Matrix Calculations: Linear Equations

Aleks Kissinger (and Herman Geuvers)

Institute for Computing and Information Sciences
Radboud University Nijmegen

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First, some admin...

**Lectures**

- Weekly, 2 hours, on Tuesdays 10:45
- Presence not compulsory...
  - But if you are going to come, actually be here! (This means laptops shut, phones away.)
- The course material consists of:
  - these slides, available via the web
  - *Linear Algebra* lecture notes by Bernd Souvignier (‘LNBS’)
- Course URL:
  
  (Link exists in blackboard, under ‘course information’).
- Generally, things appear on course website (and not on blackboard!). Check there before you ask a question.
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Assignments

- Handing in is compulsory, average must be $\geq 5$
  - Assignments must be done individually
- Werkcollege on Friday, 10:45.
  - Presence not compulsory
  - Answers (for old assignments) & Questions (for new ones)
- There is a separate Exercises web-page (see URL on course web-page).
- Schedule:
  - New assignments on the web on Tuesday
  - Next exercise meeting (Friday) you can ask questions
  - Hand-in: **Monday before noon**, handwritten or typed, on paper in the delivery boxes (or via other means in agreement with your assistant).
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Schedule notes

- There will be **no lecture** on February 9, on account of Carnival
- But there **will** be a werkcollege this Friday
- But there **won’t** be werkcollege’s on 12/2 or 25/3.
- Hand in your first assignment by **Friday** 12/2 (not Monday).
- This is all very confusing. But at least it’s on the website. 😊
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Exercise Classes

- 4 Assistants:
  - Sander Uijlen, HG00.086
  - Bart Gruppen, HG01.028
  - Abdullahi Ali, HG00.310
  - Michiel de Bondt, HG00.308

- Each assistant has a delivery box on the ground floor of the Mercator 1 building
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There are 4 exercise classes

- in which class you are will be determined by your “strength”
- you will be asked to rate yourself (“self-assessment”)

\[ ++ / + / 0 / - / -- \]

- please do this seriously: it is in your own interest to be in the appropriate group
- rough guideline: ++ for $\geq 7$ at WiskundeB, + for $\geq 6$ at WiskundeB, etc.
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Registering for the exercise classes

- On Bb and the course website you’ll find a link to a page where you can register for the exercise class and rate yourself.
- Registration **must** be done by tomorrow (Wednesday) at 17:30. (Do it today, if possible.)
- Exercise class membership will be communicated by Thursday via Bb.
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**Examination**

- Final mark is computed from:
  - Average of markings of assignments: $a$
  - Written exam (April 4): $e$
  - Final mark: $f = e + \frac{a}{10}$.
- Both $a$ and $e$ must be $\geq 5$ to pass.
- Second chance for written exam shortly thereafter.
  - you keep the outcome (average) of the assignments.
- If you fail again, you must start all over next year (including re-doing new exercises, and additional requirements).
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If you fail more than twice . . .

- Additional requirements will be imposed
- You will have to talk to the study advisor
  - if you have not done so yet, make an appointment (this also holds for KI students)
  - compulsory: presence at lectures, exercise meetings, handing in of all exercises!
Next, some advice...

How to pass this course

• Learn by doing, not just staring at the slides (or video, or lecturer)
• Pro tip: exam questions will look a lot like the exercises
• Give this course the time it needs!
• 3ec means $3 \times 28 = 84$ hours in total
  • Let’s say 20 hours for exam
  • 64 hours for 8 weeks means: 8 hours per week!
  • 4 hours in lecture and werkcollege leaves...
  • ...another 4 hours for studying & doing exercises
• Coming up-to-speed is your own responsibility
  • if you feel like you are missing some background knowledge: use Wim Gielen’s notes...or even wikipedia
Finally, on to the good stuff...

Q: What is matrix calculation all about?

linear algebra

A: It depends on who you ask...
What is linear algebra all about?

To a mathematician: linear algebra is the mathematics of **geometry** and **transformation**...

It asks: How can we represent a problem in 2D, 3D, 4D (or infinite-dimensional!) space, and **transform** it into a solution?
What is linear algebra all about?

To an engineer: linear algebra is about **numerics**...

