Formal Reasoning 2019
Solutions Test Block 1: Propositional and Predicate Logic
(23/09/19)

1. Consider the propositional formula:

$$\neg a \lor b \leftrightarrow \neg b \lor a$$

(a) Give the form of this formula according to the formal grammar from the course notes. (10 points)

$$((\neg a \lor b) \leftrightarrow (\neg b \lor a))$$

(b) Give the truth table of this formula. (10 points)

a	b	$\neg a$	$\neg b$	$\neg a \lor b$	$\neg b \lor a$	$\neg a \lor b \leftrightarrow \neg b \lor a$
0	0	1	1	1	1	1
0	1	1	0	1	0	0
1	0	0	1	0	1	0
1	1	0	0	1	1	1

2. We use the dictionary:

R	it rains
W	I get wet

(a) Give a propositional formula f_1 corresponding to the English sentence: *I get wet if it rains.* (5 points)

$$f_1 := R \to W$$

(b) Give a propositional formula f_2 corresponding to the English sentence: *I only get wet if it rains.* (5 points)

$$f_2 := W \to R$$

(c) Give a propositional formula f_3 corresponding to the English sentence: *I get wet, if and only if it rains.* (5 points)

$$f_3 := W \leftrightarrow R$$

(d) Does $f_1 \land f_2 \equiv f_3$ hold? Explain your answer. (10 points)
Yes, it holds. The statement that $f_1 \land f_2$ is logically equivalent to f_3 can be defined as the corresponding columns in the truth table being equal:

R	W	f_1 $R \to W$	f_2 $W \to R$	$f_1 \land f_2$ $(R \to W) \land (W \to R)$	f_3 $W \leftrightarrow R$
0	0	1	1	**1**	**1**
0	1	1	0	**0**	**0**
1	0	0	1	**0**	**0**
1	1	1	1	**1**	**1**

3. A formula f is called *valid* when $\models f$, and *satisfiable* when $\not\models \neg f$. Give an example of a propositional formula that is satisfiable but not valid, and explain why it has these properties. (10 points)

Take for example the formula $f := a$. We have the following truth table:

	f	$\neg f$
a	a	$\neg a$
0	**0**	**1**
1	**1**	**0**

The column of f contains a zero, so $\not\models f$, which means that f is not valid, but the column of $\neg f$ also contains a zero, so $\not\models \neg f$, which means that f *is* satisfiable.

(In a truth table 'satisfiable' means 'has a one', while 'valid' means 'all ones'.)

Common mistakes:

- Too often, people thought that being valid or logically true, depends on the situation in the actual real world. But it does not, it is really just a formal game with truth tables or models.
- Some people gave an example in predicate logic, where a propositional formula was requested.
- Some people gave an example f for which $\neg f$ was not valid.
- There were quite some wrong phrasings with respect to logically true, indicating that the author didn't know the concept exactly:
  - It is always logically true...
  - It is never logically true...
  - It is sometimes logically true...
  - It is logically true whenever $a = 1$...
- Again about phrasing: given a concrete formula, the truth table can have zeroes or ones. If you have the formula, you know whether it actually has zeroes and/or ones!

4. We use the dictionary:

N	the natural numbers
z	the number zero
$S(x, y)$	y is the *successor* of x, that is, $y = x + 1$

(a) Give a formula of predicate logic with equality that gives the meaning of the English sentence: *zero is not a successor of a natural number.* (10 points)

$$\neg\exists x \in N\, S(x, z)$$

or

$$\forall x \in N\, \neg S(x, z)$$

Common mistakes:

- Many people interchanged the negation and the quantifier, so had for instance:
$$\exists x \in N \, \neg S(x, z)$$
This just means that there is *some* x of which zero is not the successor (but then zero still can be the successor of a different number).
- This was not a mistake, but people thought that 'predicate logic with equality' meant that you *have* to use equality in your formula. This is not true, you *may* use it, but 'predicate logic without equality' is a subsystem of 'predicate logic with equality', so you do not need to. We guess this was the reason that many students had the correct
$$\forall x, y \in N \, (S(x, y) \to y \neq z)$$
or variants of this.
- Some people thought that you need $\exists z \in N$ to be allowed to refer to z. It is the other way around: z is a constant, while a quantifier binds a variable. For this reason z is already available without a quantifier, and cannot be used as the variable in a quantifier.
- Many people thought that they needed to exclude the case that x and z are equal. But if z is not a successor of a number, it also is not a successor of itself.
- Many people write a conjunction sign after the existential quantifier:
$$\neg \exists x \in N \wedge S(x, z)$$
Probably this is caused by sentences like '... there exists an x *and* that x satisfies ... ' But it is a syntax error: the connective $\wedge$ has to stand between two formulas and $\exists x \in N$ is not a formula in itself. Similarly, we saw the incorrect:
$$\forall x \in N \to \neg S(x, z)$$
- People sometimes write $\forall$ and $\exists$ upside down, so as A and E. We didn't see the mirrored negation sign this year, though.
- Some people wrote $\mathbb{N}$ and 0 for N and z, confusing the formula symbols with the intended interpretation.

