Type Theory and Coq 2012

23-01-2013

Write your name on each paper that you hand in. Each subexercise is worth 5 points, 10 points are free, and the final mark is the number of points divided by 10. Write proofs, terms and types in this test according to the conventions of Femke's course notes. Good luck!

1. (a) Prove the formula

$$(a \to a \to c) \to (b \to a) \to (b \to c)$$

   in minimal propositional logic. Indicate whether the proof has any detours.

   (b) Give the lambda term of Church-style simple type theory that corresponds to this proof.

2. (a) Prove the formula

$$a \to \forall b.\, (\forall c.\ a \to c) \to b$$

   in second order propositional logic.

   (b) Give the lambda term of $\lambda 2$ that corresponds to this proof, and give its type.

3. The rules for the eight systems from the Barendregt cube are given by the following table:

$\lambda{\to}$	$\mathcal{R} = \{(*,*)\}$
λP	$\mathcal{R} = \{(*,*), (*,\Box)\}$
$\lambda 2$	$\mathcal{R} = \{(*,*), \qquad\quad (\Box,*)\}$
$\lambda P2$	$\mathcal{R} = \{(*,*), (*,\Box), (\Box,*)\}$
$\lambda\underline{\omega}$	$\mathcal{R} = \{(*,*), \qquad\qquad\qquad\quad (\Box,\Box)\}$
$\lambda P\underline{\omega}$	$\mathcal{R} = \{(*,*), (*,\Box), \qquad\quad (\Box,\Box)\}$
$\lambda\omega$	$\mathcal{R} = \{(*,*), \qquad\quad (\Box,*), (\Box,\Box)\}$
λC	$\mathcal{R} = \{(*,*), (*,\Box), (\Box,*), (\Box,\Box)\}$

   in which (s_1, s_2) is an abbreviation of (s_1, s_2, s_2).

Furthermore, the PTS product and abstraction rules are:

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma,\, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.\, B \;:\; s_3}\ (s_1, s_2, s_3) \in \mathcal{R}$$

$$\frac{\Gamma,\, x : A \vdash M : B \qquad \Gamma \vdash \Pi x : A.\, B \;:\; s}{\Gamma \vdash \lambda x : A.\, M \;:\; \Pi x : A.\, B}$$

Finally we have the typings:

$$\begin{array}{lcl} \mathsf{nat} & : & * \\ \mathsf{vec} & : & \mathsf{nat} \to * \end{array}$$

For each of the following three terms, list in which of the systems from the Barendregt cube the term is typable:

(a)

$$\mathsf{nat} \to \mathsf{nat}$$

(b)

$$\lambda a : *.\ a \to a$$

(c)

$$\Pi n : \mathsf{nat}.\ \mathsf{vec}\ n$$

4. (a) Consider the Coq definition

```
Inductive nat : Set :=
| O : nat
| S : nat -> nat.
```

Give the *dependent* induction principle `nat_ind` of this type.

(b) Give the normal form of the term

```
nat_ind P c f (S (S O))
```

that uses the principle from the previous exercise. In this term the variables `P`, `c`, `f` and `n` are variables from the context.

(c) Give the *non-dependent* induction principle that corresponds to the induction principle from 4(a).

5. (a) Consider the Coq definition

```
Inductive le (n : nat) : nat -> Prop :=
| le_n : le n n
| le_S : forall m : nat, le n m -> le n (S m).
```

Give the *non-dependent* induction principle `le_ind` of this type. (Hint: first determine the *dependent* induction principle, and then remove the dependence on the elements of `le n m` in the predicate.)

(b) Prove that $1 \leq 2$, i.e., give an inhabitant of

```
le (S O) (S (S O))
```

where `le` is the type from the previous exercise.

6. Which of the following four inductive definitions are allowed by Coq? For the definitions that are not allowed, explain what requirement is not satisfied.

(a)
```
Inductive T1 : Type :=
| b1 : T1
| c1 : (T1 -> T1) -> T1.
```

(b)
```
Inductive T2 (A : Type) : Type :=
| b2 : T2 A
| c2 : T2 (A -> A) -> T2 A.
```

(c)
```
Inductive T3 (A : Type) : Type :=
| b3 : T3 A
| c3 : T3 A -> T3 (A -> A).
```

(d)
```
Inductive T4 : Type :=
| b4 : T4
| c4 : (nat -> T4) -> T4.
```

7. We recursively define an operation M^* on untyped lambda terms:

$$\begin{array}{rcll} x^* & := & x & \\ (\lambda x.M)^* & := & \lambda x.M^* & \\ ((\lambda x.M)N)^* & := & M^*[x := N^*] & \\ (MN)^* & := & M^*N^* & \qquad \text{when } MN \text{ is not a beta redex} \end{array}$$

and we inductively define a relation $M \Rightarrow N$ on untyped lambda terms:

$$x \Rightarrow x$$

$$\frac{M \Rightarrow M'}{\lambda x.M \Rightarrow \lambda x.M'}$$

$$\frac{M \Rightarrow M' \quad N \Rightarrow N'}{MN \Rightarrow M'N'}$$

$$\frac{M \Rightarrow M' \quad N \Rightarrow N'}{(\lambda x.M)N \Rightarrow M'[x := N']}$$

(a) State the diamond property for this relation $M \Rightarrow N$.

(b) What is the relation between the M^* operation and the $M \Rightarrow N$ relation that allows one to prove this property? (Note that the exercise does *not* ask you to prove that this relation holds.)