Deriving natural deduction rules from truth tables

(Or: How to define If-then-else as a constructive connective)

Herman Geuvers and Tonny Hurkens

Institute for Computing and Information Science
Radboud University
Nijmegen, NL

7th Indian Conference on Logic and its Applications
Indian Institute of Technology, Kanpur, India
January 2017

Classical and Constructive Logic

Classically, the meaning of a propositional connective is fixed by its truth table. This immediately implies

- consistency,
- a decision procedure,
- completeness (w.r.t. Boolean algebra's).

Constructively (following the Brouwer-Heyting-Kolmogorov interpretation), the meaning of a connective is fixed by explaining what a proof is that involves the connective.
Basically, this explains the introduction rule(s) for each connective, from which the elimination rules follow (Prawitz)
By analysing constructive proofs we then also get

- consistency (from proof normalization),
- a decision procedure (from the subformula property),
- completeness (w.r.t. Heyting algebra's and Kripke models).

This talk/paper

- Derive natural deduction rules for a connective from its truth table definition.
    - Also works for constructive logic. (Classical case known from Milne.)
    - Gives natural deduction rules for a connective "in isolation"
    - Also gives (constructive) rules for connectives that haven't been studied so far, like if-then-else.
- We study constructive if-then-else. (With $\bot$ and $\top$ it is functionally complete.)
- We give a general Kripke model for these constructive connectives. (Sound and Complete)
- We define a general notion of cut-elimination for these constructive connectives.

Standard form for natural deduction rules

$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n \qquad \Gamma, B_1 \vdash D \quad \dots \quad \Gamma, B_m \vdash D}{\Gamma \vdash D}$$

If the conclusion of a rule is $\Gamma \vdash D$, then the hypotheses of the rule can be of one of two forms:

1. $\Gamma, B \vdash D$: we are given extra data B to prove D from Γ. We call B a Casus.
2. $\Gamma \vdash A$: instead of proving D from Γ, we now need to prove A from Γ. We call A a Lemma.

One obvious advantage: we don't have to give the Γ explicitly, as it can be retrieved:

$$\frac{\vdash A_1 \quad \dots \quad \vdash A_n \qquad B_1 \vdash D \quad \dots \quad B_m \vdash D}{\vdash D}$$

Some well-known constructive rules

Rules that follow this format:

$$\frac{\vdash A \vee B \qquad A \vdash D \qquad B \vdash D}{\vdash D}\ \vee\text{-el} \qquad \frac{\vdash A \wedge B \qquad A \vdash D}{\vdash D}\ \wedge\text{-el}$$

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\ \wedge\text{-in}$$

Rule that does not follow this format:

$$\frac{A \vdash B}{\vdash A \to B}\ \to\text{-in}$$

Natural Deduction rules from truth tables

Let c be an n-ary connective c with truth table t_c.
Each row of t_c gives rise to an elimination rule or an introduction rule for c. (We write $\varphi = c(A_1, \ldots, A_n)$.)

A_1	$\ldots$	A_n	φ
p_1	$\ldots$	p_n	0

$$\mapsto \quad \frac{\vdash \varphi \ldots \vdash A_j \text{ (if } p_j = 1) \ldots A_i \vdash D \text{ (if } p_i = 0) \ldots}{\vdash D}\,\text{el}$$

constructive intro

A_1	$\ldots$	A_n	φ
q_1	$\ldots$	q_n	1

$$\mapsto \quad \frac{\ldots \vdash A_j \text{ (if } q_j = 1) \ldots A_i \vdash \varphi \text{ (if } q_i = 0) \ldots}{\vdash \varphi}\,\text{in}^i$$

classical intro

A_1	$\ldots$	A_n	φ
r_1	$\ldots$	r_n	1

$$\mapsto \quad \frac{\varphi \vdash D \ldots \vdash A_j \text{ (if } r_j = 1) \ldots A_i \vdash D \text{ (if } r_i = 0) \ldots}{\vdash D}\,\text{in}^c$$

Examples

Constructive rules for $\wedge$ (3 elim rules and one intro rule):

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D} \wedge\text{-el}_a \qquad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D} \wedge\text{-el}_b$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D} \wedge\text{-el}_c \qquad \frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}$$

- These rules can be shown to be equivalent to the well-known constructive rules.
- These rules can be optimized to 3 rules.

Examples

Rules for $\neg$: 1 elimination rule and 1 introduction rule.

