Type theory and proof assistants
answers

1.

$$\lambda x : a \to b \to c.\ \lambda y : b.\ \lambda z : a.\, xzy$$

(This term corresponds to the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a \to b \to c^x] \quad [a^z]}{b \to c}\,E\!\to \qquad [b^y]}{c}\,E\!\to}{a \to c}\,I[z]\!\to}{b \to a \to c}\,I[y]\!\to}{(a \to b \to c) \to b \to a \to c}\,I[x]\!\to$$

but this proof is not part of the answer.)

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash x : a \to b \to c \quad \Gamma \vdash z : a}{\Gamma \vdash xz : b \to c} \qquad \Gamma \vdash y : b}{\Gamma \vdash xzy : c}}{x : a \to b \to c,\, y : b \vdash (\lambda z : a.\, xzy) : a \to c}}{x : a \to b \to c \vdash (\lambda y : b.\, \lambda z : a.\, xzy) : b \to a \to c}}{(\lambda x : a \to b \to c.\ \lambda y : b.\ \lambda z : a.\, xzy) : (a \to b \to c) \to b \to a \to c}$$

where we used the abbreviation $\Gamma := x : a \to b \to c,\, y : b,\, z : a$.

3.

$$\dfrac{\dfrac{\dfrac{[A^y]}{A \to A}\,I[y]\!\to \qquad [A^x]}{A}\,E\!\to}{A \to A}\,I[x]\!\to$$

$$\dfrac{[A^x]}{A \to A}\,I[x]\!\to$$

This corresponds to the reduction

$$\lambda x : A.\, (\lambda y : A. y)\, x \ \to_\beta\ \lambda x : A.\, x$$

4.

$$\Pi A : *.\, \texttt{list}\ A\ \texttt{0}$$

In Coq notation this is

```
forall A : Set, list A O
```

5. ```
   forall P : tree -> Prop,
     P leaf ->
     (forall t1 : tree, P t1 -> forall t2 : tree, P t2 ->
      P (node t1 t2)) ->
     forall t : tree, P t
   ```

6. $$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{A : * \vdash A : *}}{A : *,\, x : A \vdash x : A} \qquad \dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{A : * \vdash A : *} \qquad \dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{A : * \vdash A : *} \quad \dfrac{\dfrac{}{\vdash * : \Box}}{A : * \vdash A : *}}{A : *,\, x : A \vdash A : *}}{A : * \vdash A \to A : *}}{A : * \vdash (\lambda x : A.\, x) : A \to A}$$

7. $$\begin{array}{c} \lambda a : *.\, \lambda x : (\Pi c : *.\, (a \to c) \to c).\, x a\, (\lambda y : a.\, y) \\ : \\ \Pi a : *.\, (\Pi c : *.\, (a \to c) \to c) \to a \end{array}$$

   (This term corresponds to the proof

   $$\dfrac{\dfrac{\dfrac{\dfrac{[\forall c.\, (a \to c) \to c^x]}{(a \to a) \to a}\, E\forall \qquad \dfrac{[a^y]}{a \to a}\, I[y]{\to}}{a}\, E{\to}}{(\forall c.\, (a \to c) \to c) \to a}\, I[x]{\to}}{\forall a.\, (\forall c.\, (a \to c) \to c) \to a}\, I\forall$$

   but this proof is not part of the answer.)

8. $$\mathbb{N} := \Pi a : *.\, a \to (a \to a) \to a$$
   $$2 := \lambda a : *.\, \lambda z : a.\, \lambda s : a \to a.\, s\, (s z)$$

9.

system	*judgments*
$\lambda{\to}$	1, 2, 4
λP	1, 2, 4, 5
$\lambda 2$	1, 2, 3, 4

10. ```
    Inductive even : nat -> Prop :=
    | even_O : even O
    | even_SS : forall n : nat, even n -> even (S (S n)).
    ```

11. $$\Phi_{rmeven}(X) := \{0\} \cup \{n+2 \mid n \in X\}$$

    Φ_{even} is order-preserving means that $\Phi_{\text{even}}(X) \subseteq \Phi_{\text{even}}(Y)$ when $X \subseteq Y$.

12. Take for L the lattice from the previous exercise and for Φ

$$\Phi(X) = \mathbb{N} \setminus X$$

$$H = \{X \mid X \subseteq \mathbb{N} \setminus X\} = \{\emptyset\}$$

$$\bigvee_L H = \bigvee_L \{\emptyset\} = \emptyset$$

13. Type checking: given Γ, M and A, determine whether $\Gamma \vdash M : A$ is a derivable judgment.

    Type synthesis: given Γ and M, determine whether an A exists such that $\Gamma \vdash M : A$ is a derivable judgment, and if so, find one.

    Type inhabitation: given Γ and A, determine whether an M exists such that $\Gamma \vdash M : A$ is a derivable judgment, and if so, find one.

    The first two are decidable for λP, while the last is not decidable.

14. If the last step of the derivation was a λ rule

$$\dfrac{\dfrac{\ldots}{\Gamma,\, x : B,\, y : C \vdash N : \ldots} \qquad \ldots}{\Gamma,\, x : B \vdash (\lambda y : C.\, N) : \ldots}$$

    one would like to use the induction hypothesis for the derivation of $\Gamma,\, x : B,\, y : C \vdash N : \ldots$ to obtain a derivation

$$\dfrac{\dfrac{\ldots}{\Gamma,\, y : C[x := P] \vdash N[x := P] : \ldots} \qquad \ldots}{\Gamma \vdash (\lambda y : C.\, N)[x := P] : \ldots}$$

    However, this does not work, because there x is not the last variable in the context.

    The way to solve this problem is to use induction loading and instead prove

$$\begin{array}{rl} & \Gamma,\, x : B,\, \Delta \vdash M : A \quad \text{and} \quad \Gamma \vdash P : B \\ \text{then} & \Gamma, \Delta[x := P] \vdash M[x := P] : A[x := P] \end{array}$$

    The lemma of the exercise is the special case of this where Δ is the empty context.

15. There are two cases:

- If $A = a$, then we know that $M[x := N]\vec{P}$ is strongly normalizing, and hence M and the terms in $\vec{P}$ are also strongly normalizing. This means that there are only finitely many reduction steps possible in $(\lambda x.\, M)\, N\vec{P}$ which do not contract the redex. Once we contract the redex, we get a reduct of $M[x := N]\vec{P}$ (by applying the reductions of M, N and $\vec{P}$ that we already did to $M[x := N]\vec{P}$) and we are in a reduct of a strongly normalizing term. Which cannot have infinitely many reductions.
- If $A = B \to C$ then we know that

  $$\forall P' \in [\![B]\!].\, M[x := N]\vec{P}P' \in [\![C]\!]$$

  and need to show that

  $$\forall P' \in [\![B]\!].\, (\lambda x.M)\, N\vec{P}P' \in [\![C]\!]$$

  But that immediately follows from the induction hypothesis, because we are doing induction on the structure of the type.

  (Note that we do not need the induction hypothesis for B.)