Matrix Calculations: Solutions of Systems of Linear Equations

A. Kissinger

Institute for Computing and Information Sciences
Radboud University Nijmegen

Version: spring 2017

Outline

Pivots

- A **pivot** is the first non-zero entry of a row:

$$\left(\begin{array}{ccc|c} 0 & \boxed{2} & 1 & -2 \\ \boxed{3} & 5 & -5 & 1 \\ 0 & 0 & \boxed{-2} & 2 \end{array}\right)$$

- If a row is all zeros, it **has no pivot**:

$$\left(\begin{array}{ccc|c} 0 & \boxed{2} & 1 & -2 \\ \boxed{3} & 5 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We call this a *zero row*.

Echelon form

A matrix is in **Echelon form** if:

1. All of the rows with pivots occur before zero rows, and
2. Pivots always occur to the right of previous pivots

$$\left(\begin{array}{cccc|c} \boxed{3} & 2 & 5 & -5 & 1 \\ 0 & 0 & \boxed{2} & 1 & -2 \\ 0 & 0 & 0 & \boxed{-2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \checkmark$$

$$\left(\begin{array}{cccc|c} \boxed{3} & 2 & 5 & -5 & 1 \\ 0 & 0 & \boxed{2} & 1 & -2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \boxed{-2} & 2 \end{array}\right) ☠ \qquad \left(\begin{array}{cccc|c} \boxed{3} & 2 & 5 & -5 & 1 \\ 0 & 0 & \boxed{\mathbf{4}} & -2 & 2 \\ 0 & \boxed{\mathbf{2}} & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) ☠ \qquad \left(\begin{array}{cccc|c} \boxed{3} & 2 & 5 & -5 & 1 \\ 0 & 0 & \boxed{\mathbf{4}} & -2 & 2 \\ 0 & 0 & \boxed{\mathbf{2}} & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) ☠$$

Points and polynomials

Here's a really useful thing about polynomials:

Theorem

For any n points in a plane, there exists a unique polynomial of degree $n-1$ which hits them all.

That is: given points $(x_1, y_1), \ldots, (x_n, y_n)$, there is precisely one polynomial function of the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1}$$

with $f(x_i) = y_i$ for all $i \leq n$.

NB. No two points should be on the same vertical line!

- The data fitting problem is: given the points (x_i, y_i) obtained from some experiment, find the $a_0, \ldots, a_{n-1}$
- This can be done with what we have seen so far!

Data fitting example

- Suppose we have 3 points $(1,6)$, $(2,3)$ and $(3,2)$
- we wish to find $f(x) = a_0 + a_1 x + a_2 x^2$ that hits them all
- The requirements $f(1) = 6$, $f(2) = 3$ and $f(3) = 2$ yield:

$$\begin{aligned} a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 &= 6 \\ a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 &= 3 \\ a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 &= 2 \end{aligned}$$

- The augmented matrix and its Echelon form are:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 2 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

- Its solution is $a_2 = 1$, $a_1 = -6$ en $a_0 = 11$, ie. $(11, -6, 1)$
- and so the required function if $f(x) = 11 - 6x + x^2$. ✓

Unique solutions

From the first lecture:

Theorem

A system of equations in n variables has a unique solution if and only if its Echelon form has n pivots.

Example (□ denotes a pivot)

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1 \end{aligned} \quad \text{gives} \quad \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} \boxed{1} & 1 & 3 \\ 0 & \boxed{1} & 1 \end{array}\right)$$

(using transformations $R_2 := R_2 - R_1$ and $R_2 := -\frac{1}{2}R_2$)

Question: What if there are more solutions? Can we describe them in a generic way?

A new tool: vectors

- A vector is a list of numbers.
- We can write it like this: $(x_1, x_2, \ldots, x_n)$
- ...or as a matrix with just one column:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

(which is sometimes called a ‘column vector’).

A new tool: vectors

- Vectors are useful for lots of stuff. In this lecture, we'll use them to hold solutions.
- Since variable names don't matter, we can write this:

$$x_1 := 2 \quad x_2 := -1 \quad x_3 := 0$$

- ...more compactly as this:

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

- ...or even more compactly as this: $(2, -1, 0)$.