A	$\neg A$
0	1
1	0

Constructive:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A}\neg\text{-in}^i$$

Classical:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\neg\text{-el} \qquad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D}\neg\text{-in}^c$$

Lemma I to simplify the rules

$$\frac{\vdash A_1 \ \ldots \ \vdash A_n \quad B_1 \vdash D \ \ldots \ B_m \vdash D \quad C \vdash D}{\vdash D}$$

$$\frac{\vdash A_1 \ \ldots \ \vdash A_n \quad \vdash C \quad B_1 \vdash D \ \ldots \ B_m \vdash D}{\vdash D}$$

is equivalent to the system with these two rules replaced by

$$\frac{\vdash A_1 \ \ldots \ \vdash A_n \quad B_1 \vdash D \ \ldots \ B_m \vdash D}{\vdash D}$$

Lemma II to simplify the rules

A system with a deduction rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.

$$\frac{\vdash A_1 \;\ldots\; \vdash A_n \quad B \vdash D}{\vdash D} \qquad\qquad \frac{\vdash A_1 \;\ldots\; \vdash A_n}{\vdash B}$$

The constructive connectives

We have already seen the $\wedge, \neg$ rules. The optimised rules for $\vee, \to, \top$ and $\bot$ we obtain are:

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \vee\text{-el} \qquad \frac{\vdash A}{\vdash A \vee B}\ \vee\text{-in}_1 \qquad \frac{\vdash B}{\vdash A \vee B}\ \vee\text{-in}_2$$

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \to\text{-el} \qquad \frac{\vdash B}{\vdash A \to B} \to\text{-in}_1 \qquad \frac{A \vdash A \to B}{\vdash A \to B} \to\text{-in}_2$$

$$\frac{}{\vdash \top}\ \top\text{-in} \qquad \frac{\vdash \bot}{\vdash D}\ \bot\text{-el}$$

The rules for the classical $\to$ connective

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \to\text{-el} \qquad \frac{\vdash B}{\vdash A \to B} \to\text{-in}_1 \qquad \frac{A \vdash D \quad A \to B \vdash D}{\vdash D} \to\text{-in}_2^c$$

Derivation of Peirce's law:

$$\dfrac{\dfrac{A \vdash A}{A \vdash ((A \to B) \to A) \to A} \qquad \dfrac{\dfrac{\dfrac{(A \to B) \to A \vdash (A \to B) \to A \quad A \to B \vdash A \to B}{A \to B, (A \to B) \to A \vdash A}}{A \to B, (A \to B) \to A \vdash ((A \to B) \to A) \to A}}{A \to B \vdash ((A \to B) \to A) \to A}}{\vdash ((A \to B) \to A) \to A} \to\text{-in}_2^c$$

The "If Then Else" connective

Notation: $A \to B/C$ for if A then B else C.

p	q	r	$p \to q/r$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

The optimized constructive rules are:

$$\frac{\vdash A \to B/C \quad \vdash A}{\vdash B}\ \text{then-el} \qquad \frac{\vdash A \to B/C \quad A \vdash D \quad C \vdash D}{\vdash D}\ \text{else-el}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \to B/C}\ \text{then-in} \qquad \frac{A \vdash A \to B/C \quad \vdash C}{\vdash A \to B/C}\ \text{else-in}$$

Some facts about constructive "If Then Else"

$A \to B / C$ is logically equivalent to $(A \to B) \wedge (A \vee C)$

We have the well-known classical equivalence

$$\text{if } A \text{ then } B \text{ else } B \equiv B$$

We don't have the other well-known classical equivalences

$$\text{if (if } A \text{ then } B \text{ else } C\text{) then } D \text{ else } E \quad \not\models$$
$$\text{if } A \text{ then (if } B \text{ then } D \text{ else } E\text{) else (if } C \text{ then } D \text{ else } E\text{)}$$

$$\text{if } A \text{ then (if } B \text{ then } D \text{ else } E\text{) else (if } C \text{ then } D \text{ else } E\text{)} \quad \not\models$$
$$\text{if (if } A \text{ then } B \text{ else } C\text{) then } D \text{ else } E$$

"If Then Else" $+\top+\bot$ is functionally complete

We define the usual constructive connectives in terms of if-then-else, $\top$ and $\bot$:

$$A \dot{\vee} B := A{\to}A/B \qquad A \dot{\wedge} B := A{\to}B/A$$

$$A \dot{\to} B := A{\to}B/\top \qquad \dot{\neg}A := A{\to}\bot/\top$$

LEMMA The defined connectives satisfy the original constructive deduction rules for these same connectives.

COROLLARY The constructive connective if-then-else, together with $\top$ and $\bot$, is functionally complete.

Kripke semantics for the constructive rules

For each n-ary connective c, we assume a truth table $t_c : \{0,1\}^n \to \{0,1\}$ and the defined constructive deduction rules.

