Apartness and Distinguishing Formulas in Hennessy-Milner Logic

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Abstract. For Labelled Transition Systems, an important question is when two states in such a system are bisimilar. Here we study the dual, in the sense of logical opposite, of bisimilarity, known as "apartness". This gives a positive way of distinguishing two states (stating that they are not bisimilar). In [3] we have studied apartness (and bisimilarity) in general co-algebraic terms. As opposed to bisimilarity, which is co-inductive, apartness is an inductive notion and we have given and studied proof systems for deriving that two states are apart. In the present paper we continue the study of apartness in the light of Hennessy-Milner theorems that establish an equivalence between bisimulation and validity of (modal) formulas: two states are bisimilar if and only if they satisfy the same set of formulas. Using the apartness view, this can be dualized: two states are apart if and only there is a formula that distinguishes them. We work this out for three situations: bisimulation for labelled transition systems (LTSs), weak bisimulation for LTSs with silent (τ) steps and branching bisimulation for LTSs with silent (τ) steps. We study the equivalences with the well-known variants of Hennessy-Milner logic and show how an apartness proof gives rise to a distinguishing formula.

1 Introduction

The standard way of looking at equality of states in a Labeled Transition Systems (LTS) is indistinguishability, which is captured via the notion of bisimulation. States are observed through "destructors", which in an LTS are the transition-steps. A bisimulation is a relation that satisfies the "transfer principle": if two states are related, and we take a transition-step, then we get two new related states. Two states are bisimilar if and only if they are observationally indistinguishable, i.e. there is a bisimulation that relates them. The coinduction principle states that two states that are bisimilar (have the same observations) are equal.

In previous work [3], we have described apartness as the dual of bisimulation for systems that are defined as co-algebras. Categorically, bisimulation is described in the category of relations **Rel**, and apartness in the "fibred opposite" of **Rel**. Here, we take a more pedestrian approach and use apartness to provide a new look on some concrete known results about various forms of bisimulation. The basic idea is that two states are apart in case they are observationally

N. Jansen et al. (Eds.): Vaandrager Festschrift 2022, LNCS 13560, pp. 266–282, 2022.
https://doi.org/10.1007/978-3-031-15629-8_14

distinguishable: there is a sequence of observations that can be made on one state but not on the other. Apartness is a positive notion: two states are apart if there is a positive way to distinguish them, and being apart is the negation of being bisimilar. Bisimilarity is co-inductive: it is the union of all bisimulation relations and therefore the largest bisimulation relation (and a final co-algebra). Apartness is inductive: it is the intersection of all apartness relations and therefore the smallest apartness relation (and an initial algebra). As apartness is inductive, there is a proof system with derivation rules to derive that two states are apart.

In the present paper, we study the proof systems for deriving apartness for some concrete cases. First we look into well-known non-deterministic LTSs, where we have transitions of the form $q \to_a q'$, with q, q' states and a a label. The non-determinism means that from a state q there are multiple a-transitions possible (or none). The apartness we get here is the dual, in the sense of the logical opposite, of standard bisimulation. Then we add silent (τ) steps, which we study modulo weak bisimulation (giving rise to its dual 'weak apartness') and modulo branching bisimulation (giving rise to its dual 'branching apartness'). For each case, we give the deduction rules for deriving that two states are apart.

To argue that apartness is a fruitful way of looking at distinguishability, we establish for each of these cases a Hennessy-Milner connection with a modal logic. This is a very well-known connection between bisimulation and logic [2,6] that we now re-establish via apartness. In bisimulation terms, the Hennessy-Milner Theorem says that two states are bisimilar if and only if the same modal formulas hold in these states, where of course the notion of bisimulation and the logic for formulas depends on the type of systems under study. In terms of apartness, the Hennessy-Milner Theorem gets a more "positive flavor" saying that two states are apart if and only if there is a modal formula that distinguishes them (i.e. that holds in one state, but not in the other). So, from a proof of the apartness of two states q and p, we can derive a formula φ such that φ holds for q and $\neg\varphi$ holds for p: the formula φ gives a positive 'witness', an explanation, for the fact that q and p are distinguishable. We illustrate this with some examples.

As a matter of fact, the present paper can be seen as an "apartness footnote" [3] to the original papers by Hennessy and Milner [6], De Nicola and Vaandrager [2] and Van Glabbeek and Weijland [4], where bisimulation has been studied in various forms for various systems with motivating examples, and its properties have been established, also in terms of modal logic.

Special Thanks

We dedicate this article to Frits Vaandrager on the occasion of his 60th birthday. I have known Frits for a long time as a very respectable researcher and colleague at Radboud University. We have met for the first time at the LiCS conference of 1992 at Santa Cruz, but that was only a very brief encounter. It is nice to see that much of Frits' earlier work, on branching bisimulation for LTSs, I have recently started appreciating much more after looking at the "apartness view" of co-algebraically defined systems. It is even nicer to see that this apartness view has been inspiring for Frits and others to study algorithms for automata learning [10]. Thanks Frits for all the nice co-operations!

2 Bisimulation and Apartness for LTSs

We start from labeled transition systems over a set of actions A, and study the well-known notion of bisimulation and the (less well-known) notion of apartness for these systems.

Definition 1. *Let A be a fixed set of* actions*. A* labelled transition system over A *or LTS over A, is a pair $(S, \to)$ where S is a set of states and $\to\, \subseteq S \times A \times S$. For $(q, a, p) \in \to$, we write $q \to_a p$ and we call the LTS* image finite *in case the set $\{p \mid q \to_a p\}$ is finite for each q, a. On an LTS we define the notions of bisimulation and apartness*

1. *A relation $R \subseteq S \times S$ is a* bisimulation *if it is symmetric and it satisfies the following transfer property*
$$\frac{q_1 \to_a q_2 \qquad R(q_1, p_1)}{\exists p_2 (p_1 \to_a p_2 \wedge R(q_2, p_2))}\ (\underline{\leftrightarrow})$$
Two states $q, p \in S$ are bisimilar*, notation $q \,\underline{\leftrightarrow}\, p$, is defined by*
$$q \,\underline{\leftrightarrow}\, p := \exists R \subseteq S \times S\,(R \textit{ is a bisimulation and } R(q, p)).$$
2. *A relation $Q \subseteq S \times S$ is an* apartness *if it is symmetric and satisfies the following rule*
$$\frac{q_1 \to_a q_2 \qquad \forall p_2 \in S(p_1 \to_a p_2 \implies Q(q_2, p_2))}{Q(q_1, p_1)}\ (\mathrm{in}_{\#})$$
Two states $q, p \in S$ are apart*, notation $q \,\underline{\#}\, p$, is defined by*
$$q \,\underline{\#}\, p := \forall Q \subseteq S \times S\,(\textit{if } Q \textit{ is an apartness, then } Q(q, p)).$$

