Matrix Calculations: Eigenvalues and Eigenvectors

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Outline

Example: political swingers, part I

- We take an extremely crude view on politics and distinguish only left and right wing political supporters
- We study changes in political views, per year
- Suppose we observe, for each year:
  - 80% of lefties remain lefties and 20% become righties
  - 90% of righties remain righties, and 10% become lefties

Questions . . .

- start with a population $L = 100$, $R = 150$, and compute the number of lefties and righties after one year;
- similarly, after 2 years, and 3 years, . . .
- We can represent these computations conveniently using matrix multiplication.

Political swingers, part II

- So if we start with a population $L = 100, R = 150$, then after one year we have:
  - lefties: $0.8 \cdot 100 + 0.1 \cdot 150 = 80 + 15 = 95$
  - righties: $0.2 \cdot 100 + 0.9 \cdot 150 = 20 + 135 = 155$
- If $\begin{pmatrix} L \\ R \end{pmatrix} = \begin{pmatrix} 100 \\ 150 \end{pmatrix}$, then after one year we have:

$$\boldsymbol{P} \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix} = \begin{pmatrix} 95 \\ 155 \end{pmatrix}$$

- After two years we have:

$$\boldsymbol{P} \cdot \begin{pmatrix} 95 \\ 155 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \cdot \begin{pmatrix} 95 \\ 155 \end{pmatrix} = \begin{pmatrix} 91.5 \\ 158.5 \end{pmatrix}$$

Political swingers, part IV

The situation after two years is obtained as:

$$\mathbf{P} \cdot \mathbf{P} \cdot \begin{pmatrix} L \\ R \end{pmatrix} = \underbrace{\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \cdot \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}}_{\text{do this multiplication first}} \cdot \begin{pmatrix} L \\ R \end{pmatrix}$$

$$= \begin{pmatrix} 0.66 & 0.17 \\ 0.34 & 0.83 \end{pmatrix} \cdot \begin{pmatrix} L \\ R \end{pmatrix}$$

The situation after n years is described by the n-fold iterated matrix:

$$\mathbf{P}^n = \underbrace{\mathbf{P} \cdot \mathbf{P} \cdots \mathbf{P}}_{n \text{ times}}$$

Etc. It looks like $\mathbf{P}^{100}$ (or worse, $\lim_{n\to\infty} \mathbf{P}^n$) is going to be a real pain to calculate. ...or is it?

Diagonal matrices

- Multiplying matrices is hard ☹
- But multiplying diagonal matrices is easy!

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \cdot \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{pmatrix} = \begin{pmatrix} aw & 0 & 0 & 0 \\ 0 & bx & 0 & 0 \\ 0 & 0 & cy & 0 \\ 0 & 0 & 0 & dz \end{pmatrix}$$

- **Strategy:** find a basis $\mathcal{B}$ where our matrix $\boldsymbol{P}$ is diagonal:

$$\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}_{\mathcal{S}} \leadsto \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}_{\mathcal{B}}$$

- So transform to $\mathcal{B}$, multiply, and (if we need to) transform back:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}_{\mathcal{B}}^{100} = \begin{pmatrix} 1^{100} & 0 \\ 0 & (0.7)^{100} \end{pmatrix}_{\mathcal{B}} \approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\mathcal{B}} \leadsto \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}_{\mathcal{S}}$$

Eigenvectors and eigenvalues

This magical basis $\mathcal{B}$ consists of *eigenvectors* of a matrix.

Definition

Assume an $n \times n$ matrix $\boldsymbol{A}$.

An eigenvector for $\boldsymbol{A}$ is a non-zero vector $\boldsymbol{v} \neq 0$ for which there is an eigenvalue $\lambda \in \mathbb{R}$ with:

$$\boldsymbol{A} \cdot \boldsymbol{v} = \lambda \cdot \boldsymbol{v}$$

Example

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for $\boldsymbol{P} = \frac{1}{10}\begin{pmatrix} 8 & 1 \\ 2 & 9 \end{pmatrix}$ with eigenvalue $\lambda = 1$.

