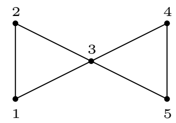
Formal Reasoning 2016
Solutions Test Block 4: Discrete Mathematics
(30/11/16)

1. Give a connected planar graph that has an Euler circuit, and in which not all vertices have degree two. Draw the graph in a planar representation, and explain why it has the required properties.

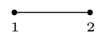


This graph has the required properties:

- It is connected since there is clearly only one component. In particular for each pair of distinct vertices x and y where $x < y$ we have a path $x \to x+1 \to x+2 \to \cdots \to y$.
- It is planar, because in the given representation there are no crossing lines.
- It has an Euler circuit because the cycle $1 \to 2 \to 3 \to 4 \to 5 \to 3 \to 1$ includes every edge exactly once.
- Not all vertices have degree two, because vertex 3 has degree four.

2. (a) Does there exist a tree that has a Hamilton path?

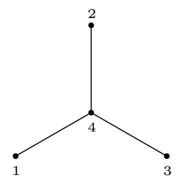
Yes, a tree is a connected graph that has no cycles, so the following graph is clearly a tree.



And the path $1 \to 2$ visits each vertex exactly once, so this is a Hamilton path.

(b) Does there exist a tree that does not have a Hamilton path?

Yes, the following graph is a tree that doesn't have a Hamilton path.



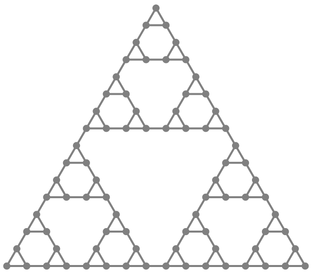
This graph has no Hamilton path. Because all vertices have to be included in any Hamilton path and the vertices 1, 2 and 3 have degree one, we know for sure that all edges $(1, 4)$, $(2, 4)$ and $(3, 4)$ must be in such a Hamilton path. But this implies that we have to visit vertex 4 at least two times, which is not allowed in a Hamilton path.

(c) Does there exist a tree that has a Hamilton circuit?

No, by definition a tree has no circuits at all, so in particular a tree has no Hamilton circuits.

Explain your answer for each of these three questions.

3. We define graphs G_n for $n \geq 1$, in which the vertices are the legal positions of the Towers of Hanoi with n disks, and the edges correspond to legal moves between those positions. For example the graph G_4 is:

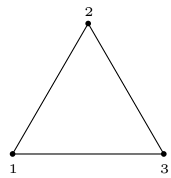


We write e_n for the number of edges in G_n. The sequence e_n satisfies the recursive equations:

$$e_1 = 3$$
$$e_{n+1} = 3e_n + 3$$

(a) How many isomorphisms are there from G_1 to G_1?

Let us first draw G_1. Obviously G_1 has three vertices, corresponding with the situation that there is one disk on peg number 1, peg number 2 or peg number 3. Note that from each situation we can go in one move to any other situation. In particular this means that K_1 is isomorphic with K_3, so G_1 looks like a triangle:



Because G_1 is basically K_3 where each vertex is connected to any other vertex, *any* bijective map of $\{1,2,3\}$ to $\{1,2,3\}$ is an isomorphism. For determining all isomorphisms we have to fix $\varphi(1)$, $\varphi(2)$ and $\varphi(3)$. So first for $\varphi(1)$ we have three choices. And then for $\varphi(2)$ we have two choices. And then for $\varphi(3)$ we only have one option left. Hence the total number of isomorphisms of G_1 to G_1 is $3 \cdot 2 \cdot 1 = 6$.

For the sake of completeness we list the isomorphisms:

$$\begin{array}{lllll}
f_1 & : & 1 \mapsto 1 & 2 \mapsto 2 & 3 \mapsto 3 \\
f_2 & : & 1 \mapsto 1 & 2 \mapsto 3 & 3 \mapsto 2 \\
f_3 & : & 1 \mapsto 2 & 2 \mapsto 1 & 3 \mapsto 3 \\
f_4 & : & 1 \mapsto 2 & 2 \mapsto 3 & 3 \mapsto 1 \\
f_3 & : & 1 \mapsto 3 & 2 \mapsto 1 & 3 \mapsto 2 \\
f_4 & : & 1 \mapsto 3 & 2 \mapsto 2 & 2 \mapsto 1
\end{array}$$

(b) Give a formula without recursion for the number of vertices in G_n.