As an immediate consequence of the definition, $q \,\underline{\#}\, p$ if and only if (q, p) is in the intersection of all apartness relations, and $\underline{\#}$ is the smallest apartness relation. It is standard that in an LTS, two states are bisimilar if and only if they are not apart, so we have
$$q \,\underline{\leftrightarrow}\, p \qquad \Longleftrightarrow \qquad \neg(q \,\underline{\#}\, p).$$
Also, apartness is an inductive notion, and so we can equivalently define q and p to be apart, $q \,\underline{\#}\, p$, if this can be *derived* using the deduction rules in Fig. reffig.aptrules. So, we can use the rules that define what an apartness relation is as the deduction rules for a proof system to derive $q \,\underline{\#}\, p$.

It should be noted that in case the LTS is image-finite, the rule above can also be written with a finite set of hypotheses:
$$\frac{q_1 \to_a q_2 \qquad \bigwedge_{\{p_2 \in S \mid p_1 \to_a p_2\}} q_2 \,\underline{\#}\, p_2}{q_1 \,\underline{\#}\, p_1}\ (\mathrm{in}_{\#})$$

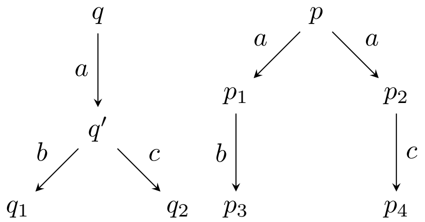
Before moving to formulas that distinguish states, we first give an example to see what a proof of apartness looks like concretely.

$$\frac{q_1 \to_a q_2 \qquad \forall p_2 \in S(p_1 \to_a p_2 \implies q_2 \mathrel{\underline{\#}} p_2)}{q_1 \mathrel{\underline{\#}} p_1}\,(\mathrm{in}_{\#})$$

$$\frac{p \mathrel{\underline{\#}} q}{q \mathrel{\underline{\#}} p}\,(\mathrm{symm})$$

Fig. 1. The deduction system for deriving $q \mathrel{\underline{\#}} p$

Example 1. We consider the LTS with actions $\{a, b, c\}$ and states and transitions as indicated in the figure.



It is well-known that q and p are not bismilar. They can be shown to be apart using the following derivation

$$\dfrac{q \to_a q' \qquad \dfrac{\dfrac{q' \to_c q_2 \quad \checkmark}{q' \mathrel{\underline{\#}} p_1} \qquad \dfrac{q' \to_b q_1 \quad \checkmark}{q' \mathrel{\underline{\#}} p_2}}{\forall p'(p \to_a p' \implies q' \mathrel{\underline{\#}} p')}}{q \mathrel{\underline{\#}} p}$$

Note that the apartness $q' \mathrel{\underline{\#}} p_1$ holds because $q' \to_c q_2$ and there is no c-transition from p_1, expressed by the check-mark. So the universal quantification $\forall p''(p_1 \to_c p'' \implies \ldots)$ is empty, and therefore holds. These are the "base cases" of the inductive definition of apartness: where we can do some transition from q but not from p, and therefore $q \mathrel{\underline{\#}} p$.

2.1 Hennessy-Milner Logic for Bisimulation

We now introduce the well-known modal logic that captures bisimulation logically and we prove the well-known Hennessy-Milner theorem using apartness.

Definition 2. *Given a set of actions A, we define the Hennessy-Milner logic for A, HML_A by the following set of formulas φ, where $a \in A$.*

$$\varphi ::= \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle \varphi.$$

Let $(S, \to)$ be an LTS over A. For $q \in S$ and φ a formula of HML_A, we define the notion φ holds in state q, *notation $q \models \varphi$, as follows, by induction on φ.*

- $q \models \top$ *always holds.*
- $q \models \neg\varphi$ *if* $q \not\models \varphi$*.*
- $q \models \varphi_1 \wedge \varphi_2$ *if* $q \models \varphi_1$ *and* $q \models \varphi_2$*.*
- $q \models \langle a \rangle \varphi$ *if there is a* q' *such that* $q \rightarrow_a q'$ *and* $q' \models \varphi$*.*

For $(S, \rightarrow)$ *an LTS over* A*,* $q, p \in S$*, and* $\varphi \in HML_A$*, we say that* φ distinguishes q, p *if* $q \models \varphi$ *and* $p \models \neg\varphi$*.*

The well-known Hennessy-Milner theorem [2,6] states that $q \leftrightarrow p$ if and only if $\forall \varphi(q \models \varphi \Leftrightarrow p \models \varphi)$. We prove the apartness analogon of this, where we compute a *distinguishing formula* from an apartness proof.

Proposition 1. *Given an image-finite LTS* $(S, \rightarrow)$ *over* A*, and* $q, p \in S$*, we have*

$$q \mathrel{\underline{\#}} p \qquad \Longleftrightarrow \qquad \exists \varphi(q \models \varphi \wedge p \models \neg\varphi).$$

Proof. ($\Rightarrow$) by induction on the proof of $q \mathrel{\underline{\#}} p$.

- If the last applied rule is symm, then by IH we have φ that distinguishes p, q, and therefore $\neg\varphi$ distinguishes q, p.
- If the last applied rule is $(\text{in}_\#)$, then we have

$$\frac{q \rightarrow_a q' \qquad \bigwedge_{\{p' \in S \mid p \rightarrow_a p'\}} q' \mathrel{\underline{\#}} p'}{q \mathrel{\underline{\#}} p}\,(\text{in}_\#)$$

  where the conjunction is over a finite set of formulas, say $\{p' \in S \mid p \rightarrow_a p'\} = \{p_1, \ldots, p_n\}$. By IH we have φ_i $(1 \leq i \leq n)$ such that φ_i distinguishes q', p_i. Now we take $\varphi := \langle a \rangle \bigwedge_{1 \leq i \leq n} \varphi_i$ and we have
  1. $q \models \varphi$: $q \rightarrow_a q'$ with $q' \models \varphi_i$ for every i, so $q \models \langle a \rangle \bigwedge_{1 \leq i \leq n} \varphi_i$.
  2. $p \models \neg\varphi$: for each p' with $p \rightarrow_a p'$ there is an i with $p' \models \neg\varphi_i$, and therefore $p' \models \neg \bigwedge_{1 \leq i \leq n} \varphi_i$. So $p \models \neg\langle a \rangle \bigwedge_{1 \leq i \leq n} \varphi_i$.