Two basic results

Lemma

An eigenvector has at most one eigenvalue

Proof: Assume $\boldsymbol{A} \cdot \boldsymbol{v} = \lambda_1 \boldsymbol{v}$ and $\boldsymbol{A} \cdot \boldsymbol{v} = \lambda_2 \boldsymbol{v}$. Then:

$$0 = \boldsymbol{A} \cdot \boldsymbol{v} - \boldsymbol{A} \cdot \boldsymbol{v} = \lambda_1 \boldsymbol{v} - \lambda_2 \boldsymbol{v} = (\lambda_1 - \lambda_2)\boldsymbol{v}$$

Since $\boldsymbol{v} \neq 0$ we must have $\lambda_1 - \lambda_2 = 0$, and thus $\lambda_1 = \lambda_2$. ☺

Lemma

If $\boldsymbol{v}$ is an eigenvector, then so is $a \cdot \boldsymbol{v}$, for each $a \neq 0$.

Proof: If $\boldsymbol{A} \cdot \boldsymbol{v} = \lambda \boldsymbol{v}$, then:

$$\begin{aligned} \boldsymbol{A} \cdot (a\boldsymbol{v}) &= a(\boldsymbol{A} \cdot \boldsymbol{v}) \qquad \text{since matrix application is linear} \\ &= a(\lambda \boldsymbol{v}) = (a\lambda)\boldsymbol{v} = (\lambda a)\boldsymbol{v} = \lambda(a\boldsymbol{v}). \end{aligned}$$

☺

Finding eigenvectors and eigenvalues

- We seek a eigenvector $\boldsymbol{v}$ and eigenvalue $\lambda \in \mathbb{R}$ with $\boldsymbol{A} \cdot \boldsymbol{v} = \lambda \boldsymbol{v}$
- That is: λ and $\boldsymbol{v}$ ($\boldsymbol{v} \neq 0$) such that $(\boldsymbol{A} - \lambda \cdot \boldsymbol{I}) \cdot \boldsymbol{v} = 0$
- Thus, we seek λ for which the system of equations corresponding to the matrix $\boldsymbol{A} - \lambda \cdot \boldsymbol{I}$ has a non-zero solution
- Hence we seek $\lambda \in \mathbb{R}$ for which the matrix $\boldsymbol{A} - \lambda \cdot \boldsymbol{I}$ does not have n pivots in its echelon form
- This means: we seek $\lambda \in \mathbb{R}$ such that $\boldsymbol{A} - \lambda \cdot \boldsymbol{I}$ is not-invertible
- So we need: $\det(\boldsymbol{A} - \lambda \cdot \boldsymbol{I}) = \mathbf{0}$
- This can be seen as an equation, with λ as variable
- This $\det(\boldsymbol{A} - \lambda \cdot \boldsymbol{I})$ is called the characteristic polynomial of the matrix A

Eigenvalue example I

- **Task**: find eigenvalues of matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$
- $\boldsymbol{A} - \lambda \cdot \boldsymbol{I} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{pmatrix}$
- Thus:

$$\begin{aligned}
\det(A - \lambda \cdot I) = 0 \iff & \begin{vmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{vmatrix} = 0 \\
\iff & (1-\lambda)(3-\lambda) - 5 \cdot 3 = 0 \\
\iff & \lambda^2 - 4\lambda - 12 = 0 \\
\iff & (\lambda - 6)(\lambda + 2) = 0 \\
\iff & \lambda = 6 \;\text{ or }\; \lambda = -2.
\end{aligned}$$

Recall: quadratic formula

- Consider a second-degree (quadratic) equation

$$ax^2 + bx + c = 0 \qquad\qquad (\text{for } a \neq 0)$$

- Its solutions are:

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- These solutions coincide (ie. $s_1 = s_2$) if $b^2 - 4ac = 0$
- Real solutions do not exist if $b^2 - 4ac < 0$

  (But "complex number" solutions do exist in this case.)
- [Recall, if s_1 and s_2 are solutions of $ax^2 + bx + c = 0$, then we can write $ax^2 + bx + c = a(x - s_1)(x - s_2)$]

Higher degree polynomial equations

- For third and fourth degree polynomial equations there are (complicated) formulas for the solutions.
- For degree ≥ 5 no such formulas exist (proved by Abel)
- In those cases one can at most use approximations.
- In the examples in this course the solutions will typically be "obvious".