Linear combinations

- We can multiply a vector by a number to get a new vector:

$$c \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}$$

  This is called **scalar multiplication**.

- ...and we can add vectors together:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

  as long as the are the same length.

Linear combinations

Mixing these two things together gives us a **linear combination** of vectors:

$$c \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + d \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \ldots = \begin{pmatrix} cx_1 + dy_1 + \ldots \\ cx_2 + dy_2 + \ldots \\ \vdots \\ cx_n + dy_n + \ldots \end{pmatrix}$$

A set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ is called **linearly independent** if no vector can be written as a linear combination of the others.

Linear independence

- These vectors:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  are **NOT** linearly independent, because $\boldsymbol{v}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2$.

- These vectors:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

  are **NOT** linearly independent, because $\boldsymbol{v}_1 = \boldsymbol{v}_2 + 2 \cdot \boldsymbol{v}_3$.

Linear independence

- These vectors:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  are linearly *independent*. There is no way to write any of them in terms of each other.

- These vectors:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

  are linearly *independent*. There is no way to write any of them in terms of each other.

Linear independence

- These vectors:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

  are...???

- 'Eyeballing' vectors works sometimes, but we need a better way of checking linear independence!

Checking linear independence

Theorem

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if and only if, for all numbers $a_1, \ldots, a_n \in \mathbb{R}$ one has:

$$a_1 \cdot \mathbf{v}_1 + \cdots + a_n \cdot \mathbf{v}_n = \mathbf{0} \quad \textit{implies} \quad a_1 = a_2 = \cdots = a_n = 0$$

Example

The 3 vectors $(1,0,0), (0,1,0), (0,0,1)$ are linearly independent, since if

$$a_1 \cdot (1,0,0) + a_2 \cdot (0,1,0) + a_3 \cdot (0,0,1) = (0,0,0)$$

then, using the computation from the previous slide,

$$(a_1, a_2, a_3) = (0,0,0), \quad \text{so that} \quad a_1 = a_2 = a_3 = 0$$

Checking linear independence

Theorem

Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent if and only if, for all numbers $a_1, \ldots, a_n \in \mathbb{R}$ one has:

$$a_1 \cdot \boldsymbol{v}_1 + \cdots + a_n \cdot \boldsymbol{v}_n = \mathbf{0} \quad \textit{implies} \quad a_1 = a_2 = \cdots = a_n = 0$$

Proof. Another way to say the theorem is $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly dependent if and only if:

$$a_1 \cdot \boldsymbol{v}_1 + a_2 \cdot \boldsymbol{v}_2 + \cdots + a_n \cdot \boldsymbol{v}_n = \mathbf{0}$$

where some a_j are non-zero. If this is true and $a_1 \neq 0$, then:

$$\boldsymbol{v}_1 = (-a_2/a_1) \cdot \boldsymbol{v}_2 + \ldots + (-a_n/a_1) \cdot \boldsymbol{v}_n$$

The vectors are dependent (also works for any other non-zero a_j).
Exercise: prove the other direction.

Proving (in)dependence via equation solving I

- Investigate (in)dependence of $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}2\\-1\\4\end{pmatrix}$, and $\begin{pmatrix}0\\5\\2\end{pmatrix}$
- Thus we ask: are there any non-zero $a_1, a_2, a_3 \in \mathbb{R}$ with:

$$a_1 \begin{pmatrix}1\\2\\3\end{pmatrix} + a_2 \begin{pmatrix}2\\-1\\4\end{pmatrix} + a_3 \begin{pmatrix}0\\5\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

- If there is a non-zero solution, the vectors are dependent, and if $a_1 = a_2 = a_3 = 0$ is the only solution, they are independent

Proving (in)dependence via equation solving II

- Our question involves the systems of equations / matrix:

$$\begin{cases} a_1 + 2a_2 = 0 \\ 2a_1 - a_2 + 5a_3 = 0 \\ 3a_1 + 4a_2 + 2a_3 = 0 \end{cases} \quad \text{corresponding to} \quad \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(in Echelon form)