DEFINITION A Kripke model is a triple $(W, \leq, \mathrm{at})$ where W is a set of worlds, $\leq$ a reflexive, transitive relation on W and a function $\mathrm{at} : W \to \wp(\mathrm{At})$ satisfying $w \leq w' \Rightarrow \mathrm{at}(w) \subseteq \mathrm{at}(w')$.

We define the notion φ is true in world w (usually written $w \Vdash \varphi$) by defining $[\![\varphi]\!]_w \in \{0,1\}$

DEFINITION of $[\![\varphi]\!]_w \in \{0,1\}$, by induction on φ:

- (atom) if φ is atomic, $[\![\varphi]\!]_w = 1$ iff $\varphi \in \mathrm{at}(w)$.
- (connective) for $\varphi = c(\varphi_1, \ldots, \varphi_n)$, $[\![\varphi]\!]_w = 1$ iff for each $w' \geq w$, $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 1$ where t_c is the truth table of c.

$\Gamma \models \psi$:= for each Kripke model and each world w, if $[\![\varphi]\!]_w = 1$ for each φ in Γ, then $[\![\psi]\!]_w = 1$.

Kripke semantics for the constructive rules

LEMMA (Soundness) If $\Gamma \vdash \psi$, then $\Gamma \models \psi$
Proof. Induction on the derivation of $\Gamma \vdash \psi$.

For completeness we need to construct a special Kripke model.

- In the literature, the completeness of Kripke semantics is proved using *prime theories*.
- A theory is prime if it satisfies the disjunction property: if $\Gamma \vdash A \vee B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$.
- We may not have $\vee$ in our set of connective, and we may have others that "behave $\vee$-like"',
- (Later, we will generalize the disjunction property to arbitrary *n*-ary constructive connectives.)
- We apply a kind of Lindenbaum construction (also used by Milne for the classical case).

Kripke semantics for the constructive rules

DEFINITION For ψ a formula and Γ a set of formulas, we say that Γ is ψ-maximal if

- $\Gamma \nvdash \psi$ and
- for every formula $\varphi \notin \Gamma$ we have: $\Gamma, \varphi \vdash \psi$.

NB. Given ψ and Γ such that $\Gamma \nvdash \psi$, we can extend Γ to a ψ-maximal set Γ' that contains Γ.

Simple important facts about ψ-maximal sets Γ:

1. For every φ, we have $\varphi \in \Gamma$ or $\Gamma, \varphi \vdash \psi$.
2. For every φ, if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.

Completeness of Kripke semantics

DEFINITION We define the Kripke model $U = (W, \leq, \mathrm{at})$:

- $W := \{(\Gamma, \psi) \mid \Gamma \text{ is a } \psi\text{-maximal set}\}$.
- $(\Gamma, \psi) \leq (\Gamma', \psi') := \Gamma \subseteq \Gamma'$.
- $\mathrm{at}(\Gamma, \psi) := \Gamma \cap \mathrm{At}$.

LEMMA In the model U we have, for all worlds $(\Gamma, \psi) \in W$:

$$\varphi \in \Gamma \iff [\![\varphi]\!]_{(\Gamma,\psi)} = 1 \quad (\forall \varphi)$$

Proof: Induction on the structure of φ.

THEOREM If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.
Proof. Suppose $\Gamma \models \psi$ and $\Gamma \not\vdash \psi$. Then we can find a ψ-maximal superset Γ' of Γ such that $\Gamma' \not\vdash \psi$. In particular: ψ is not in Γ'. So (Γ', ψ) is a world in the Kripke model U in which each member of Γ is true, but ψ is not. Contradiction to $\Gamma \models \psi$, so $\Gamma \vdash \psi$.

A generalised disjunction property

We say that the n-ary connective c is i, j-splitting in case the truth table for c has the following shape

A_1	$\ldots$	A_i	$\ldots$	A_j	$\ldots$	A_n	$c(A_1, \ldots, A_n)$
–	$\ldots$	0	$\ldots$	0	$\ldots$	–	0
–	$\ldots$	0	$\ldots$	0	$\ldots$	–	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
–	$\ldots$	0	$\ldots$	0	$\ldots$	–	0
–	$\ldots$	0	$\ldots$	0	$\ldots$	–	0

In terms of t_c:

$$t_c(p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n) = 0$$

for all $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n \in \{0, 1\}$.

LEMMA Let c be an i, j-splitting connective.
If $\vdash c(A_1, \ldots, A_n)$, then $\vdash A_i$ or $\vdash A_j$.