Eigenvalue example II

- **Task**: find eigenvalues of $\boldsymbol{A} = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix}$
- Characteristic polynomial is $\begin{vmatrix} 3-\lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{vmatrix}$

$$
\begin{aligned}
&= (3-\lambda)\begin{vmatrix} -\lambda & 5 \\ -2 & -1-\lambda \end{vmatrix} + 12\begin{vmatrix} -1 & -1 \\ -2 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} -1 & -1 \\ -\lambda & 5 \end{vmatrix} \\
&= (3-\lambda)\Big(\lambda(1+\lambda)+10\Big) + 12\Big(1+\lambda-2\Big) + 4\Big(-5-\lambda\Big) \\
&= (3-\lambda)(\lambda^2+\lambda+10) + 12(\lambda-1) - 20 - 4\lambda \\
&= 3\lambda^2 + 3\lambda + 30 - \lambda^3 - \lambda^2 - 10\lambda + 12\lambda - 12 - 20 - 4\lambda \\
&= -\lambda^3 + 2\lambda^2 + \lambda - 2.
\end{aligned}
$$

Eigenvalue example II (cntd)

- We need to solve $-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$
- We try a few "obvious" values: $\lambda = 1$ YES!
- Reduce from degree 3 to 2, by separating $(\lambda - 1)$ in:

$$\begin{aligned} -\lambda^3 + 2\lambda^2 + \lambda - 2 &= (\lambda - 1)(a\lambda^2 + b\lambda + c) \\ &= a\lambda^3 + (b - a)\lambda^2 + (c - b)\lambda - c \end{aligned}$$

- This works for $a = -1$, $b = 1$, $c = 2$
- Now we use quadratic equation for $-\lambda^2 + \lambda + 2 = 0$
- Solutions: $\lambda = \dfrac{-1 \pm \sqrt{1 + 4 \cdot 2}}{-2} = \dfrac{-1 \pm 3}{-2}$ giving $\lambda = 2, -1$
- All three eigenvalues: $\lambda = 1, \lambda = -1, \lambda = 2$

Getting eigenvectors

- Once we have eigenvalues λ_i for a matrix $\boldsymbol{A}$ we can find corresponding eigenvectors $\boldsymbol{v}_i$, with $\boldsymbol{A} \cdot \boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$
- These $\boldsymbol{v}_i$ appear as the solutions of $(\boldsymbol{A} - \lambda_i \cdot \boldsymbol{I}) \cdot \boldsymbol{v} = 0$
  - We can make a convenient choice, using that scalar multiplications $a \cdot \boldsymbol{v}_i$ are also a solution
- Once λ is known, getting $\boldsymbol{v}$ is just a matter of solving this homogenious system:

$$(\boldsymbol{A} - \lambda \cdot \boldsymbol{I}) \cdot \boldsymbol{v} = \mathbf{0}$$

Eigenvector example I

Recall the eigenvalues $\lambda = -2, \lambda = 6$ for $\boldsymbol{A} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$

$\boxed{\lambda = -2}$ gives matrix $\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{pmatrix} 1+2 & 5 \\ 3 & 3+2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}$

- Corresponding system of equations $\begin{cases} 3x + 5y &= 0 \\ 3x + 5y &= 0 \end{cases}$
- Solution choice $x = -5, y = 3$, so $(-5, 3)$ is eigenvector (of matrix $\boldsymbol{A}$ with eigenvalue $\lambda = -2$)
- Check:

$$\begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -5+15 \\ -15+9 \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \end{pmatrix} = -2 \begin{pmatrix} -5 \\ 3 \end{pmatrix} \checkmark$$

Eigenvector example I (cntd)

$\boxed{\lambda = 6}$ gives matrix $\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{pmatrix} 1-6 & 5 \\ 3 & 3-6 \end{pmatrix} = \begin{pmatrix} -5 & 5 \\ 3 & -3 \end{pmatrix}$