- This has only 2 pivots, so multiple solutions. In particular, it has non-zero solutions, for example: $a_1 = 2, a_2 = -1, a_3 = -1$ (compute and check for yourself!)
- Thus the original vectors are dependent. Explicitly:

$$2\begin{pmatrix}1\\2\\3\end{pmatrix} + (-1)\begin{pmatrix}2\\-1\\4\end{pmatrix} + (-1)\begin{pmatrix}0\\5\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Radboud University Nijmegen

Proving (in)dependence via equation solving III

- Same (in)dependence question for: $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$
- With corresponding matrix:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -3 & 1 & -2 \end{pmatrix} \quad \text{reducing to} \quad \begin{pmatrix} 5 & 0 & -1 \\ 0 & 5 & -3 \\ 0 & 0 & -4 \end{pmatrix}$$

- Thus the only solution is $a_1 = a_2 = a_3 = 0$. The vectors are independent!

Linear independence: summary

To check linear independence of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$:

1. Write the vectors as the *columns of a matrix*
2. Convert to Echelon form
3. Count the pivots
   - (# pivots) = (# columns) means independent
   - (# pivots) < (# columns) means dependent
4. Non-zero solutions show linear dependence explicitly, e.g.

$$\boldsymbol{v}_1 - 2\boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{v}_1 = 2\boldsymbol{v}_2 - \boldsymbol{v}_3$$

General solutions

The Goal:

- Describe the space of solutions of a system of equations.
- In general, there can be infinitely many solutions, but only a few are actually 'different enough' to matter. These are called basic solutions.
- Using the basic solutions, we can write down a formula which gives us any solution: the general solution.

Example (General solution for one equation)

$$2x_1 - x_2 = 3 \quad \text{gives} \quad x_2 = 2x_1 - 3$$

So a general solution (for any c) is:

$$x_1 := c \qquad x_2 := 2c - 3$$

Linear combinations of solutions

- It is **not** the case in general that linear combinations of solutions give solutions. For example, consider:

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_2 + x_4 = 2 \end{cases} \qquad \leftrightarrow \qquad \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right)$$

- This has as solutions:

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \text{ but } \textbf{not } \boldsymbol{v}_1 + \boldsymbol{v}_2 = \begin{pmatrix} -3 \\ 3 \\ -3 \\ 1 \end{pmatrix}, 3 \cdot \boldsymbol{v}_1, \ldots$$

- The problem is this system of equations is not homogeneous, because the the **2** on the right-hand-side (RHS) of the second equation.

Homogeneous systems of equations

Definition

A system of equations is called homogeneous if it has zeros on the RHS of every equation. Otherwise it is called non-homogeneous.

- We can always squash a non-homogeneous system to a homogeneous one:

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 3 & 5 & -5 & 1 \\ 0 & 0 & -2 & 2 \end{array}\right) \quad \rightsquigarrow \quad \begin{pmatrix} 0 & 2 & 1 \\ 3 & 5 & -5 \\ 0 & 0 & -2 \end{pmatrix}$$

- The solutions will change!
- ...but they are still related. We'll see how that works soon.

Zero solution, in homogeneous case

Lemma

Each homogeneous equation has $(0, \ldots, 0)$ as solution.

Proof: A homogeneous system looks like this

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & 0 \\ & \vdots & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & 0 \end{array}$$

Consider the equation at row i:

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0$$

Clearly it has as solution $x_1 = x_2 = \cdots = x_n = 0$.
This holds for each row i.

Linear combinations of solutions

Theorem

The set of solutions of a homogeneous system is closed under linear combinations (i.e. addition and scalar multiplication of vectors).

...which means:

- if $(s_1, s_2, \ldots, s_n)$ and $(t_1, t_2, \ldots, t_n)$ are solutions, then so is: $(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$, and
- if $(s_1, s_2, \ldots, s_n)$ is a solution, then so is $(c \cdot s_1, c \cdot s_2, \ldots, c \cdot s_n)$

