Matrix Calculations: Linear maps, bases, and matrices

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Outline

From last time

- Vector spaces $V, W, \ldots$ are special kinds of sets whose elements are called *vectors*.
- Vectors can be added together, or multiplied by a real number, For $\boldsymbol{v}, \boldsymbol{w} \in V$, $a \in \mathbb{R}$:

$$\boldsymbol{v} + \boldsymbol{w} \in V \qquad\qquad a \cdot \boldsymbol{v} \in V$$

- The simplest examples are:

$$\mathbb{R}^n := \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{R}\}$$

Maps between vector spaces

We can send vectors $\boldsymbol{v} \in V$ in one vector space to other vectors $\boldsymbol{w} \in W$ in another (or possibly the same) vector space?

V, W are vector spaces, so they are sets with extra stuff (namely: $+$, $\cdot$, $\mathbf{0}$).

A common theme in mathematics: study functions $f : V \to W$ which preserve the extra stuff.

Functions

- A function f is an operation that sends elements of one set X to another set Y.
  - in that case we write $f\colon X \to Y$ or sometimes $X \xrightarrow{f} Y$
  - this f sends $x \in X$ to $f(x) \in Y$
  - X is called the domain and Y the codomain of the function f
- Example. $f(n) = \frac{1}{n+1}$ can be seen as function $\mathbb{N} \to \mathbb{Q}$, that is from the *natural* numbers $\mathbb{N}$ to the *rational* numbers $\mathbb{Q}$
- On each set X there is the identity function $\mathrm{id}\colon X \to X$ that does nothing: $\mathrm{id}(x) = x$.
- Also one can compose 2 functions $X \xrightarrow{f} Y \xrightarrow{g} Z$ to a function:

$$g \circ f\colon X \longrightarrow Z \quad \text{given by} \quad (g \circ f)(x) = g(f(x))$$

Linear maps

A linear map is a function that preserves the extra stuff in a vector space:

Definition

Let V, W be two vector spaces, and $f\colon V \to W$ a map between them; f is called linear if it preserves both:

- addition: for all $\boldsymbol{v}, \boldsymbol{v}' \in V$,

$$f(\underbrace{\boldsymbol{v} + \boldsymbol{v}'}_{\text{in } V}) = \underbrace{f(\boldsymbol{v}) + f(\boldsymbol{v}')}_{\text{in } W}$$

- scalar multiplication: for each $\boldsymbol{v} \in V$ and $a \in \mathbb{R}$,

$$f(\underbrace{a \cdot \boldsymbol{v}}_{\text{in } V}) = \underbrace{a \cdot f(\boldsymbol{v})}_{\text{in } W}$$

Linear maps preserve zero and minus

Theorem

Each linear map $f\colon V \to W$ preserves:

- *zero:* $f(\mathbf{0}) = \mathbf{0}$.
- *minus:* $f(-\boldsymbol{v}) = -f(\boldsymbol{v})$

Proof:

$$\begin{aligned} f(\mathbf{0}) &= f(0 \cdot \mathbf{0}) \\ &= 0 \cdot f(\mathbf{0}) \\ &= \mathbf{0} \end{aligned} \qquad \begin{aligned} f(-\boldsymbol{v}) &= f((-1) \cdot \boldsymbol{v}) \\ &= (-1) \cdot f(\boldsymbol{v}) \\ &= -f(\boldsymbol{v}) \end{aligned}$$

Linear map examples I

$\mathbb{R}$ is a vector space. Let's consider maps $f : \mathbb{R} \to \mathbb{R}$.
Most of them are *not linear*, like, for instance:

- $f(x) = 1 + x$, since $f(0) = 1 \neq 0$
- $f(x) = x^2$, since $f(-1) = 1 = f(1) \neq -f(1)$.

So: linear maps $\mathbb{R} \to \mathbb{R}$ can only be very simple.

Theorem

Each linear map $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = c \cdot x$, for some $c \in \mathbb{R}$.

Proof:

$$f(x) = f(x \cdot 1) = x \cdot f(1) = f(1) \cdot x = c \cdot x, \quad \text{for } c = f(1).$$

Linear map examples II

Linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ start to get more interesting:

$$s(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = \begin{pmatrix} av_1 \\ v_2 \end{pmatrix} \qquad\qquad t(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = \begin{pmatrix} v_1 \\ bv_2 \end{pmatrix}$$

...these scale a vector on the X- and Y-axis.