(b) Give a formula of predicate logic with equality that gives the meaning of the English sentence: *each natural number has exactly one successor.* (15 points)

$$\forall n \in N \, \exists n' \in N \, (S(n, n') \wedge \forall x \in N \, [S(n, x) \to x = n'])$$

- Many people used z as a variable, not noticing the clash with the constant z for zero.

- Many people forget the brackets after the quantifier. Because quantifiers bind strongly, the binding then is wrong, and the variables are used in a place that is 'out of scope'. For example

$$\forall n \in N\, \exists n' \in N\, S(n, n') \wedge \forall x \in N\, S(n, x) \to x = n'$$

  means

$$([\forall n \in N\, \exists n' \in N\, S(n, n')] \wedge [\forall x \in N\, S(n, x)]) \to x = n'$$

  and the n then is not available in $S(n, x)$, while neither the x nor the n' are available in $x = n'$.
- We many times saw the incorrect 'solution':

$$\forall n \in N \exists n', n'' \in N[(S(n, n') \wedge S(n, n'')) \to n' = n'']$$

  I do not understand what is the intention here: if there is *any* n' for which the $S(n, n')$ becomes false, this already is true. Also, this way there is an implication directly under an existential quantifier: that is almost never what you want.
- Related to this many people's incorrect solutions could be 're-paired' by changing a $\forall$ to an $\exists$ or vice versa. In the above formula, one of the two $\exists$'s really is meant to be a $\forall$, we think.
- Some people just wrote:

$$\forall n \in N\, \exists n' \in N\, S(n, n')$$

  which means 'at least one' instead of 'exactly one'. I do not know whether they thought this was sufficient, or that they just hoped to get partial points this way. (They *did* get partial points.)

5. Does the following statement hold? (10 points)

$$\models \forall x, y \in D\, (x = y \to P(x) \to P(y))$$

Explain your answer.

Yes, this holds. The statement says that the formula $\models \forall x, y \in D\, (x = y \to P(x) \to P(y))$ is logically true, which means that it holds under all interpretations in all models.

Now in any model, if the two variables x and y are equal, they clearly have exactly the same properties, and therefore, whenever $P(x)$ holds, $P(y)$ also holds, and therefore from $x = y$ the statement $P(x) \to P(y)$ follows.

Common mistakes:

- When we ask to show that something holds, you should show this in terms of the definition. In this case, that means that the answers has to be in terms of the definition of the symbol '$\models$', which means it has to mention models and interpretations. Many answers did not do this, they just started talking about $P(x)$, without specifying the interpretation(s) of $P(x)$.

- The notation for implication is right associative, which means that the statement has to be read as:

$$\models \forall x, y \in D\,(x = y \to (P(x) \to P(y)))$$

Some people got this wrong. In fact

$$\not\models \forall x, y \in D\,((x = y \to P(x)) \to P(y))$$

This can be seen by taking any model in which the domain D has at least two elements and in which $P(x)$ is not everywhere true. Then we can show the statement to be false by taking $x \neq y$ and $P(y)$ false (it does not matter whether $P(x)$ is true or not).

- Some people read $f \to g \to h$ as $(f \to g) \wedge (g \to h)$. That is not what it means.

- Many people thought this had something to do with empty domains. It does not, as the statement also will be true when the interpretation of D is empty (in that case a formula $\forall x \in D\, f$ is always true no matter what f is). So an empty domain is not a special case that needs to be considered separately here.

- Many people had truth tables. The definition of $\models$ is in terms of truth tables *only* works for propositional logic, and this is a formula of predicate logic. For this reason answering this question by just giving a truth table and claiming there are only 1's in a column is certainly wrong. (You can use truth tables to analyze what the truth values of $P(x) \to P(y)$ can be, though.)

- Worse, many people gave truth values to x and y: often they had a truth table with x and y as two columns, or had text about $x = 0$ or $x = 1$, and so on. That does not make sense at all. These variables are interpreted in D and are not truth values.

- Also, people talked about $P(x) = P(y)$. That does not make sense either. The values of $P(x)$ and $P(y)$ *are* truth values. At best, you should write $P(x) \leftrightarrow P(y)$.