It asks: Can we encode a complicated question (e.g. ‘Will my bridge fall down?’) as a big matrix and compute the answer?
A simple example...

Let’s start with something everybody knows how to do:

- Suppose I went to the pub last night, but I can’t remember how many, umm... ‘sodas’ I had.
- I remember taking out 20 EUR from the cash machine.
- Sodas cost 3 EUR.
- I discover a half-eaten kapsalon in my kitchen. That’s 5 EUR.
- I have no money left. (Typical...)

By now, most people have (hopefully) figured out I had... 5 sodas. That’s because you can solve simple **linear equations**:

\[ 3x + 5 = 20 \implies x = 5 \]
An (only slightly less) simple example

I have **two numbers in mind**, but I don’t tell you which ones

- if I add them up, the result is 12
- if I subtract, the result is 5

*Which two numbers do I have in mind?*

Now we have a **system** of linear equations, in two variables:

\[
\begin{align*}
    x + y &= 12 \\
    x - y &= 5
\end{align*}
\]

with solution \(x = 8 \frac{1}{2}, \, y = 3 \frac{1}{2}\).
An (only slightly less) simple example

Let’s try to find a solution, in general, for:

\[ \begin{align*}
  x + y &= a \\
  x - y &= b
\end{align*} \]

i.e. find the values of \( x \) and \( y \) in terms of \( a \) and \( b \).

- **Adding** the two equations yields:
  \[ a + b = (x + y) + (x - y) = 2x, \quad \text{so} \quad x = \frac{a + b}{2} \]

- **Subtracting** the two equations yields:
  \[ a - b = (x + y) - (x - y) = 2y, \quad \text{so} \quad y = \frac{a - b}{2} \]

Example (from the previous slide)

\( a = 12, \ b = 5 \), so \( x = \frac{12+5}{2} = \frac{17}{2} = 8 \frac{1}{2} \) and \( y = \frac{12-5}{2} = \frac{7}{2} = 3 \frac{1}{2} \). Yes!
A more difficult example

I have two numbers in mind, but I don’t tell you which ones!

- if I add them up, the result is 12
- if I multiply, the result is 35

Which two number do I have in mind?

It is easy to see that $x = 5$, $y = 7$ is a solution.

The system of equations however, is non-linear:

\[
\begin{align*}
x + y &= 12 \\
x \cdot y &= 35
\end{align*}
\]

This is already too difficult for this course. (If you don’t believe me, try $x^5 + x = -1$ ...on second thought, maybe wait till later.)

We only do linear equations.
### Basic definitions

**Definition (linear equation and solution)**

A *linear equation* in $n$ variables $x_1, \ldots, x_n$ is an expression of the form:

$$ a_1 x_1 + \cdots + a_n x_n = b, $$

where $a_1, \ldots, a_n, b$ are given numbers (possibly zero).

A *solution* for such an equation is given by $n$ numbers $s_1, \ldots, s_n$ such that $a_1 s_1 + \cdots + a_n s_n = b$.

**Example**

The linear equation $3x_1 + 4x_2 = 11$ has many solutions, eg. $x_1 = 1, x_2 = 2$, or $x_1 = -3, x_2 = 5$. 
A \((m \times n)\) system of linear equations consists of \(m\) equations with \(n\) variables, written as:

\[
a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n = b_m
\]

A solution for such a system consists of \(n\) numbers \(s_1, \ldots, s_n\) forming a solution for each of the equations.
Example solution

Example

Consider the system of equations

\[
\begin{align*}
    x_1 + x_2 + 2x_3 &= 9 \\
    2x_1 + 4x_2 - 3x_3 &= 1 \\
    3x_1 + x_2 + x_3 &= 8.
\end{align*}
\]

- How to find solutions, if any?
- Finding solutions requires some work.
- But checking solutions is easy, and you should always do so, just to be sure.
- Solution: \( x_1 = 1, x_2 = 2, x_3 = 3. \) ✓
Easy and hard

• General systems of equations are hard to solve. But what kinds of systems are easy?
• How about this one?

\[ x_1 = 7 \]
\[ x_2 = -2 \]
\[ x_3 = 2 \]