($\Leftarrow$) by induction on φ, where $q \models \varphi$ and $p \models \neg\varphi$.

- $\varphi = \top$ cannot occur, because $p \models \neg\top$ never holds.
- $\varphi = \neg\psi$. Then $p \models \psi$ and $q \models \neg\psi$, so by induction we have a derivation of $p \mathrel{\underline{\#}} q$. By rule (symm) we have a derivation of $q \mathrel{\underline{\#}} p$.
- $\varphi = \varphi_1 \wedge \varphi_2$. Then $q \models \varphi_1$ and $q \models \varphi_2$, and also $p \models \neg\varphi_1$ or $p \models \neg\varphi_2$. In case $p \models \neg\varphi_1$ we have, by induction, a derivation of $q \mathrel{\underline{\#}} p$, and similarly in case $p \models \neg\varphi_2$, so we are done.
- $\varphi = \langle a \rangle \psi$. We know $q \models \langle a \rangle \psi$, so let q' be such that $q \rightarrow_a q'$ and $q' \models \psi$. Also $p \models \neg\langle a \rangle \psi$, so for all p' with $p \rightarrow_a p'$ we have $p' \models \neg\psi$. By induction hypothesis we have derivations of $q' \mathrel{\underline{\#}} p'$ for all p' with $p \rightarrow_a p'$, so we have the following derivation of $q \mathrel{\underline{\#}} p$, using rule $(\text{in}_\#)$

$$\frac{q \rightarrow_a q' \qquad \forall p' \in S(p \rightarrow_a p' \implies q' \mathrel{\underline{\#}} p')}{q \mathrel{\underline{\#}} p}\,(\text{in}_\#)$$

$\square$

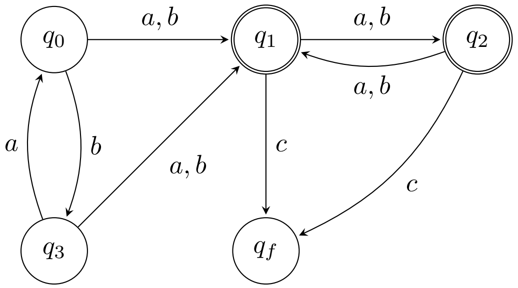
It is well-known ([6]) that image finiteness is needed for Proposition 1 to hold. This can also be observed from the proof of ($\Rightarrow$), where the image finiteness guarantees that the generated distinguishing formula contains finitely many conjunctions. So the implication ($\Rightarrow$) only holds for image finite systems, while the implication ($\Leftarrow$) holds in general.

Example 2. We continue Example 1 by giving the formula that distinguishes states q and p. It can be derived from the derivation of $q \underline{\#} p$, by following the steps in the proof of Proposition 1. The distinguishing formula is

$$\varphi := \langle a \rangle (\langle c \rangle \top \wedge \langle b \rangle \top),$$

which can be read as saying: "we can do an a-step such that after that we can do both a b-step and a c-step".

Example 3. As another example we show how we can use apartness for non-deterministic finite automata, which have also been discussed in [3]. In this example we use a special step, a c-transition (ending up in state q_f) to mimic that a state is final.



$$\dfrac{q_3 \to_a q_0 \qquad \dfrac{[q_0 \to_a q_1] \qquad \dfrac{\dfrac{q_1 \to_c q_f \qquad \dfrac{[\_] \qquad \checkmark}{\forall q'(q_0 \to_c q' \implies q_f \underline{\#} q')}}{q_1 \underline{\#} q_0}}{q_0 \underline{\#} q_1}}{\forall q'(q_0 \to_a q' \implies q_0 \underline{\#} q')}}{q_3 \underline{\#} q_0}$$

It can be shown that $q_3 \underline{\#} q_0$ by the derivation given above. In the derivation, we indicate between $[\ldots]$ all possible transitions that we need to prove a universal hypothesis of the form $\forall q'(\ldots \to q' \implies \ldots)$. Note that $q_0 \to_a q_1$ is the only a-step from q_0. The check-mark denotes the empty side-hypothesis that vacuously holds, as there is no c-step possible from q_0. The distinguishing formula computed from this derivation is $\langle a \rangle \neg \langle c \rangle \top$, saying that from q_3 one can do an a-step to a state where one cannot do a c-step, while for q_0 this is not the case.

3 Weak Bisimulation and Apartness for LTSs

We now add silent steps, or τ-steps to labeled transition systems and we study the (well-known) notion of weak bisimulation and the (less well-known) notion of weak apartness for LTSs with τ.

Definition 3. *Let A be a fixed set of* basic actions. *We denote by $A_\tau := A \cup \{\tau\}$ the set of all actions, which includes the* silent action τ. *We let α (and $\beta, \gamma, \ldots$) range over A_τ and a (and $b, c, \ldots$) range over A. A* labelled transition system with τ-steps over *A or LTS$_\tau$, is a pair $(S, \rightarrow)$ where S is a set of states and $\rightarrow \subset S \times A_\tau \times S$. For $(q, \alpha, p) \in \rightarrow$, we write $q \rightarrow_\alpha p$.*

We will be interested in the transitive reflexive closure of $\rightarrow_\tau$, which we denote by $\twoheadrightarrow_\tau$. We call the LTS$_\tau$ image finite *in case the set $\{q' \mid \exists q_1, q_2 (q \twoheadrightarrow_\tau q_1 \rightarrow_\alpha q_2 \twoheadrightarrow_\tau q'\}$ is finite for each q, α.*

On an LTS$_\tau$, we define the notions of weak bisimulation [2,6] and weak apartness. The first is well-known and the second is its dual and has been discussed in [3].

