Course: Type Theory and Coq

Exercises on Church-Rosser

All exercises are about the Church-Rosser proof from the Takahashi paper that we have discussed at the lecture. We recall the definition of $\Rightarrow$ using derivation rules:

$$\frac{}{x \Rightarrow x}\ (\beta 1) \qquad \frac{M \Rightarrow M'}{\lambda x.M \Rightarrow \lambda x.M'}\ (\beta 2)$$

$$\frac{M \Rightarrow M' \qquad N \Rightarrow N'}{M\,N \Rightarrow M'\,N'}\ (\beta 3) \qquad \frac{M \Rightarrow M' \qquad N \Rightarrow N'}{(\lambda x.M)\,N \Rightarrow M'[N'/x]}\ (\beta 4)$$

1. Consider the term $M = (\lambda x\,y.x\,x(x\,y))(\mathbf{I}\,\mathbf{I})$
   (a) Give the reduction graph of M. (You may abbreviate $\mathbf{I}\,\mathbf{I}$ to J and$\lambda x\,y.x\,x(x\,y)$ to P.)
   (b) Compute M^* and $(M^*)^*$.
   (c) Prove that $M \Rightarrow M^*$ and $M^* \Rightarrow (M^*)^*$ by giving a derivation.

2. In the definition of $\Rightarrow$, we change clause $(\beta 4)$ into

$$\frac{M \Rightarrow \lambda x.P \qquad N \Rightarrow N'}{M\,N \Rightarrow P[N'/x]}$$

   (a) Give the definition of $(-)^*$ that goes with this adapted definition of $\Rightarrow$.
   (b) Prove again (with these adpated definitions) that $M \Rightarrow N$ implies $N \Rightarrow M^*$, by doing the inductive step for case $(\beta 4)$.

3. The η-reduction rule is: $\lambda x.M\,x \rightarrow_\eta M$, if $x \notin \mathrm{FV}(M)$. In order to prove CR for $\beta\eta$ we add a clause for η-redexes to the definition of $\Rightarrow$:

$$\frac{M \Rightarrow M'}{\lambda x.M\,x \Rightarrow M'}\ x \notin \mathrm{FV}(M)$$

   (a) Show that now $(\lambda yx.y\,x)\mathbf{I} \Rightarrow \mathbf{I}$, and show that in the original definition, this is not the case.
   (b) Define $(-)^*$ for this extension to η