• ...this one’s not too shabby either:

\[ x_1 + 2x_2 - x_3 = 1 \]
\[ x_2 + 2x_3 = 2 \]
\[ x_3 = 2 \]
So, why don’t we take something hard, and transform it into something easy?

\[
\begin{align*}
2x_2 + x_3 &= -2 \\
3x_1 + 5x_2 - 5x_3 &= 1 \\
2x_1 + 4x_2 - 2x_3 &= 2
\end{align*}
\Rightarrow
\begin{align*}
x_1 + 2x_2 - x_3 &= 1 \\
x_2 + 2x_3 &= 2 \\
x_3 &= 2
\end{align*}
\Rightarrow
\begin{align*}
x_1 &= 7 \\
x_2 &= -2 \\
x_3 &= 2
\end{align*}

Sound like something linear algebra might be good for?
**Gaussian elimination** is the ‘engine room’ of all computer algebra. It was named after this guy:

Carl Friedrich Gauss (1777-1855)

(famous for inventing: like half of mathematics)
Gaussian elimination is the ‘engine room’ of all computer algebra. ...but it was probably actually invented by this guy:

![Liu Hui (ca. 3rd century AD)](image.png)
Variable names are inessential

The following programs are equivalent:

```java
for(int i=0; i<10; i++){
    P(i);
}
```

```java
for(int j=0; j<10; j++){
    P(j);
}
```

Similarly, the following systems of equations are equivalent:

\[
\begin{align*}
2x + 3y + z &= 4 \\
x + 2y + 2z &= 5 \\
3x + y + 5z &= -1
\end{align*}
\]

\[
\begin{align*}
2u + 3v + w &= 4 \\
u + 2v + 2w &= 5 \\
3u + v + 5w &= -1
\end{align*}
\]
Matrices

The essence of the system

\[
\begin{align*}
2x + 3y + z &= 4 \\
x + 2y + 2z &= 5 \\
3x + y + 5z &= -1
\end{align*}
\]

is not given by the variables, but by the numbers, written as:

**Coefficient matrix**

\[
\begin{pmatrix}
2 & 3 & 1 \\
1 & 2 & 2 \\
3 & 1 & 5
\end{pmatrix}
\]

**Augmented matrix**

\[
\begin{pmatrix}
2 & 3 & 1 & | & 4 \\
1 & 2 & 2 & | & 5 \\
3 & 1 & 5 & | & -1
\end{pmatrix}
\]
Easy and hard matrices

So, the question becomes, how to we turn a *hard* matrix:

\[
\begin{pmatrix}
0 & 2 & 1 & -2 \\
3 & 5 & -5 & 1 \\
2 & 4 & -2 & 2 \\
\end{pmatrix}
\quad \leftrightarrow \quad \left\{ \begin{array}{l}
2x_2 + x_3 = -2 \\
3x_1 + 5x_2 - 5x_3 = 1 \\
2x_1 + 4x_2 - 2x_3 = 2 \\
\end{array} \right.
\]

...into an easy one:

\[
\begin{pmatrix}
1 & 2 & -1 & 1 \\
0 & 1 & 2 & 2 \\
0 & 0 & 1 & 2 \\
\end{pmatrix}
\quad \leftrightarrow \quad \left\{ \begin{array}{l}
x_1 + 2x_2 - x_3 = 1 \\
x_2 + 2x_3 = 2 \\
x_3 = 2 \\
\end{array} \right.
\]

...or an *even easier* one:

\[
\begin{pmatrix}
1 & 0 & 0 & 7 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 2 \\
\end{pmatrix}
\quad \leftrightarrow \quad \left\{ \begin{array}{l}
x_1 = 7 \\
x_2 = -2 \\
x_3 = 2 \\
\end{array} \right.
\]
Solving equations by row operations

- Operations on equations become operations on rows, e.g.
  \[
  \begin{pmatrix}
  1 & 1 & -2 \\
  3 & -1 & 2 \\
  \end{pmatrix}
  \iff
  \begin{cases}
  x_1 + x_2 = -2 \\
  3x_1 - x_2 = 2 \\
  \end{cases}
  \]

