Chapter V

SEPARATION THEOREMS

§ 1. CHAINS ON A GRATING

1. It is convenient in the following chapters to use X^2 to denote either R^2 (the open plane) or Z^2 (the closed plane). In any one argument X^2 denotes, of course, one definite space.

A *rectangular grating*, **G**, in the open or closed plane, is formed by drawing a finite number of segments across a square, parallel to its sides. The lines go right across the square: divisions like that in Fig. 22(b) are not allowed. The grating in which the square is left undivided is not excluded.

Fig. 22

For convenience we suppose that the original square (the *frame* of the grating) has its centre at the origin and its sides parallel to the coordinate axes, so that the words "horizontal", "above", "below", etc. may be used in an obvious sense.

The 2-*cells* of G are (1) the closures of the rectangular domains into which the interior of the frame is divided, and (2) the closure of the exterior of the frame (including the point at infinity if there is one). The *edges*, or 1-*ceIls*, are the sides of the 2-cells of finite diameter, and the *vertices*, or 0-*cells*, are their corners.

Thus all cells are bounded closed sets of points, except the 2-cell outside the frame, which is closed but not bounded.†

Every edge of the grating evidently belongs to just two 2-cells, which lie on opposite sides of it.

Theorem 1·1. *If F_1 and F_2 are non-intersecting closed sets in Z^2, there exists a grating,* **G**, *no cell of which meets both F_1 and F_2.*

† To avoid confusion with another sense of "bounded", used in this chapter, this will be called the *infinite* 2-cell; and all others *finite* cells.

At least one of the closed sets, say F_1, does not contain w, and therefore lies, for some α, in the interior of the set

$$E_\alpha : [|\xi_1| \leq \alpha, |\xi_2| \leq \alpha].$$

The non-intersecting compact sets $F_2 E_\alpha$ and F_1 in R^2 are at a positive distance δ apart. Let m be an integer exceeding $4\alpha/\delta$. If the frontier of E_α is taken as frame, and m equidistant segments are drawn between, and parallel to, each pair of sides the result is evidently a grating with the required property.

In R^2 the additional condition that em at least one of the closed sets is compact is required. The proof is then as before.

Fig. 23 Fig. 24

2. A *k-chain*, C^k $(k = 0, 1, 2)$, on a grating $\mathbf{G}$, is any set of k-cells of $\mathbf{G}$. The *sum* (modulo 2) of two k-chains, C_1^k and C_2^k, is the set of k-cells that belong to one, but not both, of C_1^k and C_2^k. It is denoted by $C_1^k + C_2^k$. The complement $\mathcal{C}C^2$ of a 2-chain C^2 is the set of 2-cells of $\mathbf{G}$ not belonging to C^2. Thus if Ω^2 denotes the 2-chain containing all the 2-cells of $\mathbf{G}$, $\mathcal{C}C^2 = C^2 + \Omega^2$.

Example. 1. In Fig. 23 the set of three shaded cells is a 2-chain, and the six thick edges form a 1-chain. In Fig. 24, if C_1^2 and C_2^2 are the 2-chains whose cells are those inside the rectangles *abcd* and *pqrs*, $C_1^2 + C_2^2$ is the set of shaded 2-cells. $\mathcal{C}C_1^2 + \mathcal{C}C_2^2$ is also the set of shaded 2-cells.

Addition of k-chains is commutative and associative, and

$$\sum_1^q C_i^k = C_1^k + C_2^k + \ldots + C_q^k$$

is the set of k-cells that belong to an odd number of the chains C_i^k. Clearly for any C^k, $C^k + C^k = 0$ (the "zero chain"), and hence the equation $C_1^k + X^k = C_2^k$ is satisfied by $X^k = C_1^k + C_2^k$ and no other k-chain. Thus the k-chains on $\mathbf{G}$ form a commutative group under the operation of addition modulo 2. Some of the consequences of this fact are considered in the final section of Chapter VI, but otherwise no use is made of the language or theorems of group-theory.

The *boundary*, $\dot{C}^k$, of the k-chain C^k on $\mathbf{G}$ is (for $k = 1, 2$) the set of $(k-1)$-cells of $\mathbf{G}$ that are contained in an odd number of k-cells of C^k. (The boundary of a 0-chain is not defined.)

Example. 2. The boundary of an edge is its pair of end-points. The boundary of the 1-chain in Fig. 23 is the set of vertices a, b, c, d. The boundary of the shaded 2-chain in Fig. 24 is the set of the thickened edges.

Theorem 2·1. $(C_1^k + C_2^k)^{\cdot} = \dot{C}_1^k + \dot{C}_2^k \quad (k = 1, 2)$.

A $(k-1)$-cell A^{k-1} belonging to n_1 k-cells of C_1^k and n_2 of C_2^k belongs to $\dot{C}_i^k$ if n_i is odd, and therefore to

$$\dot{C}_1^k + \dot{C}_2^k$$

if $n_1 + n_2$ is odd. This is the condition that it belongs to $(C_1^k + C_2^k)^{\cdot}$.

