Matrix Calculations: Solutions of Systems of Linear Equations

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Outline

Solutions

When we look for solutions to a system, there are 3 possibilities:

1. A system of equations has a *single, unique solution*, e.g.

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1 \end{aligned}$$

(unique solution: $x_1 = 2, x_2 = 1$)

2. A system has *many solutions*, e.g.

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ -2x_1 + 4x_2 &= -2 \end{aligned}$$

(we have a solution whenever: $x_1 = 1 + 2x_2$)

3. A system has *no solutions*.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ 6x_1 - 4x_2 &= 6 \end{aligned}$$

(the transformation $E_2 := E_2 - 2E_1$ yields $0 = 4$.)

Solutions, geometrically

Consider systems of only two variables x, y. A linear equation $ax + by = c$ then describes a line in the plane.

For 2 such equations/lines, there are three possibilities:

1. the lines intersect in a unique point, which is the solution to both equations
2. the lines are parallel, in which case there are no joint solutions
3. the lines coincide, giving many joint solutions.

Echelon form

We can tell the difference in these 3 cases by writing the augmented matrix and tranforming to Echelon form.

Recall: A matrix is in **Echelon form** if:

1. All of the rows with pivots occur before zero rows, and
2. Pivots always occur to the right of previous pivots

$$\left(\begin{array}{cccc|c} \boxed{3} & 2 & 5 & -5 & 1 \\ 0 & 0 & \boxed{2} & 1 & -2 \\ 0 & 0 & 0 & \boxed{-2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \checkmark$$

(In)consistent systems

Definition

A system of equations is consistent (*oplosbaar*) if it has one or more solutions. Otherwise, when there are no solutions, the system is called inconsistent

Thus, for a system of equations:

nr. of solutions	**terminology**
0	inconsistent
≥ 1 (one or many)	consistent

Inconsistency and echelon forms

Theorem

A system of equations is inconsistent (non-solvable) if and only if in the echelon form of its augmented matrix there is a row with:

- *only zeros before the bar* $|$
- *a non-zero after the bar* $|$,

as in: $0\ 0\ \cdots\ 0 \mid c$, *where* $c \neq 0$.

Example

$$\begin{array}{rcl} 3x_1 - 2x_2 & = & 1 \\ 6x_1 - 4x_2 & = & 6 \end{array} \quad \text{gives} \quad \left(\begin{array}{cc|c} 3 & -2 & 1 \\ 6 & -4 & 6 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} 3 & -2 & 1 \\ 0 & 0 & 4 \end{array}\right)$$

(using the transformation $R_2 := R_2 - 2R_1$)

Unique solutions

Theorem

A system of equations in n variables has a unique solution if and only if in its Echelon form there are n pivots.

Proof. (n pivots $\implies$ unique soln., on board)

In summary: A system with n variables has an augmented matrix with n columns before the line. Its Echelon form has n pivots, so there must be *exactly* one pivot in each column. The last pivot uniquely fixes x_n. Then, since x_n is fixed, the second to last pivot uniquely fixes x_{n-1} and so on.

Unique solutions: earlier example

equations | **matrix**

$$\begin{aligned} 2x_2 + x_3 &= -2 \\ 3x_1 + 5x_2 - 5x_3 &= 1 \\ 2x_1 + 4x_2 - 2x_3 &= 2 \end{aligned} \qquad \left(\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 3 & 5 & -5 & 1 \\ 2 & 4 & -2 & 2 \end{array}\right)$$

After various transformations leads to

$$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Echelon form

There are 3 variables and 3 pivots, so there is one unique solution.

Unique solutions

So, when there are n pivots, there is 1 solution, and life is good.

Question: What if there are more solutions? Can we describe them in a generic way?

A new tool: vectors

- A vector is a list of numbers.
- We can write it like this: $(x_1, x_2, \ldots, x_n)$
- ...or as a matrix with just one column:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

(which is sometimes called a 'column vector').

A new tool: vectors

- Vectors are useful for lots of stuff. In this lecture, we'll use them to hold solutions.
- Since variable names don't matter, we can write this:

$$x_1 := 2 \quad x_2 := -1 \quad x_3 := 0$$

- ...more compactly as this:

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

- ...or even more compactly as this: $(2, -1, 0)$.

