Type Theory and Coq, 2011

17 January 2012, 10.30–12.30

Before you start, write your name, student number and study at the top of your paper. The final mark is the number of points divided by ten, where the first 10 points are free. Good luck!

1. (a) Give a proof in minimal propositional logic of the proposition

$$((((a \to b) \to a) \to a) \to b) \to b$$

(4 points)

   (b) Give the $\lambda{\to}$ term that corresponds to this proof. (3 points)

2. (a) Give a proof in minimal second order propositional logic of the proposition

$$\forall b : *. \, ((\forall a : *. \, (((a \to b) \to b) \to a)) \to (\forall a : *. \, (b \to a)))$$

(4 points)

   (b) Give the $\lambda 2$ term that corresponds to this proof. (3 points)

   (c) Give the $\lambda 2$ type of this term. (2 points)

3. (a) Give a $\lambda 2$ type that corresponds to an impredicative definition of the formula

$$A \wedge B$$

where $A : *$ and $B : *$ are $\lambda 2$ types. (3 points)

   (b) What is an elimination rule of minimal propositional logic (not one of the usual elimination rules of the conjunction) that corresponds to this definition? (3 points)

4. (a) Give the λP type derivation of the type judgment:

$$d : *, \, a : * \vdash a : *$$

For the typing rules of λP see page 5. (3 points)

   (b) Give the λP type derivation of the type judgment:

$$d : *, \, a : * \vdash d : *$$

(3 points)

(c) Give the λP type derivation of the type judgment:

$$d : *,\ a : * \vdash (a \to \Pi x : d.\, a) : *$$

You may replace subderivations for the judgments from the previous two exercises by dots. (Hint: it might be efficient not to postpone weakening until the end.) (4 points)

(d) Give a λP term that inhabits the type that is typed by this last judgment. (3 points)

(e) What is the proposition of minimal first order predicate logic that corresponds to this type, if we take d to be the domain of quantification, and a to be a proposition? (2 points)

5. (a) Give the Coq notation for the expression:

$$\lambda x : A.\, M$$

What is the motivation for this Coq notation? (2 points)

(b) Give the Coq notation for the expression:

$$\Pi x : A.\, B$$

What is the motivation for this Coq notation? (2 points)

(c) What are the Coq counterparts of the sorts $*$ and $\Box$? (3 points)

(d) What are the types in the λ-cube of $*$ and $\Box$? (2 points)

(e) What are the types of the Coq counterparts of these sorts? (3 points)

6. Consider the following six types

$$\begin{array}{l} \Pi x : a.\, b \\ \Pi x : a.\, p\, x \\ \Pi a : *.\, b \\ \Pi a : *.\, a \\ \Pi x : a.\, * \\ \Pi a : *.\, * \end{array}$$

in which $a : *$, $b : *$ and $p : a \to *$.

(a) Which of these types can also be written with $\to$ notation, instead of using Π? For the types that can be written that way, write them using $\to$ notation. (3 points)

(b) Which of these types are allowed in $\lambda\!\to$? (You do not have to give explanations, and no type derivations are required.) (2 points)

(c) Which of these types are allowed in λP? (You do not have to give explanations, and no type derivations are required.) (3 points)

(d) Which of these types are allowed in $\lambda 2$? (You do not have to give explanations, and no type derivations are required.) (3 points)

(e) What are the types of these six types in the λ-cube? (3 points)

7. (a) Give an inductive Coq definition of a type

```
boollist : Set
```

of lists of Booleans, where the Coq type of Booleans is called `bool`. (3 points)

(b) Give the induction principle for the type that you defined. (3 points)

(c) Give a recursive Coq definition of a function

```
andblist : boollist -> bool
```

that takes the conjunction of all the Booleans in the list. You can use the following functions:

```
true  : bool
false : bool
andb  : bool -> bool -> bool
```

(4 points)

8. (a) For a natural number n we define a sequence a_i as follows:

$$\begin{aligned} a_0 &= n \\ a_{i+1} &= \begin{cases} a_i/2 & \text{if } a_i \text{ is even} \\ 3\,a_i + 1 & \text{if } a_i \text{ is odd} \end{cases} \end{aligned}$$

For example, for $n = 3$, this sequence is $3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots$ Define an inductive Coq predicate

```
seq : nat -> nat -> nat -> Prop
```

where

```
seq n i a
```

expresses that `a` is the $a_{\mathtt{i}}$ from the sequence corresponding to the number `n`.