- Multiply row 1 by 3, giving:
  \[
  \begin{pmatrix}
  3 & 3 & -6 \\
  3 & -1 & 2 \\
  \end{pmatrix}
  \iff
  \begin{cases}
  3x_1 + 3x_2 = -6 \\
  3x_1 - x_2 = 2 \\
  \end{cases}
  \]

- Subtract the first row from the second, giving:
  \[
  \begin{pmatrix}
  3 & 3 & -6 \\
  0 & -4 & 8 \\
  \end{pmatrix}
  \iff
  \begin{cases}
  3x_1 + 3x_2 = -6 \\
  -4x_2 = 8 \\
  \end{cases}
  \]

- So \( x_2 = \frac{8}{-4} = -2 \). The first equation becomes:
  \[
  3x_1 - 6 = -6, \text{ so } x_1 = 0. \text{ Always check your answer.} \]
### Relevant operations & notation

<table>
<thead>
<tr>
<th></th>
<th>on equations</th>
<th>on matrices</th>
<th>LNBS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>exchange of rows</strong></td>
<td>$E_i \leftrightarrow E_j$</td>
<td>$R_i \leftrightarrow R_j$</td>
<td>$W_{i,j}$</td>
</tr>
<tr>
<td><strong>multiplication with</strong> $c \neq 0$</td>
<td>$E_i := cE_i$</td>
<td>$R_i := cR_i$</td>
<td>$V_i(c)$</td>
</tr>
<tr>
<td><strong>addition with</strong> $c \neq 0$</td>
<td>$E_i := E_i + cE_j$</td>
<td>$R_i := R_i + cR_j$</td>
<td>$O_{i,j}(c)$</td>
</tr>
</tbody>
</table>

These operations on equations/matrices:

- help to find solutions
- but do not change solutions (introduce/delete them)
The goal: rowstairs!

Definition

A matrix is in **Echelon form** (rijtrapvorm) if each row starts with strictly more zeros than the previous one.

\[
\begin{pmatrix}
1 & 2 & -1 & 1 \\
0 & 1 & 2 & 2 \\
0 & 0 & -3 & -6 \\
\end{pmatrix}
\]

e.g.  

A matrix in **reduced Echelon form** if it is in Echelon form, and each row contains at most one ‘1’ to the left of the line.

\[
\begin{pmatrix}
1 & 0 & 0 & 7 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

e.g.
Transformations example, part I

**equations**

\[
\begin{align*}
2x_2 + x_3 &= -2 \\
3x_1 + 5x_2 - 5x_3 &= 1 \\
2x_1 + 4x_2 - 2x_3 &= 2 \\
E_1 &\leftrightarrow E_3 \\
2x_1 + 4x_2 - 2x_3 &= 2 \\
3x_1 + 5x_2 - 5x_3 &= 1 \\
2x_2 + x_3 &= -2 \\
E_1 &:= \frac{1}{2} E_1 \\
x_1 + 2x_2 - 1x_3 &= 1 \\
3x_1 + 5x_2 - 5x_3 &= 1 \\
2x_2 + x_3 &= -2
\end{align*}
\]

**matrix**

\[
\begin{bmatrix}
0 & 2 & 1 & -2 \\
3 & 5 & -5 & 1 \\
2 & 4 & -2 & 2
\end{bmatrix}
\]

\[
E_1 \leftrightarrow E_3
\]

\[
\begin{bmatrix}
2 & 4 & -2 & 2 \\
3 & 5 & -5 & 1 \\
0 & 2 & 1 & -2
\end{bmatrix}
\]

\[
R_1 \leftrightarrow R_3
\]

\[
\begin{bmatrix}
1 & 2 & -1 & 1 \\
3 & 5 & -5 & 1 \\
0 & 2 & 1 & -2
\end{bmatrix}
\]

\[
R_1 := \frac{1}{2} R_1
\]
## Transformations example, part II

### Equations

\[
\begin{align*}
  x_1 + 2x_2 - 1x_3 &= 1 \\
  3x_1 + 5x_2 - 5x_3 &= 1 \\
  2x_2 + x_3 &= -2 \\
  E_2 &:= E_2 - 3E_1 \\
  x_1 + 2x_2 - 1x_3 &= 1 \\
  -x_2 - 2x_3 &= -2 \\
  2x_2 + x_3 &= -2 \\
  E_2 &:= -E_2 \\
  x_1 + 2x_2 - 1x_3 &= 1 \\
  x_2 + 2x_3 &= 2 \\
  2x_2 + x_3 &= -2
\end{align*}
\]