- Corresponding system of equations $\begin{cases} -5x + 5y &= 0 \\ 3x - 3y &= 0 \end{cases}$
- Solution choice $x = 1, y = 1$, so $(1, 1)$ is eigenvector
- Check:

$$\begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+5 \\ 3+3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \checkmark$$

Diagonalisation theorem

Theorem

Let $\boldsymbol{A}$ *be an* $n \times n$ *matrix, represented wrt. the standard basis* $\mathcal{S}$*. Assume* $\boldsymbol{A}$ *has* n *(pairwise) different eigenvalues* $\lambda_1, \ldots, \lambda_n$*, with corresponding eigenvectors* $\mathcal{B} = \{v_1, \ldots, v_n\}$*.* **Then***:*

1. *These* $v_1, \ldots, v_n$ *are linearly independent (and thus a basis)*
2. *There is an invertible basis transformation matrix* $\boldsymbol{T}_{\mathcal{B} \Rightarrow \mathcal{S}}$ *giving a diagonalisation:*

$$\boldsymbol{A} = \boldsymbol{T}_{\mathcal{B} \Rightarrow \mathcal{S}} \cdot \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \cdot \boldsymbol{T}_{\mathcal{S} \Rightarrow \mathcal{B}}$$

Thus, this diagonal matrix is the representation of $\boldsymbol{A}$ *wrt. the eigenvector basis* $\mathcal{B}$*.*

Multiple eigenvalues

- It may happen that a particular eigenvalue occurs multiple times for a matrix
  - eg. the charachterstic polynomial of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\lambda = 1$ twice as a root.
  - for this $\lambda = 1$ there are two independent eigenvectors, namely $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- In general, if an eigenvalue λ occurs n times, then there are at most n independent eigenvectors for this λ
  - linear combinations of eigenvectors with the *same* eigenvalue λ are also eigenvectors with eigenvalue λ
  - they form a subspace of dimension n: the eigenspace of λ.
  - if λ are all distinct, eigenspaces are all 1-dimensional

Applications: data processing

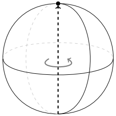
- **Problem:** suppose we have a HUGE matrix, and we want to know approximately what it looks like
- **Solution:** diagonalise it using its basis $\mathcal{B}$ of eigenvectors...then throw away all the little ones:

$$\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}_{\mathcal{B}} \approx \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{\mathcal{B}}$$

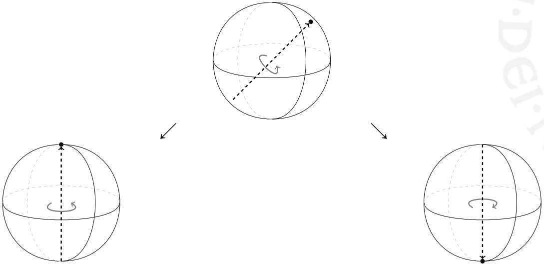
- If there are only a few **big** λ's, and lots of **little** λ's, we get almost the same matrix back
- A more sophisticated technique based on eigenvalues is called principle compent analysis (*very* common in big data applications)

Applications: quantum mechanics/computation

- quantum states can be represented by vectors, and measurements by linear maps, e.g. rotations:

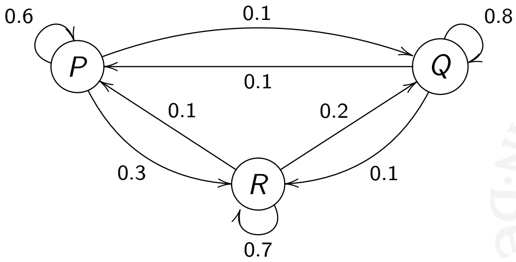


- eigenvalues represent measurement outcomes and eigenvectors represent collapse of the quantum state



Applications: probabilistic transition systems

- In probabilistic transition systems (Markov chains)



- Eigenvalues/vectors are used to make calculations more efficient, and elaborate long-term behaviour

Political swingers re-revisited, part I

- Recall the political transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 8 & 1 \\ 2 & 9 \end{pmatrix}$$