Examples of connectives with a splitting property

A	B	C	most(A, B, C)	$A \to B/C$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

- most is i, j-splitting for every i, j:
  - if $\vdash$ most(A_1, A_2, A_3), then $\vdash A_i$ or $\vdash A_j$, for any pair $i \neq j$.
- if-then-else is $1, 3$-splitting and $2, 3$-splitting (but not $1, 2$-splitting):
  - if $\vdash A \to B/C$, then $\vdash A$ or $\vdash C$ and also $\vdash B$ or $\vdash C$.
  - if $\vdash A \to B/C$, then not $\vdash A$ or $\vdash B$

Radboud University

Substituting a deduction in another

LEMMA: If $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$

If Σ is a deduction of $\Gamma \vdash \varphi$ and Π is a deduction of $\Delta, \varphi \vdash \psi$, then we have the following deduction of $\Gamma, \Delta \vdash \psi$:

$$\begin{array}{ccc} \vdots\ \Sigma & & \vdots\ \Sigma \\ \Gamma \vdash \varphi & \ldots & \Gamma \vdash \varphi \\ & \vdots\ \Pi & \\ & \Delta \vdash \psi & \end{array}$$

In Π, every application of an (axiom) rule at a leaf, deriving $\Delta' \vdash \varphi$ for some $\Delta' \supseteq \Delta$ is replaced by a copy of a deduction Σ, which is also a deduction of $\Delta', \Gamma \vdash \varphi$.

Cuts in constructive logic

Remember that the rules for c arise from rows in the truth table t_c:

A_1	$\ldots$	A_n	$c(A_1, \ldots, A_n)$
p_1	$\ldots$	p_n	0
q_1	$\ldots$	q_n	1

DEFINITION A constructive direct cut is a pattern of the following form, where $\varphi = c(A_1, \ldots, A_n)$.

$$\dfrac{\dfrac{\ldots \quad \begin{array}{c}\vdots\ \Sigma_j\\ \Gamma \vdash A_j\end{array} \quad \ldots \quad \begin{array}{c}\vdots\ \Sigma_i\\ \Gamma, A_i \vdash \varphi\end{array} \quad \ldots}{\Gamma \vdash \varphi}\,\text{in} \qquad \ldots \quad \begin{array}{c}\vdots\ \Pi_k\\ \Gamma \vdash A_k\end{array} \quad \ldots \quad \begin{array}{c}\vdots\ \Pi_\ell\\ \Gamma, A_\ell \vdash D\end{array} \quad \ldots}{\Gamma \vdash D}\,\text{el}$$

- $q_j = 1$ for A_j and $q_i = 0$ for A_i
- $p_k = 1$ for A_k and $p_\ell = 0$ for A_ℓ

Radboud University

Eliminating a direct cut (I)

The *elimination of a direct cut* is defined by replacing the deduction pattern by another one. If $\ell = j$ (for some ℓ, j), replace

$$\dfrac{\dfrac{\ldots\quad \begin{matrix}\vdots\ \Sigma_j\\ \Gamma \vdash A_j\end{matrix}\quad \ldots\quad \begin{matrix}\vdots\ \Sigma_i\\ \Gamma, A_i \vdash \varphi\end{matrix}\quad \ldots}{\Gamma \vdash \varphi}\,\text{in}\qquad \ldots\quad \begin{matrix}\vdots\ \Pi_k\\ \Gamma \vdash A_k\end{matrix}\quad \ldots\quad \begin{matrix}\vdots\ \Pi_\ell\\ \Gamma, A_\ell \vdash D\end{matrix}\quad \ldots}{\Gamma \vdash D}\,\text{el}$$

by

$$\begin{matrix} \begin{matrix}\vdots\ \Sigma_j\\ \Gamma \vdash A_j\end{matrix} \quad \ldots \quad \begin{matrix}\vdots\ \Sigma_j\\ \Gamma \vdash A_j\end{matrix} \\ \vdots\ \Pi_\ell \\ \Gamma \vdash D \end{matrix}$$

Eliminating a direct cut (II)