By induction on the number of chains it follows that

$$(C_1^k + C_2^k + \dots C_q^k)^{\cdot} = \dot{C}_1^k + \dot{C}_2^k + \dots + \dot{C}_q^k.$$

In particular the boundary of any k-chain is the sum, mod 2, of the boundaries of its k-cells.

Since $\dot{\Omega}^2 = 0$ it follows from 2·1 that, for any C^2,

$$(\mathcal{C}C^2)^{\cdot} = (C^2 + \Omega^2)^{\cdot} = \dot{C}^2.$$

A *k-cycle* is, for $k = 1, 2$, a k-chain whose boundary is zero; a 0-*cycle* is a 0-chain with an even number of 0-cells. The sum mod 2 of any set of k-cycles is a k-cycle (by 2·1, or directly for $k = 0$).

Theorem 2·2. *The boundary of any k-chain is a $(k-1)$-cycle* $(k = 1, 2)$.

The boundary of a 1-cell is two vertices—a 0-cycle. The boundary of a 2-cell is a 1-chain in which every vertex belongs to just two edges, i.e. it is a 1-cycle. From these special cases the theorem follows by addition.

3. The k-chain C^k (a finite set whose members are k-cells) is to be distinguished from the union of its k-cells, a set of points denoted by $|C^k|$ and called the *locus of C^k*, or the *set of points covered by* C^k. The functions, $\mathcal{F}$, $\mathcal{K}$ (or $^-$) and $\mathcal{J}$ are applicable to the locus $|C^k|$ but not, of course, to C^k itself. To avoid clumsy constructions, a chain (instead of, more correctly, the locus of a chain) will often be said to contain a point, to lie in a set of points, or to meet a set or another chain; but this licence will not extend to symbolic relations.†

Clearly $|C_1^k + C_2^k| \subseteq |C_1^k| \cup |C_2^k|$ in all cases. The two sets are identical if, and only if, C_1^k and C_2^k have no common k-cell. (Thus in Example 1 above, $|C_1^k| \cup |C_2^k|$ is the whole closed region bounded by the polygon *abyqrspx*, $|C_1^k + C_2^k|$ is the shaded region.) Note that whereas $\mathcal{C}|C^2|$ is an open set, $|\mathcal{C}C^2|$ is closed, and is in fact $\overline{\mathcal{C}|C^2|}$.

A k-chain C^k is, by definition, *connected* if its locus $|C^k|$ is connected. The maximal connected k-chains contained in any k-chain C^k are called the *components* of C^k. They have as loci the components of $|C^k|$.

Theorem 3·1. *If K^k is a component of C^k, $\dot{K}^k$ is the part of $\dot{C}^k$ in K^k $(k > 0)$.*

For if a cell A^{k-1} is contained in a k-cell of K^k, all the cells of C^k containing A^{k-1} are in K^k, and therefore A^{k-1} has the same parity in C^k and K^k.

The following simple deduction from 3·1 is one of the main links between the combinatorial theory of this section and the point-set notions of Part I.

Theorem 3·2. *If x and y form the boundary of a 1-chain C^1, they are connected in $|C^1|$.*

If not, the component of C^1 containing x has, by 3·1, the single odd vertex x as its boundary, contrary to 2·2.

Theorem 3·3. *For any C^2, $\mathcal{F}|C^2| = |\dot{C}^2|$.*

If $x \in |\dot{C}^2|$ then

$$x \in |C^2| \cap |\mathcal{C}C^2| = |C^2| \cap \overline{\mathcal{C}|C^2|} = \mathcal{F}|C^2|.$$

If $x \in \mathcal{F}|C^2|$, at least one 2-cell containing x is in C^2, and one is in $\mathcal{C}C^2$. Therefore a 2-cell of C^2 and one of $\mathcal{C}C^2$ have an edge in common that contains x. Thus $x \in |\dot{C}^2|$.

† Thus we shall write "x lies in C^k", but "$x \in |C^k|$".

It follows from 3·3 that Ω^2 and 0 are the only 2-cycles. Hence if $\dot{C}^2$ is connected so also is C^2, for either $C^2 = \Omega^2$ or else every component of C^2 contains a non-null component of $\dot{C}^2$.

4. It is sometimes necessary to pass from one grating, $\mathbf{G}$, to another, $\mathbf{G}^*$, by introducing additional lines; or by enlarging the frame, i.e. by extending all the lines to meet a larger square containing the original frame; or by a combination of both processes. Such a new grating $\mathbf{G}^*$ is called a *refinement* of the old. (It is convenient to agree that $\mathbf{G}$ is a refinement of itself.) A common refinement can be formed for any two gratings $\mathbf{G}_1$ and $\mathbf{G}_2$, by taking as frame any square, with sides parallel to the axes, containing the frames of $\mathbf{G}_1$, and $\mathbf{G}_2$ and as cross-lines all the lines of $\mathbf{G}_1$ and $\mathbf{G}_2$, suitably extended.