Definition 4. *Let $(S, \rightarrow)$ be an LTS$_\tau$ over A.*

1. *A relation $R \subseteq S \times S$ is a* weak bisimulation *if it is symmetric and the following two rules hold for R.*

$$\frac{q \rightarrow_\tau q' \qquad R(q,p)}{\exists p' (p \twoheadrightarrow_\tau p' \wedge R(q',p'))}\ (\text{bis}_{w\tau})$$

$$\frac{q \rightarrow_a q' \qquad R(q,p)}{\exists p', p'', p''' (p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''' \wedge R(q',p'''))}\ (\text{bis}_{w})$$

   States q, p are weakly bismilar, *notation $q \underline{\leftrightarrow}_w p$, if there exists a weak bisimulation relation R such that $R(q,p)$.*

2. *A relation $Q \subseteq S \times S$ is a* weak apartness *in case Q is symmetric and the following rules hold for Q.*

$$\frac{q \rightarrow_\tau q' \qquad \forall p' (p \twoheadrightarrow_\tau p' \implies Q(q',p'))}{Q(q,p)}\ (\text{in}_{w\tau})$$

$$\frac{q \rightarrow_a q' \qquad \forall p', p'', p''' (p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''' \implies Q(q',p'''))}{Q(q,p)}\ (\text{in}_{w})$$

   The states q and p are weakly apart, *notation $q \mathrel{\underline{\#}_w} p$, if for all weak apartness relations Q, we have $Q(q,p)$.*

Again, as an immediate consequence of the definition, $q \mathrel{\underline{\#}_w} p$ if and only if (q,p) is in the intersection of all weak apartness relations, and $\underline{\#}_w$ is the smallest weak apartness relation.

Just as in LTSs, for LTS$_\tau$s we also have that two states are weakly bisimilar if and only if they are not weakly apart, so we have

$$q \mathrel{\underline{\leftrightarrow}}_w p \qquad \Leftrightarrow \qquad \neg(q \mathrel{\#_w} p).$$

Weak apartness is an inductive notion, and so also in this case, we have a derivation system for proving $q \mathrel{\#_w} p$, using the three deduction rules of Fig. 2.

$$\frac{q \rightarrow_\tau q' \qquad \forall p' (p \twoheadrightarrow_\tau p' \implies q' \mathrel{\#_w} p')}{q \mathrel{\#_w} p}\ (\mathrm{in}_{w\tau})$$

$$\frac{q \rightarrow_a q' \qquad \forall p', p'', p''' (p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''' \implies q' \mathrel{\#_w} p''')}{q \mathrel{\#_w} p}\ (\mathrm{in}_w)$$

$$\frac{p \mathrel{\#_w} q}{q \mathrel{\#_w} p}\ (\mathrm{symm})$$

Fig. 2. The deduction system for deriving $q \mathrel{\#_w} p$

In case the LTS$_\tau$ is image-finite, the rules above can be written with a finite set of hypotheses:

$$\frac{q \rightarrow_\tau q' \qquad \bigwedge_{\{p' \mid p \twoheadrightarrow_\tau p'\}} q' \mathrel{\#_w} p'}{q \mathrel{\#_w} p}\ (\mathrm{in}_{w\tau})$$

$$\frac{q \rightarrow_a q' \qquad \bigwedge_{\{p''' \mid \exists p', p'' (p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''')\}} q' \mathrel{\#_w} p'''}{q \mathrel{\#_w} p}\ (\mathrm{in}_w)$$

3.1 Hennessy-Milner Logic for Weak Bisimulation

We now introduce the well-known modal logic that captures weak bisimulation logically and we prove the well-known Hennessy-Milner theorem [2] using weak apartness.

Definition 5. *We adapt the formulas of the logic of Definition 2 by just adding τ in the modality, so we have, given a set of actions A, the formulas of HMLτ_A given by the following set, where $\alpha \in A_\tau$.*

$$\varphi ::= \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle\alpha\rangle\varphi.$$

Let $(S, \rightarrow)$ be an LTS$_\tau$ over A. For $q \in S$ and φ a formula of HMLτ_A, we define the notion φ holds in state q, *notation $q \models_w \varphi$, as follows, by induction on φ.*

- $q \models_w \top$ *always holds.*
- $q \models_w \neg\varphi$ *if* $q \not\models_w \varphi$.
- $q \models_w \varphi_1 \wedge \varphi_2$ *if* $q \models_w \varphi_1$ *and* $q \models_w \varphi_2$.
- $q \models_w \langle a\rangle\varphi$ *if* $\exists q_1, q_2, q_3(q \twoheadrightarrow_\tau q_1 \rightarrow_a q_2 \twoheadrightarrow_\tau q_3 \wedge q_3 \models_w \varphi)$.
- $q \models_w \langle\tau\rangle\varphi$ *if* $\exists q'(q \twoheadrightarrow_\tau q' \wedge q' \models_w \varphi)$.

For $q, p \in S$, *and* $\varphi \in HML\tau_A$, *we say that* φ distinguishes q, p *if* $q \models_w \varphi$ *and* $p \models_w \neg\varphi$.

Again, the well-known Hennessy-Milner theorem states that $q \underline{\leftrightarrow}_w p$ if and only if $\forall\varphi \in \mathrm{HML}\tau_A(q \models_w \varphi \Leftrightarrow p \models_w \varphi)$. We prove the apartness analogon of this, where we compute a *distinguishing formula* from an apartness proof. For this it is useful to adapt the derivation rules for $\#_w$ a bit. This adaptation is borrowed from the "bisimulation side", where it is easily shown to be equivalent.

The rules ($\mathrm{bis}_{w\tau}$) and (bis_w) for weak bisimulation can easily seen to be equivalent to the following ones, where we replace a one-step transition by a multiple step transition. (The equivalence is standard, e.g. from [2].)

$$\frac{q \twoheadrightarrow_\tau q' \qquad R(q,p)}{\exists p'(p \twoheadrightarrow_\tau p' \wedge R(q',p'))}\,(\mathrm{bis}'_{w\tau})$$

$$\frac{q \twoheadrightarrow_\tau q_1 \rightarrow_a q_2 \twoheadrightarrow_\tau q' \qquad R(q,p)}{\exists p', p'', p'''(p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''' \wedge R(q',p'''))}\,(\mathrm{bis}'_w)$$

Therefore, by duality, taking the logical opposite, we also have the following equivalent set of rules for weak apartness.