Example

- Consider the homogeneous system $\begin{cases} 3x_1 + 2x_2 - x_3 & = & 0 \\ x_1 - x_2 & = & 0 \end{cases}$
- A solution is $x_1 = 1, x_2 = 1, x_3 = 5$, written as vector $(x_1, x_2, x_3) = (1, 1, 5)$
- Another solution is $(2, 2, 10)$
- Addition yields another solution:

$$(1, 1, 5) + (2, 2, 10) \;=\; (1 + 2, 1 + 2, 10 + 5) \;=\; (3, 3, 15).$$

- Scalar multiplication also gives solutions:

$$\begin{array}{rcl} -1 \cdot (1, 1, 5) & = & (-1 \cdot 1, -1 \cdot 1, -1 \cdot 5) \;=\; (-1, -1, -5) \\ 100 \cdot (2, 2, 10) & = & (100 \cdot 2, 100 \cdot 2, 100 \cdot 10) \;=\; (200, 200, 1000) \\ c \cdot (1, 1, 5) & = & (c \cdot 1, c \cdot 1, c \cdot 5) \;=\; (c, c, 5c) \\ & & \qquad \text{(is a solution for every } c\text{)} \end{array}$$

Proof of closure under addition

- Consider an equation $a_1x_1 + \cdots + a_nx_n = 0$
- Assume two solutions $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$
- Then $(s_1 + t_1, \ldots, s_n + t_n)$ is also a solution since:

$$\begin{aligned}
& a_1(s_1 + t_1) + \cdots + a_n(s_n + t_n) \\
= \; & \Big(a_1s_1 + a_1t_1\Big) + \cdots + \Big(a_ns_n + a_nt_n\Big) \\
= \; & \Big(a_1s_1 + \cdots + a_ns_n\Big) + \Big(a_1t_1 + \cdots + a_nt_n\Big) \\
= \; & 0 + 0 \qquad \text{since the } s_i \text{ and } t_i \text{ are solutions} \\
= \; & 0.
\end{aligned}$$

- Exercise: do a similar proof of closure under scalar multiplication

General solution of a homogeneous system

Theorem

Every solution to a homogeneous system arises from a general solution of the form:

$$(s_1, \ldots, s_n) = c_1(v_{11}, \ldots, v_{1n}) + \cdots + c_k(v_{k1}, \ldots, v_{kn})$$

for some numbers $c_1, \ldots, c_k \in \mathbb{R}$.

We call this a parametrization of our solution space. It means:

1. There is a fixed set of vectors (called basic solutions):

$$\boldsymbol{v}_1 = (v_{11}, \ldots, v_{1n}), \quad \ldots, \quad \boldsymbol{v}_k = (v_{k1}, \ldots, v_{kn})$$

2. such that every solution $\boldsymbol{s}$ is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$.
3. That is, there exist $c_1, \ldots, c_k \in \mathbb{R}$ such that

$$\boldsymbol{s} = c_1\,\boldsymbol{v}_1 + \ldots + c_k\,\boldsymbol{v}_k$$

Basic solutions of a homogeneous system

Theorem

Suppose a homogeneous system of equations in n variables has $p \leq n$ pivots. Then there are $n-p$ basic solutions $\mathbf{v}_1, \ldots, \mathbf{v}_{n-p}$.

This means that the general solution $\mathbf{s}$ can be written as a parametrization:

$$\mathbf{s} = c_1\mathbf{v}_1 + \cdots c_{n-p}\mathbf{v}_{n-p}.$$

Moreover, for any solution $\mathbf{s}$, the scalars $c_1, \ldots, c_{n-p}$ are unique.

$$(p = n) \;\Leftrightarrow\; (\text{no basic solns.}) \;\Leftrightarrow\; (\mathbf{0} \text{ is the unique soln.})$$

Finding basic solutions

- We have two kinds of variables, pivot variables and non-pivot, or free variables, depending on whether their column has a pivot:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 0 & 1 & 4 & 1 \\ 0 & 0 & \boxed{1} & 2 & 0 \end{pmatrix}$$

- The Echelon form lets us (easily) write pivot variables in terms of non-pivot variables, e.g.:

$$\begin{cases} x_1 = -x_3 - 4x_4 - x_5 \\ x_3 = -2x_4 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_4 - x_5 \\ x_3 = -2x_4 \end{cases}$$

- We can find a (non-zero) basic solution by setting exactly **one** free variable to 1 and the rest to 0.

