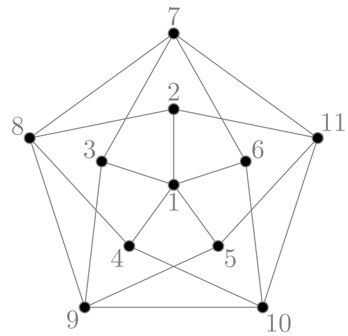
Formal Reasoning 2015
Solutions Test 4: Discrete Mathematics
(02/12/15)

1. This is the so-called Grötzsch-graph:



   (a) Does this graph have an Euler circuit? Explain your answer. (10 points)

   This is a connected graph with 11 vertices. So we may apply Euler's theorem. This theorem states that there exists an Euler circuit if and only if all vertices have an even degree. However, vertex 1 has degree 5, which is odd. So this graph has no Euler circuit.

   (b) Does this graph have a Hamilton circuit? Explain your answer. (10 points)

   Yes, it has Hamilton circuits. Take for instance $1 \to 3 \to 7 \to 6 \to 10 \to 4 \to 8 \to 9 \to 5 \to 11 \to 2 \to 1$. This circuit visits all vertices exactly once and therefore it is a Hamilton circuit.

   (c) Does this graph have an isomorphism onto itself where not every vertex is mapped onto itself? Explain your answer. (10 points)

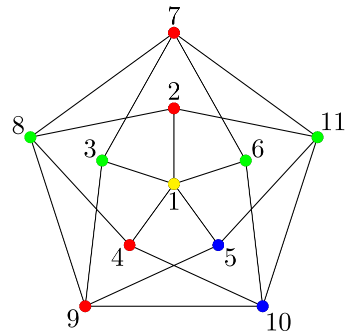
   Yes, such an isomorphism exists. Isomorphisms must keep the structure of the graph the same. By rotating the graph over 72°, the structure will still be the same, but the named vertices are on a different position. We can formalize this idea with the following isomorphism:

$$
\begin{array}{rclrclrcl}
1 & \mapsto & 1 & 2 & \mapsto & 6 & 3 & \mapsto & 2 \\
4 & \mapsto & 3 & 5 & \mapsto & 4 & 6 & \mapsto & 5 \\
7 & \mapsto & 11 & 8 & \mapsto & 7 & 9 & \mapsto & 8 \\
10 & \mapsto & 9 & 11 & \mapsto & 10 & & &
\end{array}
$$

   Because vertex 4 is mapped onto vertex 3, this mapping complies to the requirement that not all vertices are mapped onto themselves.

   (d) Give the chromatic number of this graph and give a corresponding coloring. You don't have to explain why this is indeed the chromatic number, as long as it is correct. (15 points)

   The chromatic number is 4. The figure proves that there exists a coloring with only four colors.



Not needed for this test, but only here for informational purpose. The proof that it cannot be done with three colors is as follows. Assume that there exists a coloring with three colors, yellow, red and blue and assume that vertex 1 in the middle is yellow. In particular this means that the vertices 2, 3, 4, 5 and 6 are colored with only red and blue, since they are all connected to the yellow vertex 1. Now have a look at the outer cycle 7, 8, 9, 10 and 11. Since this is a cycle with an odd number of vertices, we know that we cannot color it with only two colors. So some of these vertices must be yellow. Now we replace all of these yellow vertices in the outer cycle by the color of the corresponding inner vertices. So for instance 7 would get the color of 2. For the graph as a whole this may not lead to a proper coloring, but it should be a proper coloring if we restrict ourselves to the outer cycle. Note that vertices 8 and 11 are neighbors of 2, so they must have a different color than vertex 2. But this means that if we give vertex 7 this color, it is automatically also different from the colors of vertices 8 and 11. Furthermore, if 7 is yellow, then 8 and 11 cannot be yellow too, so we know that 8 and 11 are not changed when replacing the yellow ones (under the assumption that 7 was yellow). So what we now have created is a coloring of the outer cycle with only two colors, which is impossible. Therefore the claim that there exists a coloring with only three colors for the complete graph cannot be right.

2. (a) Give $(n+1)^3$ using Newton's Binomial Theorem and indicate in Pascal's Triangle where its coefficients can be found. (10 points)

According to Newton's Binomial Theorem this is $(n+1)^3 = n^3 + 3n^2 + 3n + 1$. Its coefficients are marked in Pascal's Triangle:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
\boxed{1} & & \boxed{3} & & \boxed{3} & & \boxed{1}
\end{array}
$$

(b) We define the following recurrence relation: (10 points)

$$
\begin{aligned}
a_0 &= 0 \\
a_{n+1} &= a_n + 3n^2 + 3n + 1 \qquad\qquad \text{for } n \geq 0
\end{aligned}
$$

Use this recurrence relation to compute a_2 and show how you found this answer.

$$
\begin{array}{lllllll}
a_0 & = & 0 & & & & \\
a_1 & = & a_0 + 3\cdot 0^2 + 3\cdot 0 + 1 & = & 0+0+0+1 & = & 1 \\
a_2 & = & a_1 + 3\cdot 1^2 + 3\cdot 1 + 1 & = & 1+3+3+1 & = & 8
\end{array}
$$

(c) Prove for the sequence from the previous question that $a_n = n^3$ for all $n \geq 0$. (15 points)

Proposition:

$a_n = n^3$ for all $n \geq 0$.

Proof by induction to n. 1

$$P(n) \;:=\; a_n = n^3$$ 2

Basis. We show that $P(0)$ holds, i.e. we show that $a_0 = 0^3$. 3

This indeed holds, because by definition we have that $a_0 = 0$ and it also holds that $0^3 = 0$. 4

Inductiestap. Let k be any natural number such that $k \geq 0$. 5

Assume that we know that $P(k)$ holds, i.e. we assume that 6
$a_k = k^3$ (Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7
$a_{k+1} = (k+1)^3$

This indeed holds, because 8

$$\begin{aligned} a_{k+1} &= a_k + 3k^2 + 3k + 1 && \text{using the recursive definition of } a_{k+1} \\ &= k^3 + 3k^2 + 3k + 1 && \text{using IH} \\ &= (k+1)^3 && \text{using Newton's Binomial Theorem} \end{aligned}$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$. 9

3. Sinterklaas[1] has a bag with six different presents for three well behaving children. In how many ways can he distribute the gifts over the children, if he wants that each child gets two presents in his or her shoe? (10 points)

For the first child Sinterklaas picks two out of the six presents. This can be done in $\binom{6}{2} = 15$ ways. For the second child Sinterklaas picks two out of the four remaining presents. This can be done in $\binom{4}{2} = 6$ ways. For the third child Sinterklaas picks the two remaining presents. This can be done in 1 way. So the total amount of different distributions is $15 \cdot 6 \cdot 1 = 90$.

[1] Sinterklaas or Sint-Nicolaas is a mythical figure with legendary, historical and folkloric origins based on Saint Nicholas. Sinterklaas is celebrated annually with the giving of gifts on 5 December, the night before Saint Nicholas Day in the Northern Netherlands and on the morning of 6 December, Saint Nicholas Day itself, in the (Roman Catholic) southern provinces, Belgium, Luxembourg and Northern France (French Flanders, Lorraine and Artois). He is the primary source of the popular Christmas icon of Santa Claus.

According to the tradition children will place their shoe near the fire place and if they have behaved well they will receive some gifts in it.