Formal Reasoning 2015
Solutions Test 5: Modal Logic
(16/12/15)

1. (a) Explain the difference in the meaning of the two sentences below using formulas of modal logic. (10 points)

   - *I know that it rains.*
   - *I know whether it rains.*

   You have to choose yourself which version of modal logic fits these sentences best and which 'dictionary' you use.

   We use epistemic logic and the simple dictionary:

R	it rains

   The sentence *I know that it rains* can easily be translated to the formula $\Box R$.

   The sentence *I know whether it rains* can be translated to the formula $(R \to \Box R) \wedge (\neg R \to \Box \neg R)$. Because knowing whether it rains (or not) is equivalent with stating that if it rains, you know that it rains and if it doesn't rain, you also know that it doesn't rain.

   A clear distinction between the two sentences is that in the first sentence it must actually rain and in the second it may not rain at all.

   (b) What is the name of the modal logic you used in item 1a? (5 points)

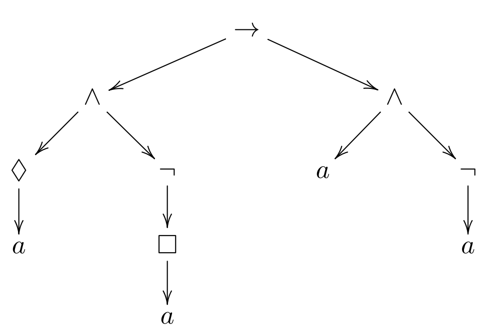
   It is epistemic logic.

   (c) What is in this logic the interpretation of $\Box f$ and $\Diamond f$? (10 points)

   The interpretation of $\Box f$ is: I know that f.
   The interpretation of $\Diamond f$ is: f is consistent with my knowledge.

2. (a) Draw the parse tree for this formula in modal logic: (10 points)

$$\Diamond a \wedge \neg \Box a \to a \wedge \neg a$$



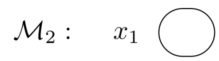
   (b) Write the formula of question 2a according to the official grammar in the course notes. (10 points)

   Following the official grammar it is: $((\Diamond a \wedge \Box a) \to (a \wedge \neg a))$.

   (c) Give a Kripke-model $\mathcal{M}_2$ such that the formula of question 2a is true, i.e. a Kripke model in which we have: (10 points)

$$\mathcal{M}_2 \vDash \Diamond a \wedge \neg \Box a \to a \wedge \neg a$$

Use the graphical representation with circles and arrows to show you model. And explain your answer.



Have a look at the following table:

$\Vdash$	a	$\Diamond a$	$\Box a$	$\neg\Box a$	$\Diamond a \land \neg\Box a$	$\neg a$	$a \land \neg a$	$\Diamond a \land \neg\Box a \to a \land \neg a$
x_1	0	0	1	0	0	1	0	1

The 'trick' is the use of the implication. If we can arrange that the formula to the left of the arrow is not true, then the whole implication is automatically true, not influenced by the inconsistency of the formula to the right of the arrow.
Hence the table proves that $x_1 \Vdash \Diamond a \land \neg\Box a \to a \land \neg a$. But because x_1 is the only world in $\mathcal{M}_2$ it now automatically holds that $\mathcal{M}_2 \Vdash \Diamond a \land \neg\Box a \to a \land \neg a$.

(d) Write model $\mathcal{M}_2$ of question 2c as triple $\langle W, R, V\rangle$. (10 points)

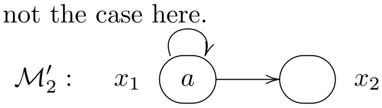
We get the triple $\langle W, R, V\rangle$ where

$$\begin{array}{rcl} W & = & \{x_1\} \\ R : W \to \mathcal{P}(W) & \text{where} & R(x_1) = \emptyset \\ V : W \to \mathcal{P}(\{a\}) & \text{where} & V(x_1) = \emptyset \end{array}$$

(e) Show that: (10 points)

$$\nvDash \Diamond a \land \neg\Box a \to a \land \neg a$$

If there is no specific model to the left of $\vDash$, it means that the formula must be true in all models. However, this is not the case here.



Now have a look at the table:

$\Vdash$	a	$\Diamond a$	$\Box a$	$\neg\Box a$	$\Diamond a \land \neg\Box a$	$\neg a$	$a \land \neg a$	$\Diamond a \land \neg\Box a \to a \land \neg a$
x_1	1	1	0	1	1	0	0	0
x_2	0	0	1	0	0	1	0	1

Becasue $x_1 \nVdash \Diamond a \land \neg\Box a \to a \land \neg a$, it holds that $\mathcal{M}'_2 \nvDash \Diamond a \land \neg\Box a \to a \land \neg a$ and hence $\nvDash \Diamond a \land \neg\Box a \to a \land \neg a$.

3. Give an LTL-formula that describes the situation in which the following two properties hold: (15 points)

- On each moment either a or b is true, but not both.
- For both a and b it holds that they are at most three times in a row true. After that, the other one should become true again.

Explain your answer. (If you can't find a single formula that describes both properties at once, try to formalize at least one of the properties.)

The first property can we transform into the equivalent property *on each moment a is true, or b is true, but $a \land b$ is not true.* We can capture this with the formula:

$$\mathcal{G}((a \lor b) \land \neg(a \land b))$$

The second property can for a be described as *on each moment it holds that if a is true on that moment, on the first moment afterwards and on the second moment afterwards, then on the third moment afterwards a will be false and b will be true.* We can capture this with the formula:

$$\mathcal{G}(a \wedge \mathcal{X}a \wedge \mathcal{X}\mathcal{X}a \rightarrow \mathcal{X}\mathcal{X}\mathcal{X}(\neg a \wedge b))$$

For b we have a similar formula of course. All together we get this formula:

$$\mathcal{G}\big((a \vee b) \wedge \neg(a \wedge b)\big) \wedge \mathcal{G}\big(a \wedge \mathcal{X}a \wedge \mathcal{X}\mathcal{X}a \rightarrow \mathcal{X}\mathcal{X}\mathcal{X}(\neg a \wedge b)\big) \wedge \mathcal{G}\big(b \wedge \mathcal{X}b \wedge \mathcal{X}\mathcal{X}b \rightarrow \mathcal{X}\mathcal{X}\mathcal{X}(\neg b \wedge a)\big)$$

Note that this formula can be simplified because the $\mathcal{G}$s can be combined and because the $\mathcal{X}\mathcal{X}\mathcal{X}\neg a$ combined with the first property already implies that $\mathcal{X}\mathcal{X}\mathcal{X}b$.

$$\mathcal{G}\big((a \vee b) \wedge \neg(a \wedge b) \wedge (a \wedge \mathcal{X}a \wedge \mathcal{X}\mathcal{X}a \rightarrow \mathcal{X}\mathcal{X}\mathcal{X}\neg a) \wedge (b \wedge \mathcal{X}b \wedge \mathcal{X}\mathcal{X}b \rightarrow \mathcal{X}\mathcal{X}\mathcal{X}\neg b)\big)$$