### Matrix

\[
\begin{pmatrix}
  1 & 2 & -1 & 1 \\
  3 & 5 & -5 & 1 \\
  0 & 2 & 1 & -2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 2 & -1 & 1 \\
  0 & -1 & -2 & -2 \\
  0 & 2 & 1 & -2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 2 & -1 & 1 \\
  0 & 1 & 2 & 2 \\
  0 & 2 & 1 & -2
\end{pmatrix}
\]
Transformations example, part III

**equations**

\[
\begin{align*}
x_1 + 2x_2 - 1x_3 &= 1 \\
x_2 + 2x_3 &= 2 \\
2x_2 + x_3 &= -2
\end{align*}
\]

\[E_3 := E_3 - 2E_2\]

\[
\begin{align*}
x_1 + 2x_2 - 1x_3 &= 1 \\
x_2 + 2x_3 &= 2 \\
-3x_3 &= -6
\end{align*}
\]

\[E_3 := -\frac{1}{3}E_3\]

**matrix**

\[
\begin{pmatrix}
1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & -2
\end{pmatrix}
\]

\[R_3 := R_3 - 2R_2\]

\[
\begin{pmatrix}
1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6
\end{pmatrix}
\]

\[R_3 := -\frac{1}{3}R_3\]

\[
\begin{pmatrix}
1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2
\end{pmatrix}
\]

Echelon (rijtrap) form
Transformations example, part IV

**equations**

\[
\begin{align*}
  x_1 + 2x_2 - 1x_3 &= 1 \\
  x_2 + 2x_3 &= 2 \\
  x_3 &= 2 \\
  E_1 &:= E_1 - 2E_2 \\
  x_1 - 5x_3 &= -3 \\
  x_2 + 2x_3 &= 2 \\
  x_3 &= 2 \\
  E_1 &:= E_1 + 5E_3, \; E_2 := E_2 - 2E_3 \\
  R_1 &:= R_1 - 2R_2 \\
  x_1 &= 7 \\
  x_2 &= -2 \\
  x_3 &= 2
\end{align*}
\]

**matrix**

\[
\begin{pmatrix}
  1 & 2 & -1 & 1 \\
  0 & 1 & 2 & 2 \\
  0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 0 & -5 & -3 \\
  0 & 1 & 2 & 2 \\
  0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 0 & 0 & 7 \\
  0 & 1 & 0 & -2 \\
  0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

Echelon form

Reduced echelon form
Gauss elimination

- Solutions can be found by mechanically applying simple rules
  - in Dutch this is called *vegen*
  - first produce *echelon form* (rijtrapvorm), then obtain single-variable equations, *reduced echelon form* (gereducteerde rijtrapvorm)
  - it is one of the two most important algorithms in virtually any computer algebra system

- Applying these operations is actually *easier on matrices*, than on the equations themselves

- You should be able to do Gauss elimination in your sleep! It is a basic technique used throughout the course.
Examples

1. \[ x_1 + x_2 = 3 \]
   \[ x_1 - x_2 = 1 \]
   has a single solution, namely \( x_1 = 2, x_2 = 1 \)

2. \[ x_1 - 2x_2 - 3x_3 = -11 \]
   \[ -x_1 + 3x_2 + 5x_3 = 15 \]
   has many solutions
   (they can be described as: \( x_1 = -x_3 - 3, x_2 = 4 - 2x_3 \), giving a solution for each value of \( x_3 \))

3. \[ 3x_1 - 2x_2 = 1 \]
   \[ 6x_1 - 4x_2 = 6 \]
   has no solutions: the transformation \( E_2 := E_2 - 2E_1 \) yields \( 0 = 4 \).
Consider systems of only two variables $x, y$. A linear equation $ax + by = c$ then describes a line in the plane.