You can use Leibniz equality, numerals, logical operations and the functions:

```
S    : nat -> nat
plus : nat -> nat -> nat
mult : nat -> nat -> nat
even : nat -> Prop
```

(5 points)

(b) A natural number is called *convergent* if it is 0, or if its sequence contains the number 1. Using the predicate from the previous exercise, give a Coq definition of a predicate

```
convergent : nat -> Prop
```

that says that its argument is convergent. (3 points)

(c) Using the predicate from the previous exercise, define a Coq formula that states that all natural numbers are convergent. (2 points)

9. (a) Give a natural deduction proof in classical minimal propositional logic with disjunction of the proposition

$$(\neg\neg a \to a) \vee \bot$$

where we used the abbreviation $\neg A := A \to \bot$. For the rules of this logic see page 6. It is sufficient to give the proof without the Γ and Δ contexts, i.e., in the style that is used for logic in Femke's course notes. You may also omit the names of the rules. (Hint: there is a proof that does not use the $E\vee$ rule and ends with an *activate* rule.) (4 points)

(b) Give the term in the $\lambda\mu{+}$ calculus that corresponds to this proof. For the rules of this calculus see also page 6. (3 points)

Typing rules of the calculi λP and $\lambda 2$

In these rules the variable s ranges over the set of sorts $\{*, \Box\}$. The product rule differs between λP and $\lambda 2$.

axiom

$$\frac{}{\vdash * : \Box}$$

variable

$$\frac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A}$$

weakening

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma,\, x : C \vdash A : B}$$

application

$$\frac{\Gamma \vdash M : \Pi x : A.\, B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[x := N]}$$

abstraction

$$\frac{\Gamma,\, x : A \vdash M : B \qquad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}$$

product (λP)

$$\frac{\Gamma \vdash A : * \qquad \Gamma,\, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

product ($\lambda 2$)

$$\frac{\Gamma \vdash A : s \qquad \Gamma,\, x : A \vdash B : *}{\Gamma \vdash \Pi x : A.\, B : *}$$

conversion

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \quad \text{where } B =_\beta B'$$

Classical minimal propositional logic with disjunction

$$\frac{}{\Gamma, A \vdash A; \Delta}\,axiom$$

$$\frac{\Gamma \vdash \perp\!\!\!\perp; A, \Delta}{\Gamma \vdash A; \Delta}\,activate \qquad \frac{\Gamma \vdash A; A, \Delta}{\Gamma \vdash \perp\!\!\!\perp; A, \Delta}\,passivate$$

$$\frac{\Gamma, A \vdash B; \Delta}{\Gamma \vdash A \to B; \Delta}\,I{\to} \qquad \frac{\Gamma \vdash A \to B; \Delta \qquad \Gamma \vdash A; \Delta}{\Gamma \vdash B; \Delta}\,E{\to}$$

$$\frac{\Gamma \vdash A; \Delta}{\Gamma \vdash A \vee B; \Delta}\,I_l{\vee} \qquad \frac{\Gamma \vdash B; \Delta}{\Gamma \vdash A \vee B; \Delta}\,I_r{\vee}$$

$$\frac{\Gamma \vdash A \vee B; \Delta \qquad \Gamma, A \vdash C; \Delta \qquad \Gamma, B \vdash C; \Delta}{\Gamma \vdash C; \Delta}\,E{\vee}$$

The typing rules of the calculus $\lambda\mu+$

$$\frac{(x : A) \in \Gamma}{\Gamma; \Delta \vdash x : A}$$

$$\frac{\Gamma; \Delta, \alpha : A \vdash c : \perp\!\!\!\perp}{\Gamma; \Delta \vdash (\mu\alpha : A.\, c) : A} \qquad \frac{\Gamma; \Delta \vdash M : A \qquad (\alpha : A) \in \Delta}{\Gamma; \Delta \vdash [\alpha]M : \perp\!\!\!\perp}$$

$$\frac{\Gamma, x : A; \Delta \vdash M : B}{\Gamma; \Delta \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma; \Delta \vdash M : A \to B \qquad \Gamma; \Delta \vdash N : A}{\Gamma; \Delta \vdash MN : B}$$

$$\frac{\Gamma; \Delta \vdash M : A}{\Gamma; \Delta \vdash \mathsf{inl}\, M : A + B} \qquad \frac{\Gamma; \Delta \vdash M : B}{\Gamma; \Delta \vdash \mathsf{inr}\, M : A + B}$$

$$\frac{\Gamma; \Delta \vdash M : A + B \qquad \Gamma, x_l : A; \Delta \vdash N_l : C \qquad \Gamma, x_r : B; \Delta \vdash N_r : C}{\Gamma; \Delta \vdash (\mathsf{case}\, M\, \mathsf{of}\, x_l : A.\, N_l \mid x_r : B.\, N_r\, \mathsf{end}) : C}$$