- Eigenvalues λ are obtained via $\det(\boldsymbol{P} - \lambda\boldsymbol{I}) = \boldsymbol{0}$:

$$(\tfrac{8}{10} - \lambda)(\tfrac{9}{10} - \lambda) - \tfrac{1}{10} \cdot \tfrac{2}{10} = \lambda^2 - \tfrac{17}{10}\lambda + \tfrac{7}{10} = 0$$

- Solutions via quadratic equation

$$\begin{aligned} \frac{1}{2}\left(\frac{17}{10} \pm \sqrt{\left(\frac{17}{10}\right)^2 - \frac{28}{10}}\right) &= \frac{1}{2}\left(\frac{17}{10} \pm \sqrt{\frac{289}{100} - \frac{280}{100}}\right) \\ &= \frac{1}{2}\left(\frac{17}{10} \pm \sqrt{\frac{9}{100}}\right) \\ &= \frac{1}{2}\left(\frac{17}{10} \pm \frac{3}{10}\right) \end{aligned}$$

- Hence $\lambda = \frac{1}{2} \cdot \frac{20}{10} = 1$ or $\lambda = \frac{1}{2} \cdot \frac{14}{10} = \frac{7}{10} = 0.7$.

Political swingers re-revisited, part II

- Compute the eigenvectors by plugging eigenvalues $\lambda = 1, 0.7$ into:

$$\begin{pmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

and find a solution to the resulting homogeneous system.

- That is, we need to solve this system, for $\lambda = 1, 0.7$:

$$\begin{cases} (0.8-\lambda)x + 0.1y = 0 \\ 0.2x + (0.9-\lambda)y = 0 \end{cases}$$

Political swingers re-revisited, part II

$\boxed{\lambda = 1}$ solve: $\begin{cases} -0.2x + 0.1y &= 0 \\ 0.2x + -0.1y &= 0 \end{cases}$ giving $(1, 2)$ as eigenvector

- Indeed $\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.8 + 0.2 \\ 0.2 + 1.8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ✓

$\boxed{\lambda = 0.7}$ solve: $\begin{cases} 0.1x + 0.1y &= 0 \\ 0.2x + 0.2y &= 0 \end{cases}$ giving $(1, -1)$ as eigenvector

- Check:

$$\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.8 - 0.1 \\ 0.2 - 0.9 \end{pmatrix} = \begin{pmatrix} 0.7 \\ -0.7 \end{pmatrix} = 0.7 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ ✓

Political swingers re-revisited, part III

- The eigenvalues 1 and 0.7 are different, and indeed the eigenvectors $(1, 2)$ and $(1, -1)$ are independent
- The coordinate-translation $\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}$ from the eigenvector basis $\mathcal{B} = \{(1, 2), (1, -1)\}$ to the standard basis $\mathcal{S} = \{(1, 0), (0, 1)\}$ consists of the eigenvectors:

$$\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

- In the reverse direction:

$$\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} = \left(\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}\right)^{-1} = \frac{1}{-1-2}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Political swingers re-revisited, part IV

We explicitly check the diagonalisation equation:

$$
\begin{aligned}
\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} &= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \tfrac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \\
&= \tfrac{1}{3} \begin{pmatrix} 1 & 0.7 \\ 2 & -0.7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \\
&= \tfrac{1}{3} \begin{pmatrix} 2.4 & 0.3 \\ 0.6 & 2.7 \end{pmatrix} \\
&= \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \\
&= \boldsymbol{P}, \quad \text{the original political transition matrix!}
\end{aligned}
$$

Political swingers re-revisited, part V

This diagonalisation $\boldsymbol{P} = \boldsymbol{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \boldsymbol{T}^{-1}$ is useful for iteration