Lemma 1. *Weak apartness, as defined in Definition 4 can equivalently be captured using the following derivation rules (where we use the set notation, as that's the one we will be using later, when we restrict to image-finite systems).*

$$\frac{q \twoheadrightarrow_\tau q' \qquad \bigwedge_{\{p' \mid p \twoheadrightarrow_\tau p'\}} q' \#_w p'}{q \#_w p}\,(\mathrm{in}'_{w\tau})$$

$$\frac{q \twoheadrightarrow_\tau q_1 \rightarrow_a q_2 \twoheadrightarrow_\tau q' \qquad \bigwedge_{\{p''' \mid \exists p', p''(p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''')\}} q' \#_w p'''}{q \#_w p}\,(\mathrm{in}'_w)$$

$$\frac{p \#_w q}{q \#_w p}\,(\mathrm{symm})$$

Proposition 2. *Given* $(S, \rightarrow)$, *an image-finite* LTS_τ *over* A, *and* $q, p \in S$, *we have*

$$q \# p \qquad \Longleftrightarrow \qquad \exists\varphi \in HML\tau_A(q \models_w \varphi \wedge p \models_w \neg\varphi).$$

Proof. ($\Rightarrow$) by induction on the proof of $q \mathrel{\underline{\#}} p$.

- If the last applied rule is symm, then by IH we have φ that distinguishes p, q, and therefore $\neg\varphi$ distinguishes q, p.
- If the last applied rule is $(\mathrm{in}'_{w\tau})$, we have

$$\frac{q \twoheadrightarrow_\tau q' \qquad \bigwedge_{\{p' \mid p \twoheadrightarrow_\tau p'\}} q' \mathrel{\underline{\#}}_w p'}{q \mathrel{\underline{\#}}_w p}\ (\mathrm{in}'_{w\tau})$$

  Say $\{p' \mid p \twoheadrightarrow_\tau p'\} = \{p_1, \ldots, p_n\}$. By induction hypothesis we have $\varphi_1, \ldots, \varphi_n$ with $q' \models_w \varphi_i$ and $p_i \models_w \neg\varphi_i$ for all i $(1 \leq i \leq n)$. Now take $\varphi := \langle\tau\rangle(\varphi_1 \wedge \ldots \wedge \varphi_n)$. Then $q \models_w \varphi$ and $p \models_w \neg\varphi$.
- If the last applied rule is (in'_w), we have

$$\frac{q \twoheadrightarrow_\tau q_1 \rightarrow_a q_2 \twoheadrightarrow_\tau q' \qquad \bigwedge_{\{p''' \mid \exists p', p''(p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''')\}} q' \mathrel{\underline{\#}}_w p'''}{q \mathrel{\underline{\#}}_w p}\ (\mathrm{in}'_w)$$

  Say $\{p''' \mid \exists p', p''(p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''')\} = \{p_1, \ldots, p_n\}$. By induction hypothesis we have $\varphi_1, \ldots, \varphi_n$ with $q' \models_w \varphi_i$ and $p_i \models_w \neg\varphi_i$ for all i $(1 \leq i \leq n)$. Now take $\varphi := \langle a\rangle(\varphi_1 \wedge \ldots \wedge \varphi_n)$. Then $q \models_w \varphi$ and $p \models_w \neg\varphi$.

($\Leftarrow$) by induction on φ, where $q \models_w \varphi$ and $p \models_w \neg\varphi$.

- The case $\varphi = \top$, $\varphi = \neg\psi$ and $\varphi = \varphi_1 \wedge \varphi_2$ are exactly the same as in the proof of Proposition 1.
- $\varphi = \langle\tau\rangle\psi$. We know $q \models_w \langle\tau\rangle\psi$, so let q' be such that $q \twoheadrightarrow_\tau q'$ and $q' \models_w \psi$. Also $p \models_w \neg\langle a\rangle\psi$, so for all p' with $p \twoheadrightarrow_\tau p'$ we have $p' \models_w \neg\psi$. By induction hypothesis we have derivations of $q' \mathrel{\underline{\#}} p'$ for all p' for which $p \twoheadrightarrow_\tau p'$, so we have the following derivation of $q \mathrel{\underline{\#}} p$, using rule $(\mathrm{in}'_{w\tau})$

$$\frac{q \twoheadrightarrow_\tau q' \qquad \bigwedge_{\{p' \mid p \twoheadrightarrow_\tau p'\}} q' \mathrel{\underline{\#}}_w p'}{q \mathrel{\underline{\#}}_w p}\ (\mathrm{in}'_{w\tau})$$

- $\varphi = \langle a\rangle\psi$. We know $q \models_w \langle a\rangle\psi$, so let q_1, q_2, q_3 be such that $q \twoheadrightarrow_\tau q_1 \rightarrow_a q_2 \twoheadrightarrow_\tau q_3$ and $q_3 \models_w \psi$. Also $p \models_w \neg\langle a\rangle\psi$, so for all p_1, p_2, p_3 with $p \twoheadrightarrow_\tau p_1 \rightarrow_a p_2 \twoheadrightarrow_\tau p_3$ we have $p_3 \models_w \neg\psi$. By induction hypothesis we have derivations of $q' \mathrel{\underline{\#}} p_3$ for all $p_3 \in \{p''' \mid \exists p', p''(p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''')\}$, so we have the following derivation of $q \mathrel{\underline{\#}} p$, using rule (in'_w)

$$\frac{q \twoheadrightarrow_\tau q_1 \rightarrow_a q_2 \twoheadrightarrow_\tau q' \qquad \bigwedge_{\{p''' \mid \exists p', p''(p \twoheadrightarrow_\tau p' \rightarrow_a p'' \twoheadrightarrow_\tau p''')\}} q' \mathrel{\underline{\#}}_w p'''}{q \mathrel{\underline{\#}}_w p}\ (\mathrm{in}'_w)$$

$\square$

4 Branching Bisimulation and Apartness for LTSs

We now study the notions of branching bisimulation and branching apartness on Labelled Transition Systems with τ-steps. So the systems we consider are still the LTS$_\tau$ systems of Definition 3, but now with a different notion of equivalence, branching bisimulation, that takes the branching structure due to the τ-steps into account. It is well-known that weak bisimulation is really weaker than branching bisimulation (if $s \underline{\leftrightarrow}_b t$, then $s \underline{\leftrightarrow}_w t$, but in general not the other way around) and similarly, weak apartness is really stronger than branching apartness (if $s \mathrel{\#_w} t$, then $s \mathrel{\#_b} t$, but in general not the other way around).

On an LTS$_\tau$, we define the notions of branching bisimulation [2,4] and branching apartness. The first is well-known and the second is its dual and has been discussed in [3].