To each k-chain, C^k, on $\mathbf{G}$ corresponds the *subdivided* k-chain C^k_* on $\mathbf{G}^*$, which is the sum of the k-chains into which the k-cells of C^k are subdivided. (0-chains are unaltered by subdivision: $C^0_* = C^0$.) A subdivided chain has the same locus as its original: $|C^k_*| = |C^k|$.

Theorem 4·1. $(C^k_1 + C^k_2)_* = C^k_{1\,*} + C^k_{2\,*}$, $(\dot{C}^k)_* = (C^k_*)^{\cdot}$.

The first relation is clear from the definitions.

The effect of subdivision on an edge is to introduce new intermediate vertices, but since they each belong to two new edges the boundary of the resulting 1-chain is still the original two endpoints. Hence, by addition, $(C^1_*)^{\cdot} = \dot{C}^1 = (\dot{C}^1)_*$. If C^2 is any 2-chain,

$$|(C^2_*)^{\cdot}| = \mathcal{F}|C^2_*| = \mathcal{F}|C^2| = |\dot{C}^2| = |(\dot{C}^2)_*|.$$

Thus $(C^2_*)^{\cdot}$ and $(\dot{C}^2)_*$ are chains on $\mathbf{G}^*$ with the same locus, and therefore identical.

Corollary. C^k_* is a k-cycle if, and only if, C^k a k-cycle.

5. Theorem 5·1. *Every* 1-*cycle, on a rectangular grating in* Z^2 *or* R^2 *is the boundary of just two* 2-*chains.*

This theorem is called the Fundamental Lemma because, in spite of its simplicity, it is the geometrical kernel of the whole of the theory in this Chapter. It is a property of the plane and sphere, and distinguishes them topologically from all other surfaces. For example, if the surface of a solid ring is divided into squares, a meridian circle, though a "1-cycle", is not the boundary of any 2-chain on the surface (Fig. 25).

The proof is by induction on the number of lines drawn across the original square to make the grating. On the grating consisting of the frame alone, the only 1-cycles are the null-set, which bounds two 2-chains (the zero-chain and Ω^2); and the square itself, which bounds the two 2-cells possessed by this grating.

Fig. 25

Fig. 26

Let the given grating, $\mathbf{G}_1$, be formed from a grating $\mathbf{G}_0$, for which the theorem may be assumed true, by the addition of a line λ across the square, which we may suppose to be parallel to the ξ_1-axis. Let Γ^1 be the given 1-cycle on $\mathbf{G}_1$. We denote by C^2 the sum of the 2-cells of $\mathbf{G}_1$, whose lower edges lie in the line λ and belong to Γ^1. The 1-cycle $\Gamma^1 + \dot{C}^2$ therefore contains no edge in λ. *It follows that it is the subdivided form of a* 1*-cycle* Γ_0^1 *on* $\mathbf{G}_0$, for $\Gamma^1 + \dot{C}^2$ contains no horizontal edge at a vertex, x, of λ, and therefore, since it is a cycle, contains both or neither of the vertical edges at x, which together make up an edge of $\mathbf{G}_0$.

By hypothesis there is a 2-chain C_0^2. on $\mathbf{G}_0$ such that $\Gamma_0^1 = \dot{C}_0^2$, and if C_1^2 is the subdivided form of C_0^2 on $\mathbf{G}_1$, $\dot{C}_1^2 = \Gamma^1 + \dot{C}^2$. Hence

$$(C_1^2 + C^2)^{\boldsymbol{\cdot}} = (\Gamma^1 + \dot{C}^2) + \dot{C}^2 = \Gamma^1.$$

The residual 2-chain $\mathcal{C}(C_1^2 + C^2)$ also has boundary Γ^1.

It has thus been shewn that Γ^1 bounds at least two 2-chains. *No* 1*-cycle bounds more than two* 2*-chains.* For if $\dot{C}_a^2 = \dot{C}_b^2$, $(C_a^2 + C_b^2)^{\boldsymbol{\cdot}} = 0$ and therefore $C_a^2 + C_b^2$ is 0 or Ω^2, i.e. $C_b^2 = C_a^2$ or $\mathcal{C}C_a^2$.

6. The convention is now introduced that if C^k is a k-chain on a grating $\mathbf{G}$, the corresponding subdivided chain on any refinement $\mathbf{G}^*$ shall be denoted by the same symbol C^k, with the addition when necessary of the words "on $\mathbf{G}^*$". Thus if C_1^k and C_2^k are chains on the gratings $\mathbf{G}_1$ and $\mathbf{G}_2$, and $\mathbf{G}^*$ is any common refinement of $\mathbf{G}_1$ and $\mathbf{G}_2$, "$C_1^k + C_2^k$ (on $\mathbf{G}^*$)" will denote $(C_1^k)_* + (C_2^k)_*$. No ambiguity can arise from the various orders in which a combination of subdivision, addition, and taking the boundary can be carried out, since, by 2·1 and 4·1 the result is always the same.