We can show these are linear by checking the two linearity equations:

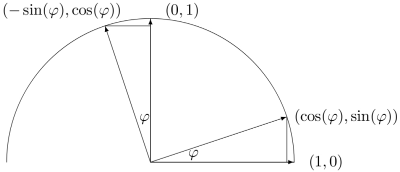
$$f(\boldsymbol{v} + \boldsymbol{w}) = f(\boldsymbol{v}) + f(\boldsymbol{w}) \qquad\qquad f(a \cdot \boldsymbol{v}) = a \cdot f(\boldsymbol{v})$$

Linear map examples III

Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = \begin{pmatrix} v_1 \cos(\varphi) - v_2 \sin(\varphi) \\ v_1 \sin(\varphi) + v_2 \cos(\varphi) \end{pmatrix}$$

This map describes rotation in the plane, with angle φ:



We can also check linearity equations.

Linear map examples IV

These extend naturally to 3D, i.e. linear maps $\mathbb{R}^3 \to \mathbb{R}^3$:

$$sx\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} av_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad sy\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} v_1 \\ bv_2 \\ v_3 \end{pmatrix} \qquad sz\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} v_1 \\ v_2 \\ cv_3 \end{pmatrix}$$

Q: How do we do rotation?

A: Keep one coordinate fixed (axis of rotation), and 2D rotate the other two, e.g.

$$rz\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} v_1 \cos(\varphi) - v_2 \sin(\varphi) \\ v_1 \sin(\varphi) + v_2 \cos(\varphi) \\ v_3 \end{pmatrix}$$

And it works!

These kinds of linear maps are the basis of all 3D graphics, animation, physics, etc.

Getting back to matrices

Q: So what is the relationship between this (cool) linear map stuff, and the (lets face it, kindof boring) stuff about matrices and linear equations from before?

A: Matrices are a convenient way to represent linear maps!

To get there, we need a new concept: *basis* of a vector space

Basis in space

- In $\mathbb{R}^3$ we can distinguish three special vectors:

$$(1,0,0) \in \mathbb{R}^3 \qquad (0,1,0) \in \mathbb{R}^3 \qquad (0,0,1) \in \mathbb{R}^3$$

- These vectors form a basis for $\mathbb{R}^3$, which means:
  1. These vectors *span* $\mathbb{R}^3$, which means each vector $(x, y, z) \in \mathbb{R}^3$ can be expressed as a linear combination of these three vectors:

$$\begin{aligned} (x, y, z) &= (x, 0, 0) + (0, y, 0) + (0, 0, z) \\ &= x \cdot (1, 0, 0) + y \cdot (0, 1, 0) + z \cdot (0, 0, 1) \end{aligned}$$

  2. Moreover, this set is as small as possible: no vectors are can be removed and still span $\mathbb{R}^3$.
- Note: condition (2) is equivalent to saying these vectors are linearly independent

Basis

Definition

Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in V$ form a basis for a vector space V if these $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$

- are linearly independent, and
- span V in the sense that each $\boldsymbol{w} \in V$ can be written as linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, namely as:

$$\boldsymbol{w} = a_1\boldsymbol{v}_1 + \cdots + a_n\boldsymbol{v}_n \quad \text{for some} \quad a_1, \ldots, a_n \in \mathbb{R}$$

- These scalars a_i are uniquely determined by $\boldsymbol{w} \in V$ (see below)
- A space V may have several bases, but the number of elements of a basis for V is always the same; it is called the dimension of V, usually written as $\dim(V) \in \mathbb{N}$.

The standard basis for $\mathbb{R}^n$

- For the space $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ there is a standard choice of basis vectors:

  $$\boldsymbol{e}_1 := (1, 0, 0 \ldots, 0),\ \boldsymbol{e}_2 := (0, 1, 0, \ldots, 0),\ \cdots, \boldsymbol{e}_n := (0, \ldots, 0, 1)$$

- $\boldsymbol{e}_i$ has a 1 in the i-th position, and 0 everywhere else.
- We can easily check that these vectors are independent and span $\mathbb{R}^n$.
- This enables us to state precisely that $\mathbb{R}^n$ is n-dimensional.