Linear combinations

- We can multiply a vector by a number to get a new vector:

$$c \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}$$

This is called **scalar multiplication**.

- ...and we can add vectors together:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

as long as the are the same length.

Linear combinations

Mixing these two things together gives us a **linear combination** of vectors:

$$c \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + d \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \ldots = \begin{pmatrix} cx_1 + dy_1 + \ldots \\ cx_2 + dy_2 + \ldots \\ \vdots \\ cx_n + dy_n + \ldots \end{pmatrix}$$

A set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ is called **linearly independent** if no vector can be written as a linear combination of the others.

Linear independence

- These vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

are **NOT** linearly independent, because $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

- These vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are **NOT** linearly independent, because $\mathbf{v}_1 = \mathbf{v}_2 + 2 \cdot \mathbf{v}_3$.

Linear independence

- These vectors:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  are linearly *independent*. There is no way to write any of them in terms of each other.

- These vectors:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

  are linearly *independent*. There is no way to write any of them in terms of each other.

Linear independence

- These vectors:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

  are... ???
- 'Eyeballing' vectors works sometimes, but we need a better way of checking linear independence!

Checking linear independence

Theorem

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if and only if, for all numbers $a_1, \ldots, a_n \in \mathbb{R}$ one has:

$$a_1 \cdot \mathbf{v}_1 + \cdots + a_n \cdot \mathbf{v}_n = \mathbf{0} \;\; \textit{implies} \;\; a_1 = a_2 = \cdots = a_n = 0$$

Example

The 3 vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ are linearly independent, since if

$$a_1 \cdot (1, 0, 0) + a_2 \cdot (0, 1, 0) + a_3 \cdot (0, 0, 1) = (0, 0, 0)$$

then, using the computation from the previous slide,

$$(a_1, a_2, a_3) = (0, 0, 0), \quad \text{so that} \quad a_1 = a_2 = a_3 = 0$$

Checking linear independence

Theorem

Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ *are linearly independent if and only if, for all numbers* $a_1, \ldots, a_n \in \mathbb{R}$ *one has:*

$$a_1 \cdot \boldsymbol{v}_1 + \cdots + a_n \cdot \boldsymbol{v}_n = \mathbf{0} \;\; \textit{implies} \;\; a_1 = a_2 = \cdots = a_n = 0$$

Proof. Another way to say the theorem is $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly dependent if and only if:

$$a_1 \cdot \boldsymbol{v}_1 + a_2 \cdot \boldsymbol{v}_2 + \cdots + a_n \cdot \boldsymbol{v}_n = \mathbf{0}$$

where some a_j are non-zero. If this is true and $a_1 \neq 0$, then:

$$\boldsymbol{v}_1 = (-a_2/a_1) \cdot \boldsymbol{v}_2 + \ldots + (-a_n/a_1) \cdot \boldsymbol{v}_n$$

The vectors are dependent (also works for any other non-zero a_j).
Exercise: prove the other direction.

Proving (in)dependence via equation solving I

- Investigate (in)dependence of $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}2\\-1\\4\end{pmatrix}$, and $\begin{pmatrix}0\\5\\2\end{pmatrix}$
- Thus we ask: are there any non-zero $a_1, a_2, a_3 \in \mathbb{R}$ with:

$$a_1 \begin{pmatrix}1\\2\\3\end{pmatrix} + a_2 \begin{pmatrix}2\\-1\\4\end{pmatrix} + a_3 \begin{pmatrix}0\\5\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

- If there is a non-zero solution, the vectors are dependent, and if $a_1 = a_2 = a_3 = 0$ is the only solution, they are independent

Proving (in)dependence via equation solving II

- Our question involves the systems of equations / matrix:

$$\begin{cases} a_1 + 2a_2 = 0 \\ 2a_1 - a_2 + 5a_3 = 0 \\ 3a_1 + 4a_2 + 2a_3 = 0 \end{cases} \quad \text{corresponding to} \quad \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(in Echelon form)