- $$\begin{aligned} \boldsymbol{P}^2 &= \boldsymbol{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \boldsymbol{T}^{-1} \cdot \boldsymbol{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \boldsymbol{T}^{-1} \\ &= \boldsymbol{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \boldsymbol{T}^{-1} \\ &= \boldsymbol{T} \cdot \begin{pmatrix} 1^2 & 0 \\ 0 & (0.7)^2 \end{pmatrix} \cdot \boldsymbol{T}^{-1} \end{aligned}$$
- $$\boldsymbol{P}^n = \boldsymbol{T} \cdot \begin{pmatrix} (1)^n & 0 \\ 0 & (0.7)^n \end{pmatrix} \cdot \boldsymbol{T}^{-1}$$
- $$\begin{aligned} \lim_{n\to\infty} \boldsymbol{P}^n &= \boldsymbol{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \boldsymbol{T}^{-1} \qquad \text{since } \lim_{n\to\infty} (0.7)^n = 0 \\ &= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \tfrac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \tfrac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \end{aligned}$$

Political swingers re-revisited, part VI

- We now have a fairly easy way to compute $\boldsymbol{P}^n \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix}$
- ...and we can see that in the limit it goes to:

$$\begin{aligned} \lim_{n\to\infty} \boldsymbol{P}^n \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix} &= \tfrac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix} \\ &= \tfrac{1}{3}\begin{pmatrix} 250 \\ 500 \end{pmatrix} = \begin{pmatrix} 83\frac{1}{3} \\ 166\frac{2}{3} \end{pmatrix} \end{aligned}$$

(This was already suggested earlier, but now we can calculate it!)

Note: we used this useful limit result

$$\lim_{n\to\infty} a^n = 0, \qquad \text{for } |a| < 1.$$

Rental car returns, part I

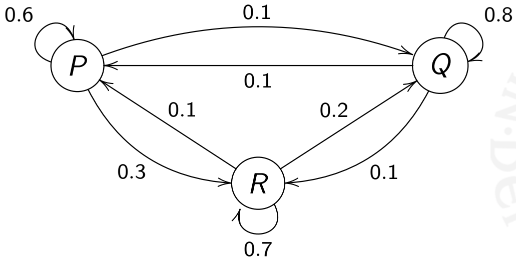
- Assume a car rental company with three locations, for picking up and returning cars, written as P, Q, R
- The weekly distribution history shows:

Location P	60% stay at P	10% go to Q	30% go to R
Location Q	10% go to P	80% stay at Q	10% go to R
Location R	10% go to P	20% go to Q	70% stay at R

Rental car returns, part II

Two possible representations of these return distributions

1. As probabilistic transition system



Rental car returns, part III

2. As a transition matrix

$$\boldsymbol{C} = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.2 \\ 0.3 & 0.1 & 0.7 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix}$$

This matrix $\boldsymbol{C}$ is called a Stochastic matrix or Markov chain:

- all entries are in the unit interval $[0, 1]$ of probabilities
- in each column, the entries add up to 1

Rental car returns, part IV

Task:

- Start from the following division of cars:

$$P = Q = R = 200 \qquad \text{ie.} \qquad \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix}$$

- Determine the division of cars after two weeks
- Determine the equilibrium division, reached as the number of weeks goes to infinity

Rental car returns, part V

- After one week we have:

$$\boldsymbol{C} \cdot \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix}$$
$$= \frac{1}{10} \begin{pmatrix} 1200 + 200 + 200 \\ 200 + 1600 + 400 \\ 600 + 200 + 1400 \end{pmatrix} = \begin{pmatrix} 160 \\ 220 \\ 220 \end{pmatrix}$$

- After two weeks we have:

$$\boldsymbol{C} \cdot \begin{pmatrix} 160 \\ 220 \\ 220 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 160 \\ 220 \\ 220 \end{pmatrix}$$
$$= \frac{1}{10} \begin{pmatrix} 960 + 220 + 220 \\ 160 + 1760 + 440 \\ 480 + 220 + 1540 \end{pmatrix} = \begin{pmatrix} 140 \\ 236 \\ 224 \end{pmatrix}$$