An alternative basis for $\mathbb{R}^2$

- The standard basis for $\mathbb{R}^2$ is $(1, 0)$, $(0, 1)$.
- But many other choices are possible, eg. $(1, 1)$, $(1, -1)$
  - **independence**: if $a \cdot (1, 1) + b \cdot (1, -1) = (0, 0)$, then:

$$\begin{cases} a + b = 0 \\ a - b = 0 \end{cases} \quad \text{and thus} \quad \begin{cases} a = 0 \\ b = 0 \end{cases}$$

  - **spanning**: each point (x, y) can written in terms of $(1, 1), (1, -1)$, namely:

$$(x, y) = \tfrac{x+y}{2}(1, 1) + \tfrac{x-y}{2}(1, -1)$$

Uniqueness of representations

Theorem

- *Suppose V is a vector space, with basis $v_1, \ldots, v_n$*
- *assume $\boldsymbol{x} \in V$ can be represented in two ways:*

$$\boldsymbol{x} = a_1\boldsymbol{v}_1 + \cdots + a_n\boldsymbol{v}_n \qquad \textit{and also} \qquad x = b_1\boldsymbol{v}_1 + \cdots + b_n\boldsymbol{v}_n$$

Then: $a_1 = b_1$ and ... and $a_n = b_n$.

Proof: This follows from independence of $v_1, \ldots, v_n$ since:

$$\begin{aligned} \mathbf{0} \;=\; \boldsymbol{x} - \boldsymbol{x} &= \big(a_1\boldsymbol{v}_1 + \cdots + a_n\boldsymbol{v}_n\big) - \big(b_1\boldsymbol{v}_1 + \cdots + b_n\boldsymbol{v}_n\big) \\ &= (a_1 - b_1)\boldsymbol{v}_1 + \cdots + (a_n - b_n)\boldsymbol{v}_n. \end{aligned}$$

Hence $a_i - b_i = 0$, by independence, and thus $a_i = b_i$. ☺

Representing vectors

- Fixing a basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ therefore gives us a *unique* way to represent a vector $\boldsymbol{v} \in V$ as a list of numbers called *coordinates*:

$$\boldsymbol{v} = a_1 \boldsymbol{v}_1 + \cdots + a_n \boldsymbol{v}_n$$

New notation: $$\boldsymbol{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_{\mathcal{B}}$$

- If $V = \mathbb{R}^n$, and $\mathcal{B}$ is the standard basis, this is just the vector itself:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

- ...but if $\mathcal{B}$ is not the standard basis, this can be different
- ...and if $V \neq \mathbb{R}^n$, a list of numbers is meaningless without fixing a basis.

What does it mean?

"The introduction of numbers as coordinates is an act of violence."
– Hermann Weyl

What does it mean?

- Space is (probably) real
- ...but coordinates (and hence bases) only exist in our head
- Choosing a basis amounts to fixing some directions in space we decide to call "**up**", "**right**", "**forward**", etc.
- Then a linear combination like:

$$\boldsymbol{v} = 5 \cdot \mathbf{up} + 3 \cdot \mathbf{right} - 2 \cdot \mathbf{forward}$$

  describes a point in space, mathematically.
- ...and it makes working with *linear maps* a *lot* easier

Linear maps and bases, example I

- Take the linear map $f((x_1, x_2, x_3)) = (x_1 - x_2,\ x_2 + x_3)$
- **Claim**: this map is entirely determined by what it does on the basis vectors $(1,0,0), (0,1,0), (0,0,1) \in \mathbb{R}^3$, namely:

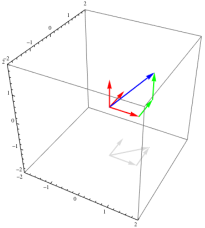
$$f((1,0,0)) = (1,0) \quad f((0,1,0)) = (-1,1) \quad f((0,0,1)) = (0,1).$$

- Indeed, using linearity:

