# Worlds Coalgebraically

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A finite array $\big(A,\ sel,\ upd\big)$ with values from the set $V$ and locations in the set $L$ is the set $A$ equipped with $sel : A \times L \to V$ and $upd : A \times L \times V \to A$ operations subject to the equations of the *theory of global state* [1], [2]:
$sel(upd(a,\ loc,\ v),\ loc) = v$, $upd(a,\ loc, sel(a,\ loc)) = a$,
$upd(upd(a,\ loc,\ v),\ loc,\ v') = upd(a,\ loc,\ v')$ and $upd(upd(a,\ loc,\ v),\ loc',\ v') = upd(upd(a,\ loc',\ v'),\ loc,\ v)$ whenever $loc \neq loc'$.

The set $A$ is isomorphic to some $R$ copies of the finite map $S = V^L$. Arrays with values from $V$ and locations from $L$ form a category $(L,\ V)$–*Array*, where an array morphism is a map in *Set*, which respects the operations in the obvious way.

In [2] we have shown that $(L,\ V)$–*Array* is equivalent to the category $Set_G$ of coalgebras of the comonad $G(X) = X^S \times S$. In the presented work we strengthen this result. The categories are *isomorphic*. The isomorphism sends an array $\big(A,\ sel,\ upd\big)$ to the coalgebra $\big(A,\ \alpha : A \to A^S \times S\big)$, where transitions $\alpha_1(a) = \overline{upd}(a,\ -)$ and observations $\alpha_2(a) = \overline{sel}(a)$, are defined by maps $\overline{upd}(a,\ (v_1,\ \ldots v_n)) = upd(\ldots\ upd(a,\ 1,\ v_1)\ \ldots, n,\ v_n)$ and $\overline{sel}(a) = (sel(a,\ 1),\ \ldots, sel(a,\ n))$.

To model fresh memory allocation one replaces the base category *Set* with the presheaf category $[\![Inj,\ Set]\!]$, where *Inj* is the category of natural numbers and injections. The presheaf category is cartesian closed with the product of presheaves defined elementwise and the exponent $Y^X$ at $n$ the set of natural transformations $Xm \times Inj'(n,\ m) \stackrel{\cdot}{\to} Ym$.

In [3] the author gives a general construction for models of indexed theories on $[\![Inj,\ Set]\!]$. There to create a model at world $n$ one needs a natural transformation $block_n : A(n+1) \stackrel{\cdot}{\to} (An)^V$ subject to a few equations.

In this work we consider a simplified dual construction for extending from global to local state, for presheaves $[\![Inj',\ Set]\!]$, where $Inj''$ is obtained from *Inj* by removing all arrows of type $1 \to n$ with $n > 2$ ("new"-arrows). As in [3], given a presheaf $A$ we consider the sequence of comodels $\big(An,\ sel_n,\ upd_n\big)$. A $V^\star$-*array* $\big(A,\ sel,\ upd\big)$ is a presheaf $A$ equipped with the natural transformations $sel : A \times L \Rightarrow V$, $upd : A \times L \times V \Rightarrow A$ whose $n$th components satisfy

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the axiomatics at world $n$. Here $V$ is the constant functor. The functor $L$ sends $n$ to $\{1, \ldots n\}$ and $Lold^i_n : \{1, \ldots, n\} \to \{1, \ldots, n+1\}$ sends $l \leq i$ to $l$, fixes the cell $i+1$ as the fresh one and sends $l > i$ to $l+1$. The functor $S$ sends $n$ to $V^n$ and $Sold^i_n : (v_1, \ldots, v_n) \mapsto (v_1, \ldots, v_i, v_0, v_{i+1}, \ldots, v_n)$, where $v_0$ is a fixed constant from the set $V$ .

Our *main result* is that *the category* $V^*$*–Array is isomorphic to* $[\![Inj',\ Set]\!]_G$. The proof is done directly by the definitions. Alternatively, one may use the fact that $\overline{upd}_n(-,-) : An \times Sn \stackrel{\cdot}{\to} An$ and the first component of the coalgebra structure map $\alpha_{1,n} : An \stackrel{\cdot}{\to} (A^S)n$ are in a 1-1 correspondence due to the adjunction. In more detail, if $a \in An$, $s \in Sm$, $f \in Inj'(n, m)$ then $\alpha_{1,n,m}(a)(s, f) = \overline{upd}_m(Af(a), s)$, and, for the inverse, if $a \in An$, $s \in Sn$ then $\overline{upd}_n(a, s) = \alpha_{1,n,n}(a)(s, \mathbf{id}_n)$.

Contrary to [3] we do not consider *block* operations and let the arrows $old^i_n$, with $0 \leq i \leq n$, of $Inj'$ create the transition to the comodel at world $n+1$. The arrow $old^i_n$ represents the map $Lold^i_n$.

To ensure the coherence of *sel* and *upd* operations in different worlds we add *transition axioms* $sel_n(Aold^n_i(a), i+1) = v_0$. The constant $v_0$ play a role of the default value which is placed into a fresh location whenever it is created.

The price for this simplification is that we cannot speak about an equational theory now, since we have equations with mixed levels of arrows.

One may define the *dual* $block_n : An \times V \stackrel{\cdot}{\to} A(n+1)$ as $block_n(a, v) = \big(\lambda\, i.upd_{n+1}(Aold^{i-1}_n(a), i, v)\big)$, where $1 \leq i \leq n+1$, and this $block_n$ will satisfy the following equations, similar to the *block*-axioms from [1]:
$upd_{n+1}(block_n(a, v)(loc), loc, v') = block_n(a, v')(loc)$,
$sel_{n+1}(block_n(a, v)(loc), loc) = v$,
$block_{n+1}(block_n(a, v)(loc), v')(Lold^{loc-1}_n(loc')) =$
$block_{n+1}(block_n(a, v')(loc'), v)(Lold^{loc'-1}_n(loc))$ whenever $loc \neq loc'$, and
$upd_{n+1}(block_n(a, v)(loc), Lold^{loc-1}_n(loc'), v') = block_n(upd_n(a, loc', v'), v)(loc)$.

One uses naturality of *sel* and *upd* to prove these equations. The connections between approaches "with" and "without" *block* still needs to be explored. In particular, one should check if any of *block* maps, satisfying the equations may be presented via *old* arrows as above.

## References


[1] G. D. Plotkin, A. J. Power. *Notions of Computation determine Monads*, Proc. FOSSACS 2002, Lecture Notes in Computer Science, **2303**, Springer-Verlag, 2002, pp. 342 - 356.

[2] A. J. Power, O. Shkaravska. *From Comodels to Coalgebras: State and Arrays*, Proc. CMCS 2004, Electronic Notes in Theoretical Computer Science, **106** (2004)

[3] A.J. Power. *Semantics for Local Computational Effects*, MFPS, submitted.