Let G be an open set and Γ^k a k-cycle on $\mathbf{G}$ in G. The cycle Γ^k *bounds in* G ($\Gamma^k \sim 0$ in G) if there is, on some refinement $\mathbf{G}^*$ of $\mathbf{G}$, a $(k+1)$-chain $C^(k+1)$ such that $|C^{k+1}| \subseteq G$ and $\dot{C}^{k+1} = \Gamma^k$ on $\mathbf{G}^*$. "Γ^k is non-bounding in G" means that $|\Gamma^k| \subseteq G$ but Γ^k does not bound in G.

Examples. 1. If A^2 is a 2-cell of $\mathbf{G}$, $\dot{A}^2 \sim 0$ in X^2 on $\mathbf{G}$ itself; but if $a \in \mathcal{J}|A^2|$ and $a \in \mathcal{C}|A^2|$ does not bound in $X^2 - (a \cup b)$ because (as will be shewn below) every 2-chain bounded by $\dot{A}^2$ contains a or b.

2. If x and y are finite points of the plane, $(x) + (y) \sim 0$ in X^2, for if $\mathbf{G}$ is a grating having x and y as vertices, $(x) + (y)$ bounds either an edge or an $\lfloor$-shaped 1-chain on $\mathbf{G}$.

It is often unnecessary to make explicit reference to a grating in the enunciation of theorems. Thus the opening phrase of the following theorem (6·1) means "If Γ_i^k is a k-cycle in G on a grating $\mathbf{G}_i$" or still more explicitly "If Γ_i^k is a k-cycle on $\mathbf{G}_i$ and $|\Gamma_i^k| \subseteq G$". Similarly if x and y are any points, "$(x) + (y) \sim 0$ in G" means "$(x) + (y)$, regarded as a 0-cycle on some grating $\mathbf{G}$ having x and y as vertices, bounds in G".

Theorem 6·1. *If Γ_i^k is a cycle in G and $\Gamma_i^k \sim 0$ in G, for*

$$i = 1, 2, \ldots, q,$$

then $\sum_1^q \Gamma_i^k \sim 0$ in G.

If $\Gamma_i^k = \dot{C}_i^{k+1}$ on $\mathbf{G}_i^*$, and if $\mathbf{G}_0$ is a common refinement of all the $\mathbf{G}_i^*$, then

$$\sum_1^q \Gamma_i^k = \Big(\sum_1^q C_i^{k+1}\Big)^{\cdot}$$

on $\mathbf{G}_0$.

Corollary. *If Γ^k is non-bounding in G, at least one of its components is non-bounding in G,* for since the components do not meet each other their (mod 2) sum is Γ^k.

Theorem 6·2. *If $\mathcal{C}G$ is connected every 1-cycle in G bounds in G.*

For if $\mathcal{C}G$ met both the 2-chains bounded by Γ^1 it would meet the common frontier, $|\Gamma^1|$, of their loci.

The relation $\Gamma_1^k + \Gamma_2^k \sim 0$ in G is also written $\Gamma_1^k \sim \Gamma_2^k$ in G, and read "Γ_1^k is homologous to Γ_2^k in G". Homology in G is an equivalence relation, for it is clearly symmetrical; it is reflexive since $\Gamma^k + \Gamma^k = 0$; and it is transitive, for if $\Gamma_1^k \sim \Gamma_2^k$ and $\Gamma_2^k \sim \Gamma_3^k$ in G, then

$$\Gamma_1^k + \Gamma_3^k = (\Gamma_1^k + \Gamma_2^k) + (\Gamma_1^k + \Gamma_2^k) \sim 0 \text{ in } G.$$

Exercise. State and prove 6·1 without making use of the convention introduced at the beginning of the paragraph.

Theorem 6·3. *A necessary and sufficient condition that the finite points x and y be connected in an open set G is that $(x) + (y) \sim 0$ in G.*

Fig. 27

Necessary. By hypothesis x and y belong to the same component, D, of G. Let E_x, be the set of all points z such that $(x) \sim (z)$ in D. *E_x is open,* for if $z \in E_x$ every point z' in a sufficiently small neighbourhood of z is connected to z by either one or two segments in D. The sum, mod 2, of this chain and the one which (by hypothesis) is bounded by x and z is a 1-chain in D bounded by x and z', i.e. $z' \in E_x$. $D - E_x$ is open, for, as before, if x and a point z', in a small neighbourhood of z of $D - E_x$, were together the boundary of a 1-chain in D it would follow that x and z bound in D, contrary to the definition of $D - E_x$. Since D is connected one of the sets E_x and $D - E_x$ must be null; and since $x \in E_x$, $D - E_x = 0$. Therefore $y \in E_x$.

Sufficient. This follows immediately from 3·2.

