Introduction to Formal Reasoning 2018
Exam
(11/01/19)

Before you read on, write your name, student number and study on the answer sheet! This exam consists of eighteen exercises and each of these exercises is worth five points. The mark for this test is the number of points divided by ten. The first ten points are free. Good luck!

1. This exercise is about the following formula of propositional logic:

$$a \to b \leftrightarrow a \vee b \wedge \neg a \to b$$

   (a) Write this formula with parentheses according to the official grammar from the course notes.

   (b) Give the full truth table of this formula.

2. Give a formula of propositional logic that formalizes the meaning of the following English sentence:

   I do not make New Year's resolutions, because if I make New Year's resolutions I do not keep them.

   Use for the dictionary:

M	I make New Year's resolutions
K	I keep my New Year's resolutions

3. Give a model v_3 of propositional logic that shows that:

$$a \to (b \to c) \not\equiv (a \to b) \to c$$

   Explain your answer.

4. Give an English sentence that corresponds to the meaning of the following formula of predicate logic with equality:

$$\forall x \in N \big[S(x) \leftrightarrow \exists y \in N \left[M(y, y, x) \wedge \forall z \in N \left[M(z, z, x) \to z = y\right]\right]\big]$$

   Use for the dictionary:

N	the domain of natural numbers
$S(x)$	x is a square number
$A(x, y, z)$	$x + y = z$
$M(x, y, z)$	$x \cdot y = z$

   Note: English sentences do not contain variable names.

5. Using the dictionary from the previous exercise, give a formula of predicate logic that formalizes the meaning of the following English sentence:

   Every natural number can be written as the sum of four squares.

6. Explain the relation between the following two notions in the semantics of predicate logic that are both written with the 'double turnstile' symbol:

$$\begin{aligned} (M, I) &\models f \\ &\models f \end{aligned}$$

In your explanation explicitly state what the symbols M, I and f stand for.

7. Give an infinite language L_7 over the alphabet $\Sigma = \{a, b\}$ such that

$$\begin{aligned} L_7^* &= \Sigma^* \\ L_7^R &\neq L_7 \\ L_7 L_7 &\neq L_7 \end{aligned}$$

Explain your answer.

8. Give a regular expression for the language:

$$L_8 := \{w \in \{a, b\}^* \mid w \text{ contains an even number of } a\text{'s and at most one } b\}$$

9. Give a context-free grammar for the language:

$$L_9 := \{w \in \{a, b\}^* \mid w = w^R\}$$

10. We define the context-free grammar G_{10}:

$$\begin{aligned} S &\to AB \\ A &\to aA \mid \lambda \\ B &\to bB \mid \lambda \end{aligned}$$

Someone proposes the property

$$P(w) := [w \text{ does not contain } ba]$$

as an invariant, in order to show that $abab \notin \mathcal{L}(G_{10})$. Is that correct? Explain your answer. *Note:* You only have to explain why this property is sufficient or not to show that $abab \notin \mathcal{L}(G_{10})$. If you think it is not, you don't have to come up with an improved property!

11. (a) Give a deterministic finite automaton M_{11} for the language

$$L_{11} := \mathcal{L}(a^* b^*)$$

So we want $L_{11} = L(M_{11})$.

(b) Give a right-linear context-free grammar G_{11} that corresponds to the automaton M_{11} from the previous sub-exercise.

12. Give a non-deterministic finite automaton M_{12} with only two states, for which any *deterministic* finite automaton that recognizes the same language has at least three states.

13. How many non-isomorphic graphs are there with four vertices and chromatic number three? Explain your answer.

14. We use recursion to define:

$$
\begin{aligned}
a_0 &= 0 \\
a_{n+1} &= a_n + 2n \qquad \text{for } n \geq 0
\end{aligned}
$$

Prove by induction that for all $n \geq 0$:

$$a_n = n^2 - n$$

15. Give the defining equations of the recursive definition of the binomial coefficients.

16. I believe in Murphy's law, which says that anything that can go wrong will go wrong. This means that if I bring an umbrella because I believe it will be going to rain, then it probably will not rain.

    Now consider the following English sentence, which expresses some variant of this:

    I believe that if I believe that it will rain, then it will not rain.

    (a) Give a modal formula that formalizes the meaning of this sentence. Use for the dictionary:

R	it will rain

    (b) What is the name of the logic in which the modal operators are interpreted as being about belief?

17. The modal logic D is the logic of serial Kripke models. A formula f is called true in the logic D if f is true in all serial Kripke models. The notation for this is $\models_D f$. Now show that:

$$\not\models_D \Box a \to a$$

    Explain your answer.

18. For each of the five mathematicians listed below, write down the notion from the course that was named after him, and for each of these notions give a *one line* description.

    As an example, a description of 'Pascal's triangle' – named after Blaise Pascal (French mathematician, physicist, inventor, writer and Catholic theologian, 1623–1662) – might be 'triangular array of the binomial coefficients'.

    (a) Sir Isaac Newton (English mathematician, physicist, astronomer, theologian and author, 1643–1727)
    (b) Sir William Rowan Hamilton (Irish mathematician, 1805–1865)
    (c) Augustus De Morgan (British mathematician and logician, 1806–1871)
    (d) Eric Temple Bell (Scottish-born mathematician and science fiction writer, who lived in the United States for most of his life, 1883–1960)
    (e) Stephen Cole Kleene (American mathematician, 1909–1994)