Computational Methods and Data Analysis 2002

Ex. 2: Solution of initial value differential equations

In this exercise you will solve the same problem with three different algorithms. In the final report you should comment about the advantages/disadvantages of each method.

Consider a sphere falling from rest in presence of air resistance. The Newton's equation of motion read:

$$m\frac{d^2x}{dt^2} = mg - kv^2 \tag{1}$$

First we examine only the behaviour of the velocity. In this case we have a first-order differential equation for the rate of change of velocity

$$m\frac{dv}{dt} = mg - kv^2 \tag{2}$$

where $m = 10^{-2}$ kg is the mass of the object, v the velocity, $g = 9,8$ ms^{-2} is the acceleration due to gravity and $k = 10^{-4}$ kg/m is the drag coefficient. If you divide both sides by m and call v=y(t) one sees that this is of the form

$$\frac{dy}{dt} = f(y,t) \tag{3}$$

with $f(y,t) = g - k/mv^2$

You know what to expect, the sphere will be first accelerated by gravity but with increasing velocity the friction term will eventually lead to a motion with constant velocity v_{lim}. This simple problem has been chosen for the following reasons:

- it has a simple analytical form for v_{lim} obtained by setting $dv/dt = 0$ to compare to the numerical solution. It has also an analytical solution for $v(t)$ which can be used to check the numerical error. The analytical solution is

$$v(t) = \sqrt{\frac{mg}{k}}\frac{exp(2\sqrt{\frac{kg}{m}}t - 1)}{exp(2\sqrt{\frac{kg}{m}}t + 1)} \tag{4}$$

  where we have used the initial values $v(t=0) = 0$. This solution is easily obtained by defining a variable $w = \sqrt{k/(mg)}v$ and by rewriting $1/(1-w^2) = 1/2(1/(1-w) + 1/(1+w))$. You can try and check it.

- there are different regimes of the motion (linear, rapidly varying, constant) and it will be clear that algorithms with adjustable time step are particularly suited in such situations.

- one can calculate both velocity and position of the falling body by reducing the second order equation of motion to a system of two coupled first order differential equations. In this case we call $y_1(t) = v$, $y_2(t) = x$ and we get a system of two coupled differental equations.

$$\frac{dy_1}{dt} = f(y,t) \tag{5}$$

$$\frac{dy_2}{dt} = y_1 \tag{6}$$

  It is up to you to decide whether to solve first the problem for a single equations (for v) and then the coupled system of two equations (for v and x) or write directly the general solution for a system of equations. Use $x(t=0) = 0$ as initial condition.

In writing the program, keep the function to integrate the equations in a separate file.

a) Use the Euler and Runge Kutta method of order 2 and 4 to calculate the velocity for $0 < t < 10$ s distributed over a uniform grid with spacing δt. In order to find an appropriate value for the time step δt check the results against the expected behaviour and value of v_{lim}.

As already mentioned the behaviour of the sphere goes through different regimes and integration over a uniform grid in time is obviously not the most efficient choice. The purpose of the following two exercises is to apply other algorithms where the time step is chosen at every step in order to keep the precision constant. You have to implement at least one of the two.

b) Use the Runge Kutta method with adaptive step size to find the velocity of the sphere for $0 < t < 10$ s with tolerance 10^{-3} and 10^{-4}. Compare the solutions and the steps used in the calculation. Optional: generalize to a system of two coupled equations to calculate velocity and position as a function of time.

c) Same as in b) but using the Runge Kutta Fehlberg method.