- This has only 2 pivots, so multiple solutions. In particular, it has non-zero solutions, for example: $a_1 = 2, a_2 = -1, a_3 = -1$

  (compute and check for yourself!)
- Thus the original vectors are dependent. Explicitly:

$$2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Proving (in)dependence via equation solving III

- Same (in)dependence question for: $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$
- With corresponding matrix:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -3 & 1 & -2 \end{pmatrix} \quad \text{reducing to} \quad \begin{pmatrix} 5 & 0 & -1 \\ 0 & 5 & -3 \\ 0 & 0 & -4 \end{pmatrix}$$

- Thus the only solution is $a_1 = a_2 = a_3 = 0$. The vectors are independent!

Linear independence: summary

To check linear independence of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

1. Write the vectors as the *columns of a matrix*
2. Convert to Echelon form
3. Count the pivots
   - (# pivots) = (# columns) means independent
   - (# pivots) < (# columns) means dependent
4. Non-zero solutions show linear dependence explicitly, e.g.

$$\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \quad \Longrightarrow \quad \mathbf{v}_1 = 2\mathbf{v}_2 - \mathbf{v}_3$$

General solutions

The Goal:

- Describe the space of solutions of a system of equations.
- In general, there can be infinitely many solutions, but only a few are actually 'different enough' to matter. These are called basic solutions.
- Using the basic solutions, we can write down a formula which gives us any solution: the general solution.

Example (General solution for one equation)

$$2x_1 - x_2 \;=\; 3 \quad \text{gives} \quad x_2 = 2x_1 - 3$$

So a general solution (for any c) is:

$$x_1 := c \qquad x_2 := 2c - 3$$

Linear combinations of solutions

- It is **not** the case in general that linear combinations of solutions give solutions. For example, consider:

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_2 + x_4 = 2 \end{cases} \quad \leftrightarrow \quad \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right)$$

- This has as solutions:

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \text{ but } \textbf{not } \boldsymbol{v}_1 + \boldsymbol{v}_2 = \begin{pmatrix} -3 \\ 3 \\ -3 \\ 1 \end{pmatrix}, 3 \cdot \boldsymbol{v}_1, \ldots$$

- The problem is this system of equations is not homogeneous, because the the **2** on the right-hand-side (RHS) of the second equation.

Homogeneous systems of equations

Definition

A system of equations is called homogeneous if it has zeros on the RHS of every equation. Otherwise it is called non-homogeneous.

- We can always squash a non-homogeneous system to a homogeneous one:

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 3 & 5 & -5 & 1 \\ 0 & 0 & -2 & 2 \end{array}\right) \rightsquigarrow \begin{pmatrix} 0 & 2 & 1 \\ 3 & 5 & -5 \\ 0 & 0 & -2 \end{pmatrix}$$

- The solutions will change!
- ...but they are still related. We'll see how that works soon.

Zero solution, in homogeneous case

Lemma

Each homogeneous equation has $(0, \ldots, 0)$ as solution.

Proof: A homogeneous system looks like this

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Consider the equation at row i:

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0$$

Clearly it has as solution $x_1 = x_2 = \cdots = x_n = 0$.
This holds for each row i.

Linear combinations of solutions

Theorem

The set of solutions of a homogeneous system is closed under linear combinations (i.e. addition and scalar multiplication of vectors).

...which means:

- if $(s_1, s_2, \ldots, s_n)$ and $(t_1, t_2, \ldots, t_n)$ are solutions, then so is: $(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$, and
- if $(s_1, s_2, \ldots, s_n)$ is a solution, then so is $(c \cdot s_1, c \cdot s_2, \ldots, c \cdot s_n)$

