10/01/18)

1. We have the following dictionary to talk about a Kripke model $\mathcal{M}$:

domains:	W	the worlds of $\mathcal{M}$
	A	the atomic propositions
constants:	x_0	the world x_0
	a	the atomic proposition a
predicates:	$R(x, y)$	y is a successor of x
	$V(x, y)$	the atomic proposition y is true in world x

Translate to formulas of predicate logic using this dictionary:

(a) $x_0 \Vdash \Box a \to \Diamond a$ (15 points)

This statement means:

If proposition a is true in all successors of world x_0, then there must be a successor of x_0 where proposition a holds.

If we translate this into a formula of predicate logic with the given dictionary we get:

$$(\forall y \in W\, (R(x_0, y) \to V(y, a))) \to (\exists z \in W\, (R(x_0, z) \land V(z, a)))$$

(b) $\mathcal{M}$ is serial (15 points)

This statement means:

Each world in $\mathcal{M}$ has at least one successor.

This can be expressed by the formula:

$$(\forall w \in W\, (\exists v \in W\, R(w, v)))$$

2. Let a graph $G = \langle V, E\rangle$ and a node $v_0 \in V$ be given. (20 points)

From this we recursively define sets $V_n \subseteq V$ for all $n \geq 0$ by:

$$V_0 = \{v_0\}$$
$$V_{n+1} = V_n \cup \{v \mid v \text{ is a neighbor of a node in } V_n\} \qquad \text{for all } n \geq 0$$

Prove with induction that for all $n > 0$, if there is a path of length n from v_0 to a node v, then $v \in V_n$.

Proposition: 0

If there is a path of length n from v_0 to a node v, then $v \in V_n$ for all $n \geq 1$.

Proof by induction on n. 1

We first define our predicate P as:

$P(n)$:= if there is a path of length n from v_0 to a node v, then $v \in V_n$ 2

Base Case. We show that $P(1)$ holds, i.e. we show that 3

if there is a path of length 1 from v_0 to a node v, then $v \in V_1$

This indeed holds, because 4

a path of length 1 between the nodes v_0 and v implies that there is an edge $(v_0, v) \in E$ and hence $v \in V_1$ because v is a neighbor of v_0 which is in V_0.

Induction Step. Let k be any natural number such that $k \geq 1$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6
if there is a path of length k from v_0 to a node v, then $v \in V_k$
(Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7
if there is a path of length $k+1$ from v_0 to a node v, then $v \in V_{k+1}$

This indeed holds, because if there is a path of length $k+1$ from v_0 to v, then there must be a node v_k such that there exists a path of length k from v_0 to v_k and an edge (v_k, v). The Induction Hypothesis now tells us that $v_k \in V_k$. And because of the edge (v_k, v) we know that v_k and v are neighbors, so $v \in V_{k+1}$. 8

Hence it follows by induction that $P(n)$ holds for all $n \geq 1$. 9

3. Prove that (20 points)

$$\left\{ {n \atop n-2} \right\} = \binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$$

for $n \geq 4$ using a combinatorial argument.

By definition $\left\{ {n \atop n-2} \right\}$ is the number of ways to distribute n distinguishable objects into $n-2$ indistinguishable boxes, where none of the boxes may be empty. So basically there are two possibilities:

- There is one box with three objects and all other $n-3$ boxes contain exactly one object. There are $\binom{n}{3}$ ways to choose the three objects that will be in the same box and because the boxes are indistinguishable there is only one way that we can distribute the remaining $n-3$ objects over the remaining $n-3$ boxes. So there are $\binom{n}{3}$ distributions of this type.
- There are two boxes with each two objects and all other $n-4$ boxes contain exactly one object. There are $\binom{n}{2}$ ways to choose the first two objects that will be together in a box. Then there are $\binom{n-2}{2}$ ways to choose the second two objects that will be together in a box. And there is only one way that we can distribute the remaining $n-4$ objects over the remaining $n-4$ boxes. However, we have counted all options twice, because the first and the second box can be swapped. So there are $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$ distributions of this type.

Hence the total number of different distributions is $\binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$, which was what we had to prove.

Note that all binomial coefficients used in this proof are well defined because $n \geq 4$.

4. Show that if $w \in L(M)$, where M is a deterministic finite automaton with n states and $|w| \geq n$, then one can write w as the concatenation (20 points)

$$w = uvu'$$

where $v \neq \lambda$ and such that $uv^k u' \in L(M)$ for all $k \geq 0$.

Hint: Consider the states that M subsequently goes through when processing w.

Assume that $|w| = m$ for some $m \in \mathbb{N}$. Processing w goes symbol by symbol and each symbol corresponds with a transition from a state to a (possibly other) state. So processing a word of length m requires m transitions, which means that there are exactly $m + 1$ states involved, since we always start in the initial state before processing any symbols. Because $|w| = m \geq n$ we get that $m + 1 > n$. Obviously, if an automaton only has n states and there are more than n states needed for processing a word, this means that there must be at least one state q that will be visited twice. (Formally this is based upon the well known pigeonhole principle: if you have $N + 1$ pigeons and N holes, then at least one hole should contain two pigeons.) Hence the production of w looks like:

$$q_0 \to q_1 \to \cdots \to q \to \cdots \to q \to \cdots \to q_f$$

where q_0 is the initial state and q_f a final state. This means that we can write $w = uvu'$ where u is the part of w that is being processed from q_0 to q, v the part that is being processed from q to q and u' the part that is being processed from q to q_f. Because there is at least one processing step in $q \to \cdots \to q$, we know that $v \neq \lambda$.

The claim that $uv^k u' \in L(M)$ for all $k \in \mathbb{N}$ now easily follows from the observation that k represents the number of times we do the loop $q \to \cdots \to q$.