Definition 6. *Given* $(S, \rightarrow)$*, an LTS*$_\tau$ *over* A *(see Definition 3), a relation* $R \subseteq S \times S$ *is a* branching bisimulation relation *if the following rules hold for* R*.*

$$\frac{q \rightarrow_\tau q' \qquad R(q,p)}{R(q',p) \vee \exists p', p''(p \twoheadrightarrow_\tau p' \rightarrow_\tau p'' \wedge R(q,p') \wedge R(q',p''))}\ (\text{bis}_{b\tau})$$

$$\frac{q \rightarrow_a q' \qquad R(q,p)}{\exists p', p''(p \twoheadrightarrow_\tau p' \rightarrow_a p'' \wedge R(q,p') \wedge R(q',p''))}\ (\text{bis}_b)$$

$$\frac{R(q,p)}{R(p,q)}\ (\text{symm})$$

The states q, p *are* branching bisimilar*, notation* $q \underline{\leftrightarrow}_b p$ *if and only if there exists a branching bisimulation relation* R *such that* $R(q,p)$*.*

We say that $Q \subseteq S \times S$ *is a* branching apartness *in case the following rules hold for* Q*.*

$$\frac{q \rightarrow_\tau q' \qquad Q(q',p) \qquad \forall p', p''(p \twoheadrightarrow_\tau p' \rightarrow_\tau p'' \implies Q(q,p') \vee Q(q',p''))}{Q(q,p)}\ (\text{in}_{b\tau})$$

$$\frac{q \rightarrow_a q' \qquad \forall p', p''(p \twoheadrightarrow_\tau p' \rightarrow_a p'' \implies Q(q,p') \vee Q(q',p''))}{Q(q,p)}\ (\text{in}_b)$$

$$\frac{Q(p,q)}{Q(q,p)}\ (\text{symm})$$

The states q *and* p *are* branching apart*, notation* $q \mathrel{\#_b} p$*, if for all branching apartness relations* Q*, we have* $Q(q,p)$*.*

Again, as an immediate consequence of the definition, $q \mathrel{\#_b} p$ if and only if (q,p) is in the intersection of all branching apartness relations, and $\#_b$ is the smallest branching apartness relation.

Just as for weak bisimulation and weak apartness we also have that two states are branching bisimilar if and only if they are not branching apart, so we have

$$q \mathrel{\underline{\leftrightarrow}_b} p \qquad \iff \qquad \neg(q \mathrel{\underline{\#}_b} p).$$

Being branching apart is the smallest branching apartness relation, so is an inductive definition that we can define using a derivation system. We can capture $q \mathrel{\underline{\#}_b} p$ using the derivation rules of Fig. 3, where we use a conjunction because in the following we will be studying branching apartness for image-finite systems.

$$\frac{q \to_\tau q' \qquad q' \mathrel{\underline{\#}_b} p \qquad \bigwedge_{\{p,p'' \mid p \twoheadrightarrow_\tau p' \to_\tau p''\}} q \mathrel{\underline{\#}_b} p' \lor q' \mathrel{\underline{\#}_b} p''}{q \mathrel{\underline{\#}_b} p}\ (\mathrm{in}_{b\tau})$$

$$\frac{q \to_a q' \qquad \bigwedge_{\{p',p'' \mid p \twoheadrightarrow_\tau p' \to_a p''\}} q \mathrel{\underline{\#}_b} p' \lor q' \mathrel{\underline{\#}_b} p''}{q \mathrel{\underline{\#}_b} p}\ (\mathrm{in}_b)$$

$$\frac{p \mathrel{\underline{\#}_b} q}{q \mathrel{\underline{\#}_b} p}\ (\mathrm{symm})$$

Fig. 3. The deduction system for deriving $q \mathrel{\underline{\#}_b} p$

4.1 Hennessy-Milner Logic for Branching Bisimulation

We now introduce the modal logic that captures branching bisimulation. The logic is an adaptation of the logic HMLτ_A with an "until" operator instead of a simple unary modality. We also state the well-known Hennessy-Milner theorem using apartness:

$$q \mathrel{\underline{\#}_b} p \iff \exists\varphi(q \models_b \varphi \land p \models_b \neg\varphi),$$

of which we only prove the $(\Rightarrow)$ case, which produces a distinguishing formula from an apartness proof. We will illustrate this with some examples.

Of course the $(\Leftarrow)$ implication above also holds, and it can be proven by contra-position, by proving $q \mathrel{\underline{\leftrightarrow}_b} p \implies \forall\varphi(q \models_b \varphi \implies p \models_b \varphi)$, a proof of which can be found e.g. in [2]. It would be nice to prove it directly, by induction on φ, similar to the proofs of Propositions 1 and 2, but that turns out to be difficult, and we have not yet been able to establish a direct proof.

Definition 7. *We define HMLτb_A by the following set of formulas, given a set of actions A (where $\alpha \in A_\tau$):*

$$\varphi ::= \top \mid \neg\varphi \mid \varphi_1 \land \varphi_2 \mid \varphi_1\langle\alpha\rangle\varphi_2.$$

Let $(S, \to)$ *be an* LTS_τ *over* A. *For* $q \in S$ *and* φ *a formula of* $HML\tau b_A$, *we define the notion* φ holds in state q, *notation* $q \models_b \varphi$, *as follows, by induction on* φ.

- $q \models_b \top$ *always holds.*
- $q \models_b \neg\varphi$ *if* $q \not\models \varphi$.
- $q \models_b \varphi_1 \wedge \varphi_2$ *if* $q \models \varphi_1$ *and* $q \models \varphi_2$.
- $q \models_b \varphi\langle a\rangle\psi$ *if there are states* $q_1, \ldots, q_n, q_{n+1}$ *such that* $q = q_1 \to_\tau \ldots \to_\tau q_n \to_a q_{n+1} \wedge \forall i(1 \le i \le n)\ q_i \models \varphi \wedge q_{n+1} \models \psi$.
- $q \models_b \varphi\langle\tau\rangle\psi$ *if* $q \models \psi$ *or there are states* $q_1, \ldots, q_n, q_{n+1}$ *such that* $q = q_1 \to_\tau \ldots \to_\tau q_n \to_\tau q_{n+1} \wedge \forall i(1 \le i \le n)\ q_i \models \varphi \wedge q_{n+1} \models \psi$.

For $q, p \in S$, *and* $\varphi \in HML\tau b_A$, *we say that* φ distinguishes q, p *if* $q \models_b \varphi$ *and* $p \models_b \neg\varphi$.

Again, the well-known Hennessy-Milner theorem states that $q \underline{\leftrightarrow}_b p$ if and only if $\forall\varphi \in \mathrm{HML}\tau b_A(q \models_b \varphi \Leftrightarrow p \models_b \varphi)$. We state the apartness analogon of this, where we compute a *distinguishing formula* from an apartness proof.

