Matrix Calculations

Sample Problems for Orthogonality

Exercises with solutions

1. Consider vectors $a = (-1, 3, -7)$, $b = (2, -1, 5)$, $c = (0, 1, 5)$. Compute the following:

   (a) $(2\langle a, b\rangle - \langle c, b\rangle)(2\langle b, c\rangle + \langle b, a\rangle)$

   (b) Length of a and length of b

   (c) Distance between a and b

   (d) Angle between a and b (give the cosine of the angle)

Begin Solutions . . . .

(a) $\langle a, b\rangle = \langle b, a\rangle = -2 - 3 - 35 = -40$
$\langle c, b\rangle = \langle b, c\rangle = 0 - 1 + 25 = 24$
$(2\langle a, b\rangle - \langle c, b\rangle)(2\langle b, c\rangle + \langle b, a\rangle) = (2 \times (-40) - 24)(2 \times (24) + (-40)) = = (-104) \times (8) = -832$

(b) $\|a\| = \sqrt{1 + 9 + 49} = \sqrt{59}$
$\|b\| = \sqrt{4 + 1 + 25} = \sqrt{30}$

(c) $\|a - b\| = \|(-3, 4, -12)\| = \sqrt{9 + 16 + 144} = \sqrt{169} = 13.$

(d) $\cos(\theta) = \frac{\langle a,b\rangle}{\|a\|\|b\|} = \frac{-40}{\sqrt{59}\sqrt{30}}$

End Solutions . . . .

2. (a) For $v = (2, 3, -6)$ in the direction of $w = (1, -5, 2)$, find a vector u that is orthogonal to both v and w in $\mathbb{R}^3$. Check your results!

Begin Solutions . . . .

(a) Let $u = (x, y, z)$. We need to find vector such that, $\langle v, u\rangle = 0$ and $\langle w, u\rangle = 0$.
Thus, we have a system of equations:

$$\begin{aligned} x - 5y + 2z &= 0 \\ 2x + 3y - 6z &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & -5 & 2 \\ 2 & 3 & -6 \end{pmatrix} = (R_2 := R_2 - 2R_1) = \begin{pmatrix} 1 & -5 & 2 \\ 0 & 13 & -10 \end{pmatrix}$$

Thus, $y = \frac{10}{13}z$, $x = 5y - 2z = 5 \times \frac{9}{13}z - 2z = \frac{50}{13}z - 2z = \frac{24}{13}z$
We take $z = 13$, *so* $u = (24, 10, 13)$
Check:
$\langle v, u\rangle = 0$ *because* $\langle (2, 3, -6), (24, 10, 13)\rangle = 48 + 30 - 78 = 0$
$\langle w, u\rangle = 0$ *because* $\langle (1, -5, 2), (24, 10, 13)\rangle = 24 - 50 + 26 = 0$

End Solutions . . . .

3. The subspace $V \subseteq \mathbb{R}^4$ is spanned by the vectors:

$$v_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \qquad v_2 = \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \qquad v_3 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

(a) Transform the basis $\{v_1, v_2, v_3\}$ into an orthogonal one.

(b) Get rid of the fractions in the orthogonal bases and demonstrate that the resulting vectors are indeed orthogonal. Why is it possible to do get rid of the fractions?

Begin Solutions . . . .

(a) $v_1' = v_1$

$$v_2' = v_2 - \frac{\langle v_2, v_1'\rangle}{\langle v_1', v_1'\rangle} v_1' = \begin{pmatrix}0\\1\\1\\1\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-2\\1\\1\\3\end{pmatrix}$$

$$v_3' = v_3 - \frac{\langle v_3, v_1'\rangle}{\langle v_1', v_1'\rangle} v_1' - \frac{\langle v_3, v_2'\rangle}{\langle v_2', v_2'\rangle} v_2' = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{3}{3}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} - \frac{(-\frac{2}{3}+\frac{1}{3}+\frac{1}{3}+1)}{(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}+1)}\begin{pmatrix}-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\\1\end{pmatrix} =$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} - \frac{3}{5}\begin{pmatrix}-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\\1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2\\-1\\-1\\2\end{pmatrix} \text{ (leave as this or) } = \begin{pmatrix}\frac{2}{5}\\-\frac{1}{5}\\-\frac{1}{5}\\\frac{2}{5}\end{pmatrix}$$

So an orthogonal basis is

$$\left(\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \frac{1}{3}\begin{pmatrix}-2\\1\\1\\3\end{pmatrix}, \frac{1}{5}\begin{pmatrix}2\\-1\\-1\\2\end{pmatrix}\right)$$

(b)

$$v_1' = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \qquad v_2' = \begin{pmatrix}-2\\1\\1\\3\end{pmatrix}, \qquad v_3' = \begin{pmatrix}2\\-1\\-1\\2\end{pmatrix}$$

This is allowed, because (i) if we multiply any of the base vectors by a scalar ($\neq 0$), they still form a basis and (ii) if $\langle v, w\rangle = 0$, then $\langle av, w\rangle = 0$ for any scalar a, so they remain orthogonal.

Check:

$\langle v_1', v_2'\rangle = -2 + 1 + 1 + 0 = 0$

$\langle v_1', v_3'\rangle = 2 - 1 - 1 + 0 = 0$

$\langle v_2', v_3'\rangle = -4 - 1 - 1 + 6 = 0$

End Solutions . . . .

4. Let p be a parameter and let $U \subseteq \mathbb{R}^3$ be the subspace spanned by the independent vectors $v_1 = (4, -p, -p^2)$ and $v_2 = (2p, p^2, p)$.

(a) Determine the values of p for which (v_1, v_2) forms an orthogonal basis of U. (Describe all possible values and argue that these are the only ones.)

(b) Choose one basis that you've found in (a), and turn it into an orthonormal basis of U.

Begin Solutions ....

(a) We need $\langle v_1, v_2 \rangle = 0$ and v_1 and v_2 independent. So

$$\langle v_1, v_2 \rangle = 0 \Leftrightarrow 8p - p^3 - p^3 = 0 \Leftrightarrow 2p(4 - p^2) = 0 \Leftrightarrow p = 0 \vee p = 2 \vee p = -2$$

- *If $p = 0$, then $v_2 = (0, 0, 0)$ and v_1 and v_2 are not independent, so $p = 0$ is incorrect.*
- *If $p = 2$, then $v_1 = (4, -2, -4)$, $v_2 = (4, 4, 2)$ and v_1 and v_2 are independent, so $p = 2$ is correct.*
- *If $p = -2$, then $v_1 = (4, 2, -4)$, $v_2 = (4, 4, -2)$ and v_1 and v_2 are independent, so $p = -2$ is correct.*

So the values of p for which (v_1, v_2) forms an orthogonal basis of U are $p = 2$ and $p = -2$

(b) An orthonormal basis is found by normalizing these vectors. This is done by dividing the base vectors by their length. For w any of the 4 vectors above: $\|w\| = \sqrt{16 + 4 + 16} = \sqrt{36} = 6$.
So the correct answers are

- *For $p = 2$: The basis found in (a) is $((4, -2, -4), (4, 4, 2))$. After normalization: $((\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}), (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}))$.*
- *For $p = -2$: The basis found in (a) is $((4, 2, -4), (4, 4, -2))$. After normalization: $((\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}), (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}))$.*

End Solutions ....