We start by making a list with the values we can see in the picture of G_4:

n	number of vertices in G_n
1	3
2	9
3	27
4	81

So the direct formula seems to be 3^n.

By using recursion we can give another argument why this should be the proper formula: if we go from G_n to G_{n+1} it means that we have to add one new disk that is larger than all other disks. This implies that this disk can only be at the bottom of the pegs. Furthermore, any legal position of G_n can be extended to put a larger disk below the existing disks on any of the three pegs. So here we see that for each extra disk, the number of legal positions is multiplied by three, giving 3^n for the game with n disks.

(c) Compute the number of edges in the graph G_4 using the recursive equations that were given above. Include your intermediate results.

$$\begin{array}{lllllllll} e_1 & = & 3 & & & & & & \\ e_2 & = & 3 \cdot e_1 + 3 & = & 3 \cdot 3 + 3 & = & 9 + 3 & = & 12 \\ e_3 & = & 3 \cdot e_2 + 3 & = & 3 \cdot 12 + 3 & = & 36 + 3 & = & 39 \\ e_4 & = & 3 \cdot e_3 + 3 & = & 3 \cdot 39 + 3 & = & 117 + 3 & = & 120 \end{array}$$

(d) Prove by induction that $e_n = \frac{1}{2}(3^{n+1} - 3)$ for all $n \geq 1$. In this proof you may use the recursive equations that were given above.

Proposition: 0

$e_n = \frac{1}{2}(3^{n+1} - 3)$ for all $n \geq 1$.

Proof by induction on n. 1

We first define our predicate P as:

$$P(n) \quad := \quad e_n = \frac{1}{2}(3^{n+1} - 3)$$ 2

Base Case. We show that $P(1)$ holds, i.e. we show that 3

$e_1 = \frac{1}{2}(3^{1+1} - 3)$

This indeed holds, because by definition $e_1 = 3$ and 4

$$\frac{1}{2}(3^{1+1} - 3) = \frac{1}{2}(3^2 - 3) = \frac{1}{2}(9 - 3) = \frac{1}{2} \cdot 6 = 3$$

Induction Step. Let k be any natural number such that $k \geq 1$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6

$e_k = \frac{1}{2}(3^{k+1} - 3)$ (Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7

$e_{k+1} = \frac{1}{2}(3^{k+1+1} - 3)$

This indeed holds, because 8

$$
\begin{array}{rcll}
e_{k+1} & = & 3e_k + 3 & \text{(by definition of } e_{k+1}\text{)} \\
& \overset{\text{IH}}{=} & 3 \cdot \frac{1}{2}(3^{k+1} - 3) + 3 & \text{(by applying the IH)} \\
& = & \frac{1}{2}(3^{k+1+1} - 9) + 3 & \text{(elementary algebra)} \\
& = & \frac{1}{2} \cdot 3^{k+1+1} - 4\frac{1}{2} + 3 & \text{(elementary algebra)} \\
& = & \frac{1}{2} \cdot 3^{k+1+1} - 1\frac{1}{2} & \text{(elementary algebra)} \\
& = & \frac{1}{2}(3^{k+1+1} - 3) & \text{(elementary algebra)}
\end{array}
$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 1$. 9

4. Calculate $(x+y)^7$ according to the binomial theorem, with the coefficients as explicit numbers. Indicate where the relevant binomial coefficients occur, both in your answer as well as in Pascal's triangle.

The general version of Newton's binomial theory states that

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

So if we use $n = 7$ we get:

$$
\begin{array}{ll}
(x+y)^7 & \\
= & \binom{7}{0}x^7 + \binom{7}{1}x^6y + \binom{7}{2}x^5y^2 + \binom{7}{3}x^4y^3 + \binom{7}{4}x^3y^4 + \binom{7}{5}x^2y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7 \\
= & 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7
\end{array}
$$

The coefficients are marked in Pascal's triangle below:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
\boxed{1} & & \boxed{7} & & \boxed{21} & & \boxed{35} & & \boxed{35} & & \boxed{21} & & \boxed{7} & & \boxed{1}
\end{array}
$$