Proposition 3. *Given* $(S, \to)$, *an image-finite* LTS_τ *over* A, *and* $q, p \in S$, *we have*

$$q \mathrel{\underline{\#}_b} p \qquad \Longleftrightarrow \qquad \exists\varphi \in HML\tau b_A(q \models_b \varphi \wedge p \models_b \neg\varphi).$$

Proof. The Proposition is of course a corollary of the bisimulation version, which is just the contra-positive, and which is proved, e.g. in [2]. We only prove ($\Rightarrow$) by induction on the proof of $q \mathrel{\underline{\#}_b} p$.

- If the last applied rule is symm, then by IH we have φ that distinguishes p, q, and therefore $\neg\varphi$ distinguishes q, p.
- If the last applied rule is (in_b), then we have

$$\frac{q \to_a q' \qquad \bigwedge_{\{p',p'' \mid p \twoheadrightarrow_\tau p' \to_a p''\}} q \mathrel{\underline{\#}_b} p' \vee q' \mathrel{\underline{\#}_b} p''}{q \mathrel{\underline{\#}_b} p}\ (\mathrm{in}_b)$$

  where the conjunction is over a finite set of formulas, say that $\{(p', p'') \mid p \twoheadrightarrow_\tau p' \to_a p''\} = \{(p_1, r_1) \ldots, (p_m, r_m)\}$, so the pairs (p_j, r_j) are the states for which we have $p \twoheadrightarrow_\tau p_j \to_a r_j$. By IH we have for each j $(1 \le j \le m)$ a φ_j such that φ_j distinguishes q and p_j ($q \models_b \varphi_j$, $p_j \models_b \neg\varphi_j$), or a ψ_j such that ψ_j distinguishes q' and r_j ($q' \models_b \psi_j$, $r_j \models_b \neg\psi_j$). Now we take

$$\begin{aligned} \Phi &:= \bigwedge_{1\le j\le m} \varphi_j, \\ \Psi &:= \bigwedge_{1\le j\le m} \psi_j, \\ \varphi &:= \Phi\langle a\rangle\Psi. \end{aligned}$$

We have

1. $q \models_b \varphi$: For $q \to_a q'$ we have $q \models \Phi$ and $q' \models_b \Psi$.
2. $p \models_b \neg\varphi$: let $p_1, \ldots, p_n, p_{n+1}$ be such that $p = p_1 \to_\tau \ldots \to_\tau p_n \to_a p_{n+1}$. We know by induction hypothesis that for some j, $p_n \models_b \neg\varphi_j$ (and then $p_n \models_b \neg\Phi$) or $p_{n+1} \models_b \neg\psi_j$ (and then $p_{n+1} \models_b \neg\Psi$). So $\exists i \leq n(p_i \models_b \neg\Phi)$ or $p_{n+1} \models_b \neg\Psi$, which what we needed to prove.

– If the last applied rule is $(\mathrm{in}_{b\tau})$, then we have

$$\frac{q \to_\tau q' \qquad q' \mathrel{\underline{\#}_b} p \qquad \bigwedge_{\{p,p'' \mid p \twoheadrightarrow_\tau p' \to_\tau p''\}} q \mathrel{\underline{\#}_b} p' \vee q' \mathrel{\underline{\#}_b} p''}{q \mathrel{\underline{\#}_b} p}\,(\mathrm{in}_{b\tau})$$

where the conjunction is over a finite set of formulas, say that $\{(p', p'') \mid p \twoheadrightarrow_\tau p' \to_\tau p''\} = \{(p_1, r_1) \ldots, (p_m, r_m)\}$, so the pairs (p_j, r_j) are the states for which we have $p \twoheadrightarrow_\tau p_j \to_\tau r_j$. By IH we have a φ_0 for which $q' \models_b \varphi_0$ and $p \models_b \neg\varphi_0$. Also by IH we have for each j $(1 \leq j \leq m)$ a φ_j such that $q \models_b \varphi_j$ and $p_j \models_b \neg\varphi_j$, or a ψ_j such that $q' \models_b \psi_j$ and $r_j \models_b \neg\psi_j$. Now we take

$$\begin{aligned}
\Phi &:= \bigwedge_{1 \leq j \leq m} \varphi_j,\\
\Psi &:= \varphi_0 \wedge \bigwedge_{1 \leq j \leq m} \psi_j,\\
\varphi &:= \Phi\langle\tau\rangle\Psi.
\end{aligned}$$

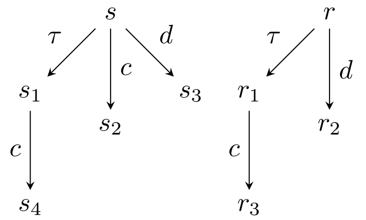
We have

1. $q \models_b \varphi$: For $q \to_\tau q'$ we have $q \models \Phi$ and $q' \models_b \Psi$.
2. $p \models_b \neg\varphi$: $p \models_b \neg\Psi$ (by $p \models_b \neg\varphi_0$) and for $p_1, \ldots, p_n, p_{n+1}$ with $p = p_1 \to_\tau \ldots \to_\tau p_n \to_a p_{n+1}$ we know by induction hypothesis that for some j: $p_n \models_b \neg\varphi_j$ (and then $p_n \models_b \neg\Phi$) or $p_{n+1} \models_b \neg\psi_j$ (and then $p_{n+1} \models_b \neg\Psi$). So $\exists i \leq n(p_i \models_b \neg\Phi)$ or $p_{n+1} \models_b \neg\Psi$, which what we needed to prove. $\square$

4.2 Examples

We now give some examples of how to compute a distinguishing formula from an apartness proof.

Example 4. The first example is a well-known LTS_τ with two states that are not branching bisimilar and we give the proof of their branching apartness and compute the distinguishing formula from that proof.



We give a derivation of $s \underline{\#}_b r$, where we indicate between $[\ldots]$ all possible transitions that we need to prove a hypothesis for (just one in the case of the c-step; none in the case of the d-step).