Theorem 6·4. *If x and y do not lie on $|\Gamma_1^1|$ or $|\Gamma_2^1|$, at least one of the 1-cycles Γ_1^1, Γ_2^1, $\Gamma_1^1 + \Gamma_2^1$ bounds in $X^2 - (x \cup y)$.*

Let C_i^2 be the 2-chain, on $\mathbf{G}_i$ say, that contains x and is bounded by Γ_i^1 $(i = 1, 2)$. If, for $i = 1$ or 2, y is also in $|C_i^2|$ then $\Gamma_i^1 = (\mathcal{C}C_i^2)^{\cdot} \sim 0$ in $X^2 - (x \cup y)$. If y is in neither $|C_1^2|$ nor $|C_2^2|$ it is not in $|C_1^2 + C_2^2|$. Since every cell containing x belongs to both C_1^2 and C_2^2, x is not in $|C_1^2 + C_2^2|$. Thus

$$\Gamma_1^1 + \Gamma_2^1 = (C_1^2 + C_2^2)^{\cdot} \sim 0 \text{ in } X^2 - (x \cup y).$$

7. *Change of notation.* It has so far been possible to discuss the properties of 0-, 1- and 2-chains to some extent together, but from now on their parts in the theory are different. To avoid a multiplicity of suffixes it is convenient to adopt new notations. We use

A for a 2-cell, K for a 2-chain;
α for a 1-cell, γ for a 1-cycle, κ for a general 1-chain;
italic letters x, y, a, b, etc. for vertices.

General 0-chains are not often referred to, and are then denoted, as before, by C^0, 0-cycles by Γ^0.

The 0-chain (x) is always to have its brackets, e.g. in $(x) \sim (y)$. This prevents the possibility of confusion between the 0-chain $(x)+(y)$ and the vector sum of points $x+y$.

8. Two 1-chains, κ_1 and κ_2, on a grating have *general intersection* if they cross at every common vertex, that is to say if (i) no common vertex lies on the outer frame of the grating, and (ii) at each common vertex the two edges parallel to $\xi_1 = 0$ belong to one 1-chain, and the other two edges to the other.

Theorem 8·1. *If a* 1*-cycle* γ *and a* 1*-chain* κ *with boundary* $(x)+(y)$ *have general intersection,* $\gamma \sim 0$ *in* $X^2 - (x \cup y)$ *if, and only if, the number,* n*, of crossings is even.*

Let κ_1 and κ_2 be the subchains of κ in K_1 and K_2, the 2-chains bounded by γ. Each common vertex of κ and γ belongs to just one edge of κ_1, and therefore $\dot{\kappa}_1$ contains all these vertices. If n is odd, the 0-cycle $\dot{\kappa}_1$ cannot consist precisely of this odd set of vertices, and therefore κ_1 contains one of the vertices x, y—the only other available odd vertices—and κ_2 contains the other. If n is even, $\dot{\kappa}_1$, contains both or neither of x and y, and κ_2 neither or both, accordingly.

Corollary. *Two* 1*-cycles,* γ_1 *and* γ_2*, with general intersection have an even number of crossings.*

Theorem 8·2. If, for $i = 1, 2$, κ_i is a 1-chain with boundary $(x_i) + (y_i)$; if κ_1 and κ_2 have general intersection, with an odd number of crossings; and if the continuum F contains x_1 and y_1 but does not meet κ_2; then $F \cup |\kappa_1|$ separates x_2 and y_2.

If not let κ_3 bea 1-chain in $\mathcal{C}(F \cup |\kappa_1|)$ with boundary $(x_2)+(y_2)$. The 1-cycle $\kappa_2 + \kappa_3$ has general intersection with κ_1, the crossings being the same as those of κ_1 and κ_2 and therefore odd in number. Therefore x_1 and y_1 belong to different 2-chains bounded by $\kappa_2+\kappa_3$. But this is impossible, for they are joined by the continuum F, meeting neither κ_2 nor κ_3.

These theorems can be made the basis of alternative proofs of the separation theorems that occupy the rest of this chapter. More general forms of 8·1 and 8·2 are proved in Chapter VII (8·5 Corollary).

§ 2. ALEXANDER'S LEMMA AND JORDAN'S THEOREM

9. A theorem ("Alexander's Lemma")[16] will now be proved which gives an answer to a question that often arises in the course of this chapter. Suppose it is given that two points can be joined by a path not meeting a closed set F_1, and also by a path not meeting a closed set F_2; in what circumstances can they be joined by a path meeting neither F_1 nor F_2? Or, in terms of open sets, given that $(x) \sim (y)$ in G_1 and in G_2, in what circumstances does $(x) \sim (y)$ in G_1G_2?

The lemma can be stated in two ways, which are easily seen to be equivalent, by taking G_1 and G_2 in the first version to be the complements of F_1 and F_2 in the second.

Theorem 9·1·1. *Let G_1 and G_2 be open sets in the closed plane Z^2. If the points x and y bound the 1-chains κ_1 in G_1 and κ_2 in G_2, and if $\kappa_1 + \kappa_2$ in $G_1 \cup G_2$, then $(x) \sim (y)$ in G_1G_2.*

Theorem 9·1·2. *Let F_1 and F_2 be closed sets in the closed plane Z^2. If the points x and y bound the 1-chains κ_i not meeting F_i (for $i = 1, 2$) and if $\kappa_1 + \kappa_2 \sim 0$ in $Z^2 - F_1F_2$ then x and y are not separated by $F_1 \cup F_2$.*

We shall prove the second version, 9·1·2, which, in spite of its less direct form, is more readily applicable in most cases.