Finding basic solutions

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \end{array}$$
$$\begin{pmatrix} \boxed{1} & 0 & 1 & 4 & 1 \\ 0 & 0 & \boxed{1} & 2 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = -2x_4 - x_5 \\ x_3 = -2x_4 \end{cases}$$

5 variables and 2 pivots gives us $5 - 2 = 3$ basic solutions:

$$\begin{array}{ccc} x_2 := 1 & x_2 := 0 & x_2 := 0 \\ x_4 := 0 & x_4 := 1 & x_4 := 0 \\ x_5 := 0 & x_5 := 0 & x_5 := 1 \end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_4 - x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

General Solution

Now, any solution to the system is obtainable as a linear combination of basic solutions:

$$x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2x_4 - x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \end{pmatrix}$$

Picking solutions this way guarantees **linear independence**.

Finding basic solutions: technique 2

- Keep all columns with a pivot,
- One-by-one, keep only the *i*-th non-pivot columns (while removing the others), and find a (non-zero) solution
- (this is like setting all the other free variables to zero)
- Add 0's to each solution to account for the columns (i.e. free variables) we removed

General solution and basic solutions, example

- For the matrix: $\begin{pmatrix} \boxed{1} & 1 & 0 & 4 \\ 0 & 0 & \boxed{2} & 2 \end{pmatrix}$
- There are 4 columns (variables) and 2 pivots, so $4 - 2 = 2$ basic solutions
- First keep only the first non-pivot column:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{with chosen solution} \quad (x_1, x_2, x_3) = (1, -1, 0)$$

- Next keep only the second non-pivot column:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \end{pmatrix} \quad \text{with chosen solution} \quad (x_1, x_3, x_4) = (4, 1, -1)$$

- The general 4-variable solution is now obtained as:

$$c_1 \cdot (1, -1, 0, 0) + c_2 \cdot (4, 0, 1, -1)$$

General solutions example, check

We double-check that any vector:

$$\begin{aligned} & c_1 \cdot (4, 0, 1, -1) + c_2 \cdot (1, -1, 0, 0) \\ &= (4 \cdot c_1, 0, 1 \cdot c_1, -1 \cdot c_1) + (1 \cdot c_2, -1 \cdot c_2, 0, 0) \\ &= (4c_1 + c_2, -c_2, c_1, -c_1) \end{aligned}$$

gives a solution of:

$$\begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 0 & 2 & 2 \end{pmatrix} \quad \text{i.e. of} \quad \begin{cases} x_1 + x_2 + 4x_4 & = 0 \\ 2x_3 + 2x_4 & = 0 \end{cases}$$

Just fill in $x_1 = 4c_1 + c_2,\ x_2 = -c_2,\ x_3 = c_1,\ x_4 = -c_1$

$$\begin{aligned} (4c_1 + c_2) - c_2 + 4 \cdot -c_1 &= 0 \\ 2c_1 - 2c_1 &= 0 \end{aligned} \qquad \checkmark$$

Summary of homogeneous systems

Given a homogeneous system in n variables:

- A basic solution is a non-zero solution of the system.
- If there are n pivots in its echelon form, there is no basic solution, so only $\mathbf{0} = (0, \ldots, 0)$ is a solution.
- Basic solutions are not unique. For instance, if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ give basic solutions, so do $\boldsymbol{v}_1 + \boldsymbol{v}_2$, $\boldsymbol{v}_1 - \boldsymbol{v}_2$, and any other linear combination.
- If there are $p < n$ pivots in its Echelon form, it has $n - p$ linearly independent basic solutions.

Non-homogeneous case: subtracting solutions

Theorem

The difference of two solutions of a **non-homogeneous** *system is a solution for the associated* **homogeneous** *system.*

More explicitly: given two solutions $(s_1, \ldots, s_n)$ *and* $(t_1, \ldots, t_n)$ *of a* **non-homogeneous** *system, the difference* $(s_1 - t_1, \ldots, s_n - t_n)$ *is a solution of the associated* **homogeneous** *system.*

Proof: Let $a_1x_1 + \cdots + a_nx_n = b$ be the equation. Then:

$$\begin{aligned} & a_1(s_1 - t_1) + \cdots + a_n(s_n - t_n) \\ = \;& \big(a_1s_1 - a_1t_1\big) + \cdots + \big(a_ns_n - a_nt_n\big) \\ = \;& \big(a_1s_1 + \cdots + a_ns_n\big) - \big(a_1t_1 + \cdots + a_nt_n\big) \\ = \;& b - b \qquad \text{since the } s_i \text{ and } t_i \text{ are solutions} \\ = \;& 0. \end{aligned}$$