For 2 such equations/lines, there are three possibilities:

1. the lines intersect in a **unique point**, which is the solution to both equations
2. the lines are **parallel**, in which case there are no joint solutions
3. the lines **coincide**, giving many joint solutions.
A system of equations is **consistent** (oplosbaar) if it has one or more solutions. Otherwise, when there are no solutions, the system is called **inconsistent**.

Thus, for a system of equations:

<table>
<thead>
<tr>
<th>nr. of solutions</th>
<th>terminology</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>inconsistent</td>
</tr>
<tr>
<td>≥ 1</td>
<td>consistent</td>
</tr>
</tbody>
</table>

(one or many)
Pivots and Echelon form

Definition

A pivot (Dutch: spil or draaipunt) is the first non-zero element of a row in a matrix.

Echelon form therefore means each pivot must occur (strictly) to the right of the pivot on the previous row.
Pivots and echelon form, examples

Example (● = pivot)

\[
\begin{pmatrix}
\bullet & \ast & \ast \\
0 & \bullet & \ast \\
0 & 0 & \bullet
\end{pmatrix},
\begin{pmatrix}
\bullet & \ast & \ast & \ast \\
0 & \bullet & \ast & \ast \\
0 & 0 & \bullet & \ast \\
0 & 0 & 0 & \bullet
\end{pmatrix},
\begin{pmatrix}
\bullet & \ast & \ast & \ast & \ast \\
0 & \bullet & \ast & \ast & \ast \\
0 & 0 & \bullet & \ast & \ast \\
0 & 0 & 0 & \bullet & \ast \\
0 & 0 & 0 & 0 & \bullet
\end{pmatrix}
\]

Non-examples:

\[
\begin{pmatrix}
\bullet & \ast & \ast \\
0 & \bullet & \ast \\
\bullet & 0 & \ast
\end{pmatrix},
\begin{pmatrix}
0 & \bullet & \ast \\
0 & \bullet & \ast \\
0 & 0 & 0 & \bullet
\end{pmatrix},
\begin{pmatrix}
\bullet & \ast & \ast & \ast & \ast \\
0 & \bullet & \ast & \ast & \ast \\
0 & 0 & \bullet & \ast & \ast \\
0 & 0 & 0 & \bullet & \ast \\
0 & 0 & 0 & 0 & \bullet
\end{pmatrix}
\]
Inconsistency and echelon forms

**Theorem**

A system of equations is **inconsistent** (non-solvable) if and only if in the echelon form of its augmented matrix there is a row with:

- only zeros before the bar |
- a non-zero after the bar |

as in: $0 \ 0 \ \cdots \ 0 \mid c$, where $c \neq 0$.

**Example**

\[
\begin{align*}
3x_1 - 2x_2 &= 1 \\
6x_1 - 4x_2 &= 6
\end{align*}
\]

gives
\[
\begin{pmatrix}
3 & -2 & 1 \\
6 & -4 & 6
\end{pmatrix}
\]

and
\[
\begin{pmatrix}
3 & -2 & 1 \\
0 & 0 & 4
\end{pmatrix}
\]

(using the transformation $R_2 := R_2 - 2R_1$)
Unique solutions

**Theorem**

A system of equations in \( n \) variables has a unique solution if and only if in its echelon form there are \( n \) pivots.

**Example (\( \square \) denotes a pivot)**

\[
\begin{align*}
    x_1 + x_2 &= 3 \\
    x_1 - x_2 &= 1
\end{align*}
\]

gives

\[
\begin{pmatrix}
    1 & 1 & 3 \\
    1 & -1 & 1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
    1 & 1 & 3 \\
    0 & 1 & 1
\end{pmatrix}
\]

(using transformations \( R_2 := R_2 - R_1 \) and \( R_2 := -\frac{1}{2}R_2 \))
Unique solutions: earlier example

\[
\begin{align*}
2x_2 + x_3 &= -2 \\
3x_1 + 5x_2 - 5x_3 &= 1 \\
2x_1 + 4x_2 - 2x_3 &= 2
\end{align*}
\]

After various transformations leads to

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= 1 \\
x_2 + 2x_3 &= 2 \\
x_3 &= 2
\end{align*}
\]

There are 3 variables and 3 pivots, so there is one unique solution.