By hypothesis, one of the 2-chains bounded by $\kappa_1 + \kappa_2$ say K on a grating $\mathbf{G}$, does not meet F_1F_2, and hence $|K|F_1$ and $|K|F_2$ do not meet. Let $\mathbf{G}^*$ be a refinement of $\mathbf{G}$ such that no cell meets both $|K|F_1$ and $|K|F_2$, and let K_1 be the set of 2-cells of K^* (=

K as subdivided on $\mathbf{G}^*$) that meet F_1. Consider the 1-chain $\kappa_0 = \kappa_2 + \dot{K}_1$ on $\mathbf{G}^*$. (In Fig. 28 K_1 is horizontally shaded, and κ_0 is the thickened lines.) Since $\dot{K}_1$ is a cycle $\dot{\kappa}_0 = \dot{\kappa}_2$, i.e. κ_0 is bounded by x and y.

Fig. 28

κ_0 *does not meet* $F_1 \cup F_2$. By the definition of κ_2, $|\kappa_2| \cup F_2 = 0$, and since all cells of K_1 meet F_1, $|K_1|F_2 = 0$. Therefore κ_0 does not meet F_2. The set F_1 does not meet $K + K_1$, nor a fortiori its boundary $\kappa_1 + \kappa_2 + \dot{K}_1$. Since $|\kappa_1|F_1 = 0$, and

$$|\kappa_0| = |\kappa_2 + \dot{K}_1| = |\kappa_1 + (\kappa_1 + \kappa_2 + \dot{K}_1)| \subseteq |\kappa_1| \cup |\kappa_1 + \kappa_2 + \dot{K}_1|,$$

it follows that κ_0 does not meet F_1.

This completes the proof.

Corollary. *In 9·1·1 the vertex-pair* $(x) + (y)$ *may be replaced by any* 0*-cycle* Γ^0 *bounding* κ_1 *in* G_1 *and* κ_2 *in* G_2. *The proof is unaffected.*

In the *open plane* the condition "F_1 or F_2 is bounded" must be added, to ensure the existence of the grating $\mathbf{G}^*$; but this extra condition is not necessary in the case $F_1F_2 = 0$, for then K can always be chosen to be finite.

Theorem 9-2. *If the common part of the two closed sets* F_1 *and* F_2 *in* Z^2 *is connected, two points which are connected in* $\mathcal{C}F_1$ *and* $\mathcal{C}F_2$ *are connected in* $\mathcal{C}(F_1 \cup F_2)$. (Holds in R^2 if $F_1F_2 = 0$.)

By a preliminary topological mapping of the whole space, which does not disturb separation properties, the points may both be made finite. They then bound chains κ_i not meeting F_i, and since F_1F_2 is connected $\kappa_1 + \kappa_2$ bounds in $\mathcal{C}(F_1F_2)$.

Corollary 1. *If* $\mathcal{C}F_1$, $\mathcal{C}F_2$ *and* F_1F_2 *are connected,* $\mathcal{C}(F_1 \cup F_2)$ *is connected.* The extra condition "F_1 or F_2 is bounded" is again required in R^2, for 9·2 and Corollary 1, except when $F_1F_2 = 0$. (*Example*: F_1 and F_2 are the positive ξ_1- and ξ_2-axes, including the origin in both cases.)

Corollary 2. *If* D_1 *and* D_2 *are domains in* X^2 $(= Z^2$ *or* $R^2)$ *such that* $D_1 \cup D_2 = X^2$, D_1D_2 *is connected.*

Theorem 9·3. *If a closed set F is contained in a domain D in X^2, and $D_1, D_1, \ldots$ are the components of $\mathcal{C}F$, the components of $D - F$ are $DD_1, DD_2, \ldots$, provided $F \neq X^2$.*

DD_i is not null. For $\mathcal{C}F - D_i$ is open, since it is the union of components of $\mathcal{C}F$ (IV. 6·2), and therefore if DD_i were null

$$D_i \mid (\mathcal{C}F - D_i) \cup D$$

would be a partition of $\mathcal{C}F \cup D$, $= X^2$, which is impossible.

DD_i is connected. If x and y are any two of its points they are connected in D and in $\mathcal{C}F$, and $F \cap \mathcal{C}D = 0$. Therefore, by Theorem 9·2, x and y are connected in $D\mathcal{C}F$, i.e. they are joined by a continuum in D not meeting F, and therefore lying in D_i. Thus x and y are connected in DD_i.

Since the points of DD_i and DD_j $(i \neq j)$, are clearly not connected in $D - F$ the theorem is proved.

It follows that $\mathcal{C}F$ and $D - F$ have the same number of components.

Fig. 29

Theorems 9·2 and 9·3 do not hold on surfaces in general. For example, on the ring-surface in Fig. 29 the circle γ determines one domain in the complete surface, but two in the white domain.

Exercise. If the closed set F in X^2 has the components $F_1, F_2, \ldots, F_p$ (p finite), and if each F_i has a finite (positive) number, n_i, of residual domains, then $\mathcal{C}F$ has $\sum_{i=1}^{p}(n_i - 1) + 1$ components. [Induction on p.]

