Matrix Calculations

Assignment 3, Tuesday, February 14, 2017

Exercise teachers. Recall the following split-up of students:

teacher	lecture room	email
John van de Wetering	HG00.065	`wetering@cs.ru.nl`
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The delivery boxes are located in the Mercator 1 building on the ground floor (where the Computer Science department ICIS is located).

Handing in your answers: There are two options, *depending on your exercise class teacher*:

1. Delivery box (default): Put your solutions in the appropriate delivery box (see above). Before putting your solutions in the box make sure:
   - your name and student number are written clearly on the document.
2. E-mail (if your teacher agrees): Send your solutions by e-mail to your exercise class teacher (see above) with subject '*assignment 3*'. This e-mail should only contain a single PDF document as attachment (unless explicitly stated otherwise). Before sending an e-mail make sure:
   - the file is a PDF document that is well readable
   - your name is part of the filename (for example MyName_assignment-3.pdf)
   - your name and student number are included in the document (since they will be printed)

Deadline: Monday, February 20, 12:00 sharp!

Goals: After completing these exercises successfully you should be able to determine the (in)dependence of vectors, determine whether certain sets form vector spaces (or subspaces), and prove (non)linearity of maps.
The total number of points is 20.

1. **(3 points)** Check if the following vectors are linearly dependent/independent. Explain your answers briefly:

$$\text{(i)} \quad \left(\begin{pmatrix} 5 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix} \right) \qquad \text{(ii)} \quad \left(\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -14 \\ 20 \\ 7 \end{pmatrix} \right)$$

2. **(2 points)** Express the vector $v = (4, 3, 2) \in \mathbb{R}^3$ as a linear combination of the following vectors:

$$v_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 3 \\ -6 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix}$$

3. **(5 points)** Which of the following subsets of $\mathbb{R}^n$ are subspaces?

(i) $S \subseteq \mathbb{R}^3$ defined by $S = \{(x, 2x, 3x) \mid x \in \mathbb{R}\}$.

(ii) $S \subseteq \mathbb{R}^2$ defined by $S = \{(x, x + y) \mid x, y \in \mathbb{R}\}$

(iii) $S \subseteq \mathbb{R}^2$ defined by $S = \{(x, x + 1) \mid x \in \mathbb{R}\}$

(iv) For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, $S \subseteq \mathbb{R}^n$ defined by: $\{a \cdot \boldsymbol{v} + b \cdot \boldsymbol{w} \mid a, b \in \mathbb{R}\}$.

(v) $\mathbb{N} \subseteq \mathbb{R}$

If a set is a subspace, prove it. If it is not, give an argument why not.

4. **(4 points)** Prove explicitly that the following maps are linear.

   (i) $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f((x, y, z)) = (y + z, 2x + z, 3x - y + z)$.

   (ii) $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f((x, y, z)) = (ax + by, cx + z, dx)$, for $a, b, c, d \in \mathbb{R}$.

5. **(2 points)** Show that the following maps are *not* linear

   (i) $f \colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f((x, y, z)) = (x + z, y + xz)$;

   (ii) $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f((x, y)) = (x, y + 3)$.

6. **(0 points)** *Extra exercise, for those interested*

   Prove that the following, alternative definition of a vector space is equivalent to the one given in the slides:

   Definition 1. A *vector space* $(V, +, \cdot, \mathbf{0})$ is a set V, a special element $\mathbf{0} \in V$ and operations $+, \cdot$ satisfying the following properties:

   (a) $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$

   (b) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$

   (c) $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$

   (d) $(a + b) \cdot \boldsymbol{v} = a \cdot \boldsymbol{v} + b \cdot \boldsymbol{v}$

   (e) $a \cdot (\boldsymbol{v} + \boldsymbol{w}) = a \cdot \boldsymbol{v} + a \cdot \boldsymbol{w}$

   (f) $a \cdot (b \cdot \boldsymbol{v}) = ab \cdot \boldsymbol{v}$

   (g) $1 \cdot \boldsymbol{v} = \boldsymbol{v}$

   (h) for all $\boldsymbol{v} \in V$ there exists $-\boldsymbol{v} \in V$ such that $-\boldsymbol{v} + \boldsymbol{v} = \mathbf{0}$

   That is, assuming (a)–(g), condition (h) is equivalent to the equation $0 \cdot \boldsymbol{v} = \mathbf{0}$.