Rental car returns, part VI

- For the equilibrium we first compute eigenvalues and eigenvectors of the transition matrix $\boldsymbol{C}$
- The characteristic polynomial is:

$$\begin{aligned}
&\begin{vmatrix} 0.6-\lambda & 0.1 & 0.1 \\ 0.1 & 0.8-\lambda & 0.2 \\ 0.3 & 0.1 & 0.7-\lambda \end{vmatrix} \\
&= \tfrac{1}{1000}\Big[(6-10\lambda)\Big((8-10\lambda)(7-10\lambda)-2\Big) \\
&\qquad -1\Big((7-10\lambda)-1\Big)+3\Big(2-1(8-10\lambda)\Big)\Big] \\
&= \cdots \\
&= \tfrac{1}{1000}\Big[-1000\lambda^3+2100\lambda^2-1400\lambda+300\Big] \\
&= -\lambda^3+2.1\lambda^2-1.4\lambda+0.3.
\end{aligned}$$

Rental car returns, part VII

- Next we solve $-\lambda^3 + 2.1\lambda^2 - 1.4\lambda + 0.3 = 0$.
- We seek a trivial solution; again $\lambda = 1$ works!
- Now we can write

$$-\lambda^3 + 2.1\lambda^2 - 1.4\lambda + 0.3 \;=\; (\lambda - 1)(-\lambda^2 + 1.1\lambda - 0.3)$$

- We can apply the quadratic formula to the second part:

$$\begin{aligned} \frac{-1.1 \pm \sqrt{(1.1)^2 - 4 \cdot 0.3}}{-2} &= \frac{-1.1 \pm \sqrt{1.21 - 1.2}}{-2} \\ &= \frac{-1.1 \pm \sqrt{0.01}}{-2} \\ &= \frac{-1.1 \pm 0.1}{-2} \end{aligned}$$

- This yields additional eigenvalues: $\lambda = 0.5$ and $\lambda = 0.6$.

Rental car returns, part VIII

$\boxed{\lambda = 1}$ has eigenvector $(4, 9, 7)$; indeed:

$$\boldsymbol{C} \cdot \begin{pmatrix} 4 \\ 9 \\ 7 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 9 \\ 7 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 24+9+7 \\ 4+72+14 \\ 12+9+49 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 9 \\ 7 \end{pmatrix}$$

$\boxed{\lambda = 0.6}$ has eigenvector $(0, -1, 1)$:

$$\boldsymbol{C} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -1+1 \\ -8+2 \\ -1+7 \end{pmatrix} = 0.6 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\boxed{\lambda = 0.5}$ has eigenvector $(-1, -1, 2)$:

$$\boldsymbol{C} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -6-1+2 \\ -1-8+4 \\ -3-1+14 \end{pmatrix} = 0.5 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

Rental car returns, part IX

- Now: eigenvector basis $\mathcal{B} = \{(4,9,7),(0,-1,1),(-1,-1,2)\}$ and standard basis as $\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1)\}$.
- Then we can do change-of-coordinates back-and-forth:

$$\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} = \begin{pmatrix} 4 & 0 & -1 \\ 9 & -1 & -1 \\ 7 & 1 & 2 \end{pmatrix} \qquad \boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} = \frac{1}{20}\begin{pmatrix} 1 & 1 & 1 \\ 25 & -15 & 5 \\ -16 & 4 & 4 \end{pmatrix}$$

- These translation matrices yield a diagonalisation:

$$\boldsymbol{C} = \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \cdot \boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}}$$

Rental car returns, part X

- Thus:

$$\lim_{n\to\infty} \boldsymbol{C}^n = \lim_{n\to\infty} \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \cdot \begin{pmatrix} 1^n & 0 & 0 \\ 0 & (0.6)^n & 0 \\ 0 & 0 & (0.5)^n \end{pmatrix} \cdot \boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}}$$

$$= \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} = \frac{1}{20} \begin{pmatrix} 4 & 4 & 4 \\ 9 & 9 & 9 \\ 7 & 7 & 7 \end{pmatrix}$$

- Finally, the equilibrium starting from $P = Q = R = 200$ is:

$$\frac{1}{20} \begin{pmatrix} 4 & 4 & 4 \\ 9 & 9 & 9 \\ 7 & 7 & 7 \end{pmatrix} \cdot \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix} = \begin{pmatrix} 120 \\ 270 \\ 210 \end{pmatrix}.$$