The following theorems form a partial converse of Alexander's Lemma.

Theorem 9·4·1. *Let G_1 and G_2 be open sets in X^2 such that $\mathcal{C}G_1$ and $\mathcal{C}G_2$ are connected. Let the 0-cycle Γ^0 in G_1G_2 bound the 1-chain κ_i in G_i, for $i = 1, 2$. Then if $\Gamma^0 \sim 0$ in G_1G_2, $\kappa_1 + \kappa_2 \sim 0$ in $G_1 \cup G_2$.*

Let $\Gamma_0 = \dot{\kappa}_0$, $|\kappa_0| \subseteq G_1G_2$. Then the 1-cycle $\kappa_0 + \kappa_i$ does not meet the connected set $\mathcal{C}G_i$, and therefore bounds in G_i, and all the more in $G_1 \cup G_2$. Hence

$$(\kappa_1 + \kappa_2) = (\kappa_0 + \kappa_1) + (\kappa_0 + \kappa_2) \sim 0 \text{ in } G_1 \cup G_2.$$

Theorem 9·4·2. *Let F_1 and F_2 be connected closed sets in X^2 and let the* 0*-cycle Γ_0 in $\mathcal{C}(F_1 \cup F_2)$ bound the* 1*-chain κ_i not meeting F_i $(i = 1, 2)$. Let the vertices of Γ^0 be paired in any way as*

$$(x_1, y_1), (x_2, y_2), \ldots, (x_q, y_q).$$

Then if $\kappa_1 + \kappa_2$ is non-bounding in $X^2 - F_1F_2$, at least one x_r is separated from y_r by $F_1 \cup F_2$.

If, for each vertex pair, $(x_r) + (y_r) \sim 0$ in $\mathcal{C}(F_1 \cup F_2)$, their sum $\Gamma^0 \sim 0$ in $\mathcal{C}(F_1 \cup F_2)$, and therefore $\kappa_1 + \kappa_2 \sim 0$ in $X^2 - F_1F_2$, by 9·4·1.

10. *A simple arc* (or *Jordan arc*) is a set of points homeomorphic with the closed segment $\langle 0, 1\rangle$. A *simple closed curve* (or *Jordan curve*) is a set of points homeomorphic with the circle $\xi_1^2 + \xi_2^2 = 1$ in R^2. The arc and curve are connected and compact, and are therefore closed sets in any containing space. It was shewn in IV. para. 2 that a simple arc or closed curve in R^2 or Z^2 is nowhere dense, and therefore if E is one or other of these curves,

$$\mathcal{F}E = \mathcal{F}(\mathcal{C}E) = E.$$

It is convenient in dealing with simple arcs and curves to suppose a definite topological mapping chosen in either case, and we may then speak of "the point τ of the curve", meaning in the case of the are the image of the point τ of $\langle 0, 1\rangle$, and in the case of the closed curve the image of the point with angular coordinate $2\pi\tau$ on the circle. In either caser ranges from 0 to 1, but on the closed curve the points 0 and 1 are identical. The points 0 and 1 of the simple are are called its *end-points*: it was proved on p. 62, Example 5, that they are distinguished topologically from other points of the curve.

Any point τ, other than 0 or 1, of the segment $\langle 0, 1\rangle$ divides it into two segments with τ as their only common point. It follows that if x is any point, other than an end-point, of a simple arc, the arc is the union of two simple arcs having no common point except x, which is an end-point of each. From the corresponding property of the circle it follows that if x and y are any two points of a simple closed curve, the curve is the union of two simple arcs having no common points except x and y, which are the end-points of both. It was shewn in IV. § 5 that these properties distinguish the simple arc and simple closed curve from all other compact connected spaces, and they are in fact the only properties that are used in proving the following theorems.

Fig. 30 Fig. 31

To understand the difficulties that would attend any attempt to deal with simple arcs and curves by means of linear approximations, and to appreciate the power of the "direct" method of Veblen, Brouwer and Alexander used in the following proofs, it should be borne in mind that these curves, although topologically "simple" in the sense that they have no multiple points, may from the point of view of differential geometry present highly condensed singularities. The arcs of two spirals $r = (\theta + \alpha_1)^{-\lambda}$ and $r = (\theta + \alpha_2)^{-\lambda}$ ($\lambda > 0$, $\theta > -\alpha_i$), cut off by the circle $r = \beta$ form a simple arc with its end-points on the circle. A simple closed curve can be constructed having a spiral cusp of this kind in every are, however small. The method of construction is the usual "condensation of singularities": first spiral cusps of a certain diameter ϵ_1 are introduced at regular intervals round a circle; then cusps of diameter ϵ_2 at closer intervals round this curve; and so on. It is easy to see that if the numbers ϵ_n tend to zero sufficiently fast the curves tend to a simple closed curve as limit.

Theorem 10·1. *A simple arc in the open or closed plane has a single complementary domain.*

Let L be the arc. Its complementary set is certainly not null, since L is nowhere dense (IV. 2).