Example

- Consider the homogeneous system $\begin{cases} 3x_1 + 2x_2 - x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}$
- A solution is $x_1 = 1, x_2 = 1, x_3 = 5$, written as vector $(x_1, x_2, x_3) = (1, 1, 5)$
- Another solution is $(2, 2, 10)$
- Addition yields another solution:

$$(1, 1, 5) + (2, 2, 10) = (1 + 2, 1 + 2, 10 + 5) = (3, 3, 15).$$

- Scalar multiplication also gives solutions:

$$\begin{aligned} -1 \cdot (1, 1, 5) &= (-1 \cdot 1, -1 \cdot 1, -1 \cdot 5) = (-1, -1, -5) \\ 100 \cdot (2, 2, 10) &= (100 \cdot 2, 100 \cdot 2, 100 \cdot 10) = (200, 200, 1000) \\ c \cdot (1, 1, 5) &= (c \cdot 1, c \cdot 1, c \cdot 5) = (c, c, 5c) \\ &\qquad\qquad \text{(is a solution for every } c\text{)} \end{aligned}$$

Proof of closure under addition

- Consider an equation $a_1x_1 + \cdots + a_nx_n = 0$
- Assume two solutions $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$
- Then $(s_1 + t_1, \ldots, s_n + t_n)$ is also a solution since:

$$\begin{aligned}
& a_1(s_1 + t_1) + \cdots + a_n(s_n + t_n) \\
&= \Big(a_1s_1 + a_1t_1\Big) + \cdots + \Big(a_ns_n + a_nt_n\Big) \\
&= \Big(a_1s_1 + \cdots + a_ns_n\Big) + \Big(a_1t_1 + \cdots + a_nt_n\Big) \\
&= 0 + 0 \qquad \text{since the } s_i \text{ and } t_i \text{ are solutions} \\
&= 0.
\end{aligned}$$

- Exercise: do a similar proof of closure under scalar multiplication

General solution of a homogeneous system

Theorem

Every solution to a homogeneous system arises from a general solution of the form:

$$(s_1, \ldots, s_n) = c_1(v_{11}, \ldots, v_{1n}) + \cdots + c_k(v_{k1}, \ldots, v_{kn})$$

for some numbers $c_1, \ldots, c_k \in \mathbb{R}$.

We call this a parametrization of our solution space. It means:

1. There is a fixed set of vectors (called basic solutions):

$$\boldsymbol{v}_1 = (v_{11}, \ldots, v_{1n}), \quad \ldots, \quad \boldsymbol{v}_k = (v_{k1}, \ldots, v_{kn})$$

2. such that every solution $\boldsymbol{s}$ is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$.
3. That is, there exist $c_1, \ldots, c_k \in \mathbb{R}$ such that

$$\boldsymbol{s} = c_1 \boldsymbol{v}_1 + \ldots + c_k \boldsymbol{v}_k$$

Basic solutions of a homogeneous system

Theorem

Suppose a homogeneous system of equations in n variables has $p \leq n$ pivots. Then there are $n - p$ basic solutions $\mathbf{v}_1, \ldots, \mathbf{v}_{n-p}$.

This means that the general solution $\mathbf{s}$ can be written as a parametrization:

$$\mathbf{s} = c_1\mathbf{v}_1 + \cdots c_{n-p}\mathbf{v}_{n-p}.$$

Moreover, for any solution $\mathbf{s}$, the scalars $c_1, \ldots, c_{n-p}$ are unique.

$$(p = n) \;\Leftrightarrow\; (\text{no basic solns.}) \;\Leftrightarrow\; (\mathbf{0} \text{ is the unique soln.})$$

Finding basic solutions

- We have two kinds of variables, pivot variables and non-pivot, or free variables, depending on whether their column has a pivot:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 0 & 1 & 4 & 1 \\ 0 & 0 & \boxed{1} & 2 & 0 \end{pmatrix}$$

- The Echelon form lets us (easily) write pivot variables in terms of non-pivot variables, e.g.:

$$\begin{cases} x_1 = -x_3 - 4x_4 - x_5 \\ x_3 = -2x_4 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_4 - x_5 \\ x_3 = -2x_4 \end{cases}$$

- We can find a (non-zero) basic solution by setting exactly **one** free variable to 1 and the rest to 0.