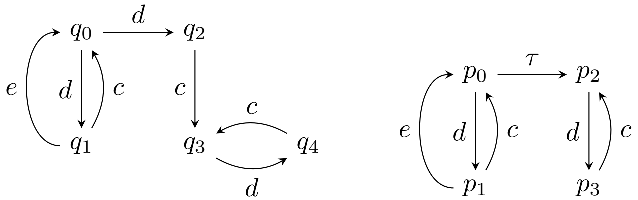
$$\dfrac{s \to_c s_2 \qquad \dfrac{[r \to_\tau r_1 \to_c r_3] \qquad \dfrac{\dfrac{s \to_d s_3 \qquad \dfrac{[\_] \quad \checkmark}{\forall r', r''(r_1 \twoheadrightarrow_\tau r' \to_d r'' \implies s \underline{\#}_b r' \lor s_3 \underline{\#}_b r'')}}{s \underline{\#}_b r_1}}{s \underline{\#}_b r_1 \lor s_2 \underline{\#}_b r_3}}{\forall r', r''(r \twoheadrightarrow_\tau r' \to_c r'' \implies s \underline{\#}_b r' \lor s_2 \underline{\#}_b r'')}}{s \underline{\#}_b r}$$

The distinguishing formula that we compute from this derivation, following the proof of Proposition 3 is

$$(\top\langle d\rangle\top)\langle c\rangle\top,$$

which holds in state s and expresses that there is a τ-path to a state where a c-step is possible, and in all states along that τ-path, a d-step is possible.

Example 5. We have the LTS given below, for which we have $q_0 \underline{\#}_b p_0$, which we prove and then compute the distinguishing formula.



A derivation of $q_0 \underline{\#}_b p_0$ is the following, where for space reasons, we singled out the sub-derivation of $q_0 \underline{\#}_b p_2$, which we call Σ. Again, we indicate between $[\ldots]$ all possible transitions that we need to prove a hypothesis for.

$$\dfrac{q_0 \to_d q_2 \qquad \dfrac{[p_0 \to_d p_1] \qquad \dfrac{\dfrac{\dfrac{p_1 \to_e p_0}{p_1 \underline{\#}_b q_2}}{q_2 \underline{\#}_b p_1}}{q_0 \underline{\#}_b p_0 \lor q_2 \underline{\#}_b p_1} \qquad [p_0 \twoheadrightarrow_\tau p_2 \to_d p_3] \qquad \dfrac{\Sigma}{q_0 \underline{\#}_b p_2 \lor q_2 \underline{\#}_b p_3}}{\forall p', p''(p_0 \twoheadrightarrow_\tau p' \to_d p'' \implies q_0 \underline{\#}_b p' \lor q_2 \underline{\#}_b p'')}}{q_0 \underline{\#}_b p_0}$$

And here is the sub-derivation Σ of $q_0 \underline{\#}_b p_2$:

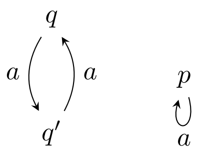
$$\Sigma := \dfrac{q_0 \to_d q_1 \qquad \dfrac{[p_2 \twoheadrightarrow_\tau p_2 \to_d p_3] \qquad \dfrac{\dfrac{q_1 \to_e q_0}{q_1 \underline{\#}_b p_3}}{q_0 \underline{\#}_b p_2 \lor q_1 \underline{\#}_b p_3}}{\forall p', p''(p_2 \twoheadrightarrow_\tau p' \to_d p'' \implies q_0 \underline{\#}_b p' \lor q_1 \underline{\#}_b p'')}}{q_0 \underline{\#}_b p_2}$$

The distinguishing formula that we compute from Σ is $\top \langle d \rangle (\top \langle e \rangle \top)$. The distinguishing formula for $q_0 \mathrel{\underline{\#}_b} p_0$ is

$$\Phi := (\top \langle d \rangle (\top \langle e \rangle \top)) \langle d \rangle \neg(\top \langle e \rangle \top)$$

We have $q_0 \models_b \Phi$ and $p_0 \models_b \neg\Phi$.

Example 6. We can also use the proof system for $\underline{\#}_b$ to establish that $q \mathrel{\underline{\leftrightarrow}_b} p$. Here is a simple example to illustrate this.



If $q \mathrel{\underline{\#}_b} p$, then there is a *shortest derivation* of $q \mathrel{\underline{\#}_b} p$, and we notice that it doesn't exist. Therefore we can conclude that $\neg q \mathrel{\underline{\#}_b} p$ and so $q \mathrel{\underline{\leftrightarrow}_b} p$. In our search for a derivation of $q \mathrel{\underline{\#}_b} p$ we have to keep track of goals that we have already encountered; the search would proceed as follows:

$$\dfrac{q \to_a q' \qquad \dfrac{\dfrac{q' \to_a q \qquad \dfrac{\text{fail}}{q' \mathrel{\underline{\#}_b} p \vee q \mathrel{\underline{\#}_b} p}}{q' \mathrel{\underline{\#}_b} p}}{q \mathrel{\underline{\#}_b} p \vee q' \mathrel{\underline{\#}_b} p}}{q \mathrel{\underline{\#}_b} p}$$

4.3 Related and Further Work

Of course, the concept of observations is well-known and tightly related to bisimulation. Korver [8] presents an algorithm that, if two states are not branching bisimilar, produces a formula in Hennessy-Milner [6] logic with until operator that distinguishes the two states. This work implicitly uses the notion of apartness without singling out its proof rules. Another work is Chow [1] on testing equivalence of states in finite state machines and more recent work is by Smetsers et al. [9], where an efficient algorithm is presented for finding a minimal separating sequence for a pair of in-equivalent states in a finite state machine. It would be interesting to see whether this work, and the idea of finding such a separating sequence, can be formulated in terms of apartness, and if the algorithms can be improved using that approach. In general it would be interesting to understand the various efficient algorithms for checking branching bisimulation [5,7] in terms of apartness. A first concrete application of apartness for studying systems has been made by Vaandrager and colleagues [10] in the development of a new automata learning algorithm.

For the meta-theoretic study of bisimulation, it sometimes pays off to go to the "dual view" of apartness, for one because apartness is an inductive notion, so we have an induction principle. There are examples of that in [3]. Also, sometime, the apartness view just gives a different, fresh, angle on bisimulation which might be fruitful. We have also seen examples where the bisimulation view works much better than the apartness view, e.g. in the proof of the reverse implication of Proposition 3, which we have not been able to establish directly (without first going from the apartness-view to the bisimulation-view). It is interesting to understand why this is the case.

Finally, we believe that apartness and the proof system for apartness may provide useful in studying more quantitative or qualitative notions of distinguishability: how "different" are two states and in which points do they differ? The latter is already established by the Hennessy-Milner formula, but one can also think of this in a more "directed sense", by studying a notion of "directed apartness" (as a dual to simulation?) and witnesses establishing that states are not simulated by others.

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