If $k = i$ (for some k, i), replace

$$\dfrac{\dfrac{\ldots \quad \begin{array}{c}\vdots\ \Sigma_j\\ \Gamma\vdash A_j\end{array} \quad\ldots\quad \begin{array}{c}\vdots\ \Sigma_i\\ \Gamma, A_i\vdash\varphi\end{array}\quad\ldots}{\Gamma\vdash\varphi}\ \text{in} \qquad \ldots\quad \begin{array}{c}\vdots\ \Pi_k\\ \Gamma\vdash A_k\end{array}\quad\ldots\quad \begin{array}{c}\vdots\ \Pi_\ell\\ \Gamma, A_\ell\vdash D\end{array}\quad\ldots}{\Gamma\vdash D}\ \text{el}$$

by

$$\dfrac{\begin{array}{c}\begin{array}{ccc}\vdots\ \Pi_k & & \vdots\ \Pi_k\\ \Gamma\vdash A_i & \ldots & \Gamma\vdash A_i\end{array}\\ \vdots\ \Sigma_i\\ \Gamma\vdash\varphi\end{array} \qquad \ldots\quad \begin{array}{c}\vdots\ \Pi_k\\ \Gamma\vdash A_i\end{array}\quad\ldots\quad \begin{array}{c}\vdots\ \Pi_\ell\\ \Gamma, A_\ell\vdash D\end{array}\quad\ldots}{\Gamma\vdash D}$$

Cuts for if-then-else (I)

The cut-elimination rules for if-then-else are the following.

(then-then)

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \begin{array}{c}\vdots\ \Sigma \\ \Gamma \vdash B\end{array}}{\Gamma \vdash A{\to}B/C}\,\text{in} \qquad \Gamma \vdash A}{\Gamma \vdash B}\,\text{el} \qquad \mapsto \qquad \begin{array}{c}\vdots\ \Sigma \\ \Gamma \vdash B\end{array}$$

(else-then)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Sigma \\ \Gamma, A \vdash A{\to}B/C\end{array} \qquad \Gamma \vdash C}{\Gamma \vdash A{\to}B/C}\,\text{in} \qquad \begin{array}{c}\vdots\ \Pi \\ \Gamma \vdash A\end{array}}{\Gamma \vdash B}\,\text{el} \qquad \mapsto \qquad \dfrac{\begin{array}{c}\begin{array}{c}\vdots\ \Pi \\ \Gamma \vdash A\end{array} \ \ldots \ \begin{array}{c}\vdots\ \Pi \\ \Gamma \vdash A\end{array} \\ \vdots\ \Sigma \\ \Gamma \vdash A{\to}B/C\end{array} \qquad \begin{array}{c}\vdots\ \Pi \\ \Gamma \vdash A\end{array}}{\Gamma \vdash B}\,\text{el}$$

Radbound University

Cuts for if-then-else (II)

(then-else)

$$\dfrac{\dfrac{\overset{\vdots\,\Sigma}{\Gamma \vdash A} \quad \Gamma \vdash B}{\Gamma \vdash A \to B/C}\,\text{in} \qquad \overset{\vdots\,\Pi}{\Gamma, A \vdash D} \qquad \Gamma, C \vdash D}{\Gamma \vdash D}\,\text{el} \quad \mapsto \quad \begin{array}{c} \overset{\vdots\,\Sigma}{\Gamma \vdash A} \;\ldots\; \overset{\vdots\,\Sigma}{\Gamma \vdash A} \\ \vdots\,\Pi \\ \Gamma \vdash D \end{array}$$

(else-else)

$$\dfrac{\dfrac{\Gamma, A \vdash A \to B/C \quad \overset{\vdots\,\Sigma}{\Gamma \vdash C}}{\Gamma \vdash A \to B/C}\,\text{in} \qquad \Gamma, A \vdash D \qquad \overset{\vdots\,\Pi}{\Gamma, C \vdash D}}{\Gamma \vdash D}\,\text{el} \quad \mapsto \quad \begin{array}{c} \overset{\vdots\,\Sigma}{\Gamma \vdash C} \;\ldots\; \overset{\vdots\,\Sigma}{\Gamma \vdash C} \\ \vdots\,\Pi \\ \Gamma \vdash D \end{array}$$

Conclusions

- Simple way to construct deduction rules for new connectives, constructively and classically
- Study connectives “in isolation”. (Without defining them and without using other connectives.)
- Generic Kripke semantics
- Correct (?) constructive reading of if-then-else:
    - Functionally complete (with $\top$ and $\bot$)
    - Proper constructive “splitting” properties

Further and Future work, Related work

Further work:

- Add rules for commuting cuts to get the “right” normal form of derivations. (Done for if-then-else.)
- Study of Normalization for cut-elimination for if-then-else.
- Curry-Howard interpretation of formulas-as-types and proofs-as-terms:
  - Proofs as programs and cut-elimination as evaluation (reduction)
  - Meaning of the new connectives as data types

Future work:

- General definition of classical cut-elimination
- Relation with other term calculi for classical logic: subtraction logic, $\lambda\mu$ (Parigot), $\bar{\lambda}\mu\tilde{\mu}$ (Curien, Herbelin).

Related work:

- Peter Milne, Jan von Plato and Sara Negri, Peter Schroeder-Heister, ...