Finding basic solutions

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 0 & 1 & 4 & 1 \\ 0 & 0 & \boxed{1} & 2 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = -2x_4 - x_5 \\ x_3 = -2x_4 \end{cases}$$

5 variables and 2 pivots gives us $5 - 2 = 3$ basic solutions:

$$\begin{array}{ccc} x_2 := 1 & x_2 := 0 & x_2 := 0 \\ x_4 := 0 & x_4 := 1 & x_4 := 0 \\ x_5 := 0 & x_5 := 0 & x_5 := 1 \end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_4 - x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

General Solution

Now, any solution to the system is obtainable as a linear combination of basic solutions:

$$x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2x_4 - x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \end{pmatrix}$$

Picking solutions this way guarantees **linear independence**.

Finding basic solutions: technique 2

- Keep all columns with a pivot,
- One-by-one, keep only the *i*-th non-pivot columns (while removing the others), and find a (non-zero) solution
- (this is like setting all the other free variables to zero)
- Add 0's to each solution to account for the columns (i.e. free variables) we removed

General solution and basic solutions, example

- For the matrix: $\begin{pmatrix} \boxed{1} & 1 & 0 & 4 \\ 0 & 0 & \boxed{2} & 2 \end{pmatrix}$
- There are 4 columns (variables) and 2 pivots, so $4 - 2 = 2$ basic solutions
- First keep only the first non-pivot column:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{with chosen solution} \quad (x_1, x_2, x_3) = (1, -1, 0)$$

- Next keep only the second non-pivot column:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \end{pmatrix} \quad \text{with chosen solution} \quad (x_1, x_3, x_4) = (4, 1, -1)$$

- The general 4-variable solution is now obtained as:

$$c_1 \cdot (1, -1, 0, 0) + c_2 \cdot (4, 0, 1, -1)$$

General solutions example, check

We double-check that any vector:

$$\begin{aligned} & c_1 \cdot (4, 0, 1, -1) + c_2 \cdot (1, -1, 0, 0) \\ = \; & (4 \cdot c_1, 0, 1 \cdot c_1, -1 \cdot c_1) + (1 \cdot c_2, -1 \cdot c_2, 0, 0) \\ = \; & (4c_1 + c_2, -c_2, c_1, -c_1) \end{aligned}$$

gives a solution of:

$$\begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 0 & 2 & 2 \end{pmatrix} \quad \text{i.e. of} \quad \begin{cases} x_1 + x_2 + 4x_4 & = \; 0 \\ 2x_3 + 2x_4 & = \; 0 \end{cases}$$

Just fill in $x_1 = 4c_1 + c_2, \; x_2 = -c_2, \; x_3 = c_1, \; x_4 = -c_1$

$$\begin{aligned} (4c_1 + c_2) - c_2 + 4 \cdot -c_1 & = \; 0 \\ 2c_1 - 2c_1 & = \; 0 \end{aligned} \qquad \checkmark$$

Summary of homogeneous systems

Given a homogeneous system in n variables:

- A basic solution is a non-zero solution of the system.
- If there are n pivots in its echelon form, there is no basic solution, so only $\mathbf{0} = (0, \ldots, 0)$ is a solution.
- Basic solutions are not unique. For instance, if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ give basic solutions, so do $\boldsymbol{v}_1 + \boldsymbol{v}_2$, $\boldsymbol{v}_1 - \boldsymbol{v}_2$, and any other linear combination.
- If there are $p < n$ pivots in its Echelon form, it has $n - p$ linearly independent basic solutions.

Non-homogeneous case: subtracting solutions

Theorem

The difference of two solutions of a **non-homogeneous** *system is a solution for the associated* **homogeneous** *system.*

More explicitly: given two solutions $(s_1, \ldots, s_n)$ *and* $(t_1, \ldots, t_n)$ *of a* **non-homogeneous** *system, the difference* $(s_1 - t_1, \ldots, s_n - t_n)$ *is a solution of the associated* **homogeneous** *system.*

Proof: Let $a_1x_1 + \cdots + a_nx_n = b$ be the equation. Then:

$$\begin{aligned}
& a_1(s_1 - t_1) + \cdots + a_n(s_n - t_n) \\
&= \big(a_1s_1 - a_1t_1\big) + \cdots + \big(a_ns_n - a_nt_n\big) \\
&= \big(a_1s_1 + \cdots + a_ns_n\big) - \big(a_1t_1 + \cdots + a_nt_n\big) \\
&= b - b \qquad \text{since the } s_i \text{ and } t_i \text{ are solutions} \\
&= 0.
\end{aligned}$$