☺

General solution for non-homogeneous systems

Theorem

Assume a non-homogeneous system has a solution given by the vector $\boldsymbol{p}$, which we call a particular solution.
Then any other solution $\boldsymbol{s}$ of the non-homogeneous system can be written as

$$\boldsymbol{s} = \boldsymbol{p} + \boldsymbol{h}$$

where $\boldsymbol{h}$ is a solution of the associated homogeneous system.

Proof: Let $\boldsymbol{s}$ be a solution of the non-homogeneous system. Then $\boldsymbol{h} = \boldsymbol{s} - \boldsymbol{p}$ is a solution of the associated homogeneous system. Hence we can write $\boldsymbol{s}$ as $\boldsymbol{p} + \boldsymbol{h}$, for $\boldsymbol{h}$ some solution of the associated homogeneous system. ☺

Example: solutions of a non-homogeneous system

- Consider the non-homogeneous system $\begin{cases} x + y + 2z &= 9 \\ y - 3z &= 4 \end{cases}$
- with solutions: $(0, 7, 1)$ and $(5, 4, 0)$
- We can write $(0, 7, 1)$ as: $(5, 4, 0) + (-5, 3, 1)$
- where:
  - $\boldsymbol{p} = (5, 4, 0)$ is a particular solution (of the original system)
  - $(-5, 3, 1)$ is a solution of the associated homogeneous system: $\begin{cases} x + y + 2z &= 0 \\ y - 3z &= 0 \end{cases}$
- Similarly, $(10, 1, -1)$ is a solution of the non-homogeneous system and

$$(10, 1, -1) \;=\; (5, 4, 0) + (5, -3, -1)$$

- where:
  - $(5, -3, -1)$ is a solution of the associated homogeneous system.

General solution for non-homogeneous systems, concretely

Theorem

The general solution of a non-homogeneous system of equations in n variables is given by a parametrization as follows:

$$(s_1, \ldots, s_n) = (p_1, \ldots, p_n) + c_1(v_{11}, \ldots, v_{1n}) + \cdots c_k(v_{k1}, \ldots, v_{kn})$$

for $c_1, \ldots, c_k \in \mathbb{R}$,
where

- $(p_1, \ldots, p_n)$ *is a particular solution*
- $(v_{11}, \ldots, v_{1n}), \ldots, (v_{k1}, \ldots, v_{kn})$ *are basic solutions of the associated homogeneous system.*
- *So* $c_1(v_{11}, \ldots, v_{1n}) + \cdots + c_k(v_{k1}, \ldots, v_{kn})$ *is a general solution for the associated homogeneous system.*

Elaborated example, part I

- Consider the non-homogeneous system of equations given by the augmented matrix in echelon form:

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

- It has 5 variables, 3 pivots, and thus $5 - 3 = 2$ basic solutions
- To find a particular solution, remove the non-pivot columns, and (uniquely!) solve the resulting system:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

- This has $(10, -11, 4)$ as solution; the orginal 5-variable system then has particular solution $(10, 0, -11, 0, 4)$.

Elaborated example, part II

- Consider the associated homogeneous system of equations:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

- The two basic solutions are found by removing each of the two non-pivot columns separately, and finding solutions:

$$\begin{array}{cccc} x_1 & x_3 & x_4 & x_5 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_5 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix} \text{ and } \begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

- We find: $(1, -2, 1, 0)$ and $(-1, 1, 0, 0)$. Adding zeros for missing columns gives: $(1, 0, -2, 1, 0)$ and $(-1, 1, 0, 0, 0)$.

Elaborated example, part III

Wrapping up: all solutions of the system

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

are of the form:

$$\underbrace{(10, 0, -11, 0, 4)}_{\text{particular sol.}} + \underbrace{c_1(1, 0, -2, 1, 0) + c_2(-1, 1, 0, 0, 0)}_{\text{two basic solutions}}.$$

This is the general solution of the non-homogeneous system.