4.4.1. The Stone Theorem. *Every open cover of a metrizable space has an open refinement which is both locally finite and σ-discrete.*

Proof. Let $\{U_s\}_{s\in S}$ be an open cover of a metrizable space X; take metric ρ on the space X and a well-ordering relation $<$ on the set S. Define inductive families $\mathcal{V}_i = \{V_{s,i}\}_{s\in S}$ of subsets of X by letting

$$V_{s,i} = \bigcup B(c, 1/2^i),$$

where the union is taken over all points $c \in X$ satisfying the following conditions:

$$s \text{ is the smallest element of } S \text{ such that } c \in U_s. \tag{1}$$

$$c \notin V_{t,j} \text{ for } j < i \text{ and } t \in S. \tag{2}$$

$$B(c, 3/2^i) \subset U_s. \tag{3}$$

It follows from the definition that the sets $V_{s,i}$ are open, and (3) implies that $V_{s,i} \subset U_s$. Let x be a point of X; take the smallest $s \in S$ such that $x \in U_s$ and a natural number i such that $B(x, 3/2^i) \subset U_s$. Clearly, we either have $x \in V_{t,j}$ for a $j < i$ and a $t \in S$ or $x \in V_{s,i}$. Hence, the union $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ is an open refinement of the cover $\{U_s\}_{s\in S}$.

We shall prove that for every i

$$\text{if } x_1 \in V_{s_1,i},\ x_2 \in V_{s_2,i} \text{ and } s_1 \neq s_2, \text{ then } \rho(x_1, x_2) > 1/2^i, \tag{4}$$

and this will show that the families $\mathcal{V}_i$ are discrete, because every $1/2^{i+1}$-ball meets at most one member of $\mathcal{V}_i$.

Let us assume that $s_1 < s_2$. By the definition of $V_{s_1,i}$ and $V_{s_2,i}$ there exist points c_1, c_2 satisfying (1)-(3) such that $x_k \in B(c_k, 1/2^i) \subset V_{s_k,i}$ for $k = 1, 2$. From (3) it follows that $B(c_1, 3/2^i) \subset U_{s_1}$ and from (1) we see that $c_2 \notin U_{s_1}$, so that $\rho(c_1, c_2) \geq 3/2^i$. Hence,

$$\rho(x_1, x_2) \geq \rho(c_1, c_2) - \rho(c_1, x_1) - \rho(c_2, x_2) > 1/2^i,$$

which proves (4).

To conclude the proof of the theorem it suffices to show that for every $t \in S$ and any pair k, j of natural numbers

$$\text{if } B(x, 1/2^k) \subset V_{t,j}, \text{ then } B(x, 1/2^{i+k}) \cup V_{s,i} = \emptyset \text{ for } i \geq j + k \text{ and } s \in S, \tag{5}$$

because for every $x \in X$ there exist k, j and t such that $B(x, 1/2^k) \subset V_{t,j}$ and thus the ball $B(x, 1/2^{j+k})$ meets at most $j + k - 1$ members of $\mathcal{V}$.

It follows from (2) that the points c in the definition of $V_{s,i}$ do not belong to V_t, j whenever $i \geq j + k$; since $B(x, 1/2^k) \subset V_{t,j}$, we have $\rho(x, c) \geq 1/2^k$ for any such c. The inequalities $j + k \geq k + 1$ and $i \geq k + 1$ imply that $B(x, 1/2^{j+k}) \cup B(c, 1/2^i) = \emptyset$, and this yields (5). $\diamondsuit$