General solution for non-homogeneous systems

Theorem

Assume a non-homogeneous system has a solution given by the vector $\boldsymbol{p}$, which we call a particular solution.
Then any other solution $\boldsymbol{s}$ of the non-homogeneous system can be written as

$$\boldsymbol{s} = \boldsymbol{p} + \boldsymbol{h}$$

where $\boldsymbol{h}$ is a solution of the associated homogeneous system.

Proof: Let $\boldsymbol{s}$ be a solution of the non-homogeneous system. Then $\boldsymbol{h} = \boldsymbol{s} - \boldsymbol{p}$ is a solution of the associated homogeneous system. Hence we can write $\boldsymbol{s}$ as $\boldsymbol{p} + \boldsymbol{h}$, for $\boldsymbol{h}$ some solution of the associated homogeneous system. ☺

Example: solutions of a non-homogeneous system

- Consider the non-homogeneous system $\begin{cases} x + y + 2z = 9 \\ y - 3z = 4 \end{cases}$
- with solutions: $(0, 7, 1)$ and $(5, 4, 0)$
- We can write $(0, 7, 1)$ as: $(5, 4, 0) + (-5, 3, 1)$
- where:
  - $\boldsymbol{p} = (5, 4, 0)$ is a particular solution (of the original system)
  - $(-5, 3, 1)$ is a solution of the associated homogeneous system: $\begin{cases} x + y + 2z = 0 \\ y - 3z = 0 \end{cases}$
- Similarly, $(10, 1, -1)$ is a solution of the non-homogeneous system and

$$(10, 1, -1) \;=\; (5, 4, 0) + (5, -3, -1)$$

- where:
  - $(5, -3, -1)$ is a solution of the associated homogeneous system.

General solution for non-homogeneous systems, concretely

Theorem

The general solution of a non-homogeneous system of equations in n variables is given by a parametrization as follows:

$$(s_1, \ldots, s_n) = (p_1, \ldots, p_n) + c_1(v_{11}, \ldots, v_{1n}) + \cdots c_k(v_{k1}, \ldots, v_{kn})$$

for $c_1, \ldots, c_k \in \mathbb{R}$,
where

- $(p_1, \ldots, p_n)$ *is a particular solution*
- $(v_{11}, \ldots, v_{1n}), \ldots, (v_{k1}, \ldots, v_{kn})$ *are basic solutions of the associated homogeneous system.*
- *So* $c_1(v_{11}, \ldots, v_{1n}) + \cdots + c_k(v_{k1}, \ldots, v_{kn})$ *is a general solution for the associated homogeneous system.*

Elaborated example, part I

- Consider the non-homogeneous system of equations given by the augmented matrix in echelon form:

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

- It has 5 variables, 3 pivots, and thus $5 - 3 = 2$ basic solutions
- To find a particular solution, remove the non-pivot columns, and (uniquely!) solve the resulting system:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

- This has $(10, -11, 4)$ as solution; the orginal 5-variable system then has particular solution $(10, 0, -11, 0, 4)$.

Elaborated example, part II

- Consider the associated homogeneous system of equations:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

- The two basic solutions are found by removing each of the two non-pivot columns separately, and finding solutions:

$$\begin{array}{cccc} x_1 & x_3 & x_4 & x_5 \end{array} \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_5 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

- We find: $(1, -2, 1, 0)$ and $(-1, 1, 0, 0)$. Adding zeros for missing columns gives: $(1, 0, -2, 1, 0)$ and $(-1, 1, 0, 0, 0)$.

Elaborated example, part III

Wrapping up: all solutions of the system

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

are of the form:

$$\underbrace{(10, 0, -11, 0, 4)}_{\text{particular sol.}} + \underbrace{c_1(1, 0, -2, 1, 0) + c_2(-1, 1, 0, 0, 0)}_{\text{two basic solutions}}.$$

This is the general solution of the non-homogeneous system.