Suppose the points x and y are separated by L. Then if a is the point $\tau = \frac{1}{2}$ of L, dividing it, say, into the two arcs L_1 and L_2, either L_1 or L_2 separates x from y. For if not, L_1 and L_2 satisfy the conditions of 9·2 ($L_1 L_2 = a$, and if the containing space is R^2 the arc, being compact, is bounded). Therefore their union, L, does not separate x and y.

A bisection argument now leads us to a contradiction. Of the two halves L_1, L_2 of L we select one that separates x and y, then a half of this half separating x and y, and so on, where a "half" of a simple arc means the image of half the pattern segment. If $L^{(n)}$ is the nth arc so chosen, the segments of which $L^{(1)}, L^{(2)}, \ldots$ are images converge to a point of $\langle 0, 1\rangle$ (since each is a half of its predecessor). Therefore, owing to the continuity of the mapping, the arcs $L^{(1)}, L^{(2)}, \ldots$ themselves converge to a point a_0 of L. For some n there is a square neighbourhood† of a_0 containing $L^{(n)}$ but not x or y. The points x and y are connected by a continuum not meeting this square, and therefore not meeting $L^{(n)}$.

Thus the assumption that x and y are separated by L has led to a contradiction.

Theorem 10·2. (Jordan's Theorem.)[17] *A simple closed curve in the open or closed plane has two complementary domains, of each of which it is the complete frontier.*

Fig. 32

Let J be the curve. If the containing space is R^2, J is bounded.

I. *$\mathcal{C}J$ has at most two components.* Let a and b be any two points of J, dividing it into the simple arcs L_1 and L_2. Suppose that x, y and z are finite points of three different residual domains of J. By 10·1 there exist 1-chains κ_1 and κ_2 bounded by x and y, such that κ_i does not meet L_i $(i = 1, 2)$. Since x and y are separated by $L_1 \cup L_2$ it follows from Alexander's Lemma that $\kappa_1 + \kappa_2$ is non-bounding in $\mathcal{C}(L_1 L_2)$, i.e. in

$$X^2 - (a \cup b).$$

Similarly, $(y) + (z)$ is the common boundary of two 1-chains κ_3 and κ_4 not meeting L_1 and L_2 respectively, and $\kappa_3 + \kappa_4$ is non-bounding in $X^2 - (a \cup b)$. Therefore, by 6·4,

$$(1) \qquad (\kappa_1 + \kappa_2) + (\kappa_3 + \kappa_4) \sim 0 \quad \text{in} \quad X^2 - (a \cup b).$$

The 1-chain $\kappa_1 + \kappa_3$ does not meet L_1 and it is bounded by $(x + y) + (y + z)$, i.e. by x and z. Similarly, $\kappa_2 + \kappa_4$ does not meet L_2 and is bounded by x and z. The sum

$$(\kappa_1 + \kappa_3) + (\kappa_2 + \kappa_4)$$

† A square neighbourhood of the point at infinity is the *exterior* of any square.

is identical with (1), and therefore bounds in $\mathcal{C}(L_1 L_2)$. Hence x and z are not separated by $L_1 \cup L_2$, contrary to the hypothesis.

II. *$\mathcal{C}J$ has at least two components.* Let a, b, L_1, L_2 be as before and let Q be a square† with sides parallel to the coordinate axes separating a from b. Since the closed sets $L_1 Q$ and $L_2 Q$ do not meet, there is a grating **G** (which may be so chosen that Q is the locus of a 1-chain σ) no cell of which meets both $L_1 Q$ and $L_2 Q$. Let κ_0 be the set of 1-cells of σ that meet L_1.

The 0-cycle $\dot{\kappa}_0$ bounds κ_0 in $\mathcal{C}L_2$ and $\kappa_0 + \sigma$ in $\mathcal{C}L_1$, and

$$\kappa_0 + (\kappa_0 + \sigma) = \sigma$$

is non-bounding in $X^2 - (a \cup b) = X^2 - L_1 L_2$. Hence by 9·4·2 a pair of vertices of $\dot{\kappa}_0$ are separated by $L_1 \cup L_2$.

III. *If D_0 and D_1 are the components of $\mathcal{C}J$, $\mathcal{F}D_0 = \mathcal{F}D_1 = J$.* In the course of proving II it was shewn that there are points of both residual domains on Q. Now a is an arbitrary point of J and Q may be chosen so that it lies in an assigned neiohbourhood of a. Therefore every point of J belongs to both $\mathcal{F}D_0$ and $\mathcal{F}D_1$. By IV. 3·1, both frontiers are contained in the frontier of $\mathcal{C}J$, i.e. in J.

This completes the proof of the whole theorem.

*

16 (p. 110). J. W. Alexander, *Trans. Amer. Math. Soc.* 23 (1922), 342.

17 (p. 115). C. Jordan was the first to state, and to attempt to prove, the theorem that bears his name. The first correct proof was given by Veblen in 1905. The text proof, which is that of Alexander, is based on Brouwer's proof (*Math. Annalen*, 1909).

† I.e. the 1-dimensional frontier.