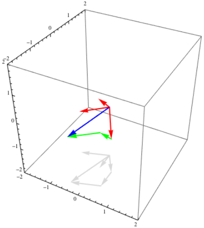
$$\begin{aligned} & f((x_1, x_2, x_3)) \\ &= f\Big((x_1,0,0) + (0,x_2,0) + (0,0,x_3)\Big) \\ &= f\Big(x_1 \cdot (1,0,0) + x_2 \cdot (0,1,0) + x_3 \cdot (0,0,1)\Big) \\ &= f\Big(x_1 \cdot (1,0,0)\Big) + f\Big(x_2 \cdot (0,1,0)\Big) + f\Big(x_3 \cdot (0,0,1)\Big) \\ &= x_1 \cdot f((1,0,0)) + x_2 \cdot f((0,1,0)) + x_3 \cdot f((0,0,1)) \\ &= x_1 \cdot (1,0) + x_2 \cdot (-1,1) + x_3 \cdot (0,1) \\ &= (x_1 - x_2,\ x_2 + x_3) \end{aligned}$$

Linear maps and bases, geometrically

"If we know how to transform any set of axes for a space, we know how to transform everything."



$\mapsto$



Linear maps and bases, example I (cntd)

- $f((x_1, x_2, x_3)) = (x_1 - x_2,\ x_2 + x_3)$ is totally determined by:

  $f((1,0,0)) = (1,0) \qquad f((0,1,0)) = (-1,1) \qquad f((0,0,1)) = (0,1)$

- We can organise this data into a 2 × 3 matrix:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

  The vector $f(\boldsymbol{v}_i)$, for basis vector $\boldsymbol{v}_i$, appears as the i-the column.

- Applying f can be done by a new kind of multiplication:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \cdot x_1 + -1 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 + x_3 \end{pmatrix}$$

Matrix-vector multiplication: Definition

Definition

For vectors $\boldsymbol{v} = (x_1, \ldots, x_n), \boldsymbol{w} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ define their inner product (or dot product) as the real number:

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle \;=\; x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

Definition

If $\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, then $\boldsymbol{w} := \boldsymbol{A} \cdot \boldsymbol{v}$

is the vector whose i-th element is the dot product of the i-th row of matrix $\boldsymbol{A}$ with the (input) vector $\boldsymbol{v}$.

Matrix-vector multiplication, explicitly

For $\boldsymbol{A}$ an $m \times n$ matrix, $\boldsymbol{B}$ a column vector of length n:

$$\boldsymbol{A} \cdot \boldsymbol{b} = \boldsymbol{c}$$

is a column vector of length m.

$$\begin{pmatrix} \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \vdots \\ a_{i1}b_1 + \cdots + a_{in}b_n \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ c_i \\ \vdots \end{pmatrix}$$

$$c_i = \sum_{k=1}^{n} a_{ik} b_k$$

Representing linear maps

Theorem

For every linear map $f : \mathbb{R}^n \to \mathbb{R}^m$, there exists an $m \times n$ matrix $\boldsymbol{A}$ where:

$$f(\boldsymbol{v}) = \boldsymbol{A} \cdot \boldsymbol{v}$$

(where "$\cdot$" is the matrix multiplication of $\boldsymbol{A}$ and a vector $\boldsymbol{v}$)

Proof. Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be the standard basis for $\mathbb{R}^n$. $\boldsymbol{A}$ be the matrix whose i-th column is $f(\boldsymbol{e}_i)$. Then:

$$\boldsymbol{A} \cdot \boldsymbol{e}_i = \begin{pmatrix} a_{11}0 + \ldots + a_{1i}1 + \ldots + a_{1n}0 \\ \vdots \\ a_{m1}0 + \ldots + a_{mi}1 + \ldots + a_{mn}0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix} = f(\boldsymbol{e}_i)$$

Since it is enough to check basis vectors and $f(\boldsymbol{e}_i) = \boldsymbol{A} \cdot \boldsymbol{e}_i$, we are done. ☺

Matrix-vector multiplication, concretely

- Recall $f((x_1, x_2, x_3)) = (x_1 - x_2,\ x_2 + x_3)$ with matrix:
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
- We can directly calculate $f((1, 2, -1)) = (1 - 2, 2 - 1) = (-1, 1)$
- We can also get the same result by matrix-vector multiplication:
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + -1 \cdot 2 + 0 \cdot -1 \\ 0 \cdot 1 + 1 \cdot 2 + 1 \cdot -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
- This multiplication can be understood as: putting the argument values $x_1 = 1, x_2 = 2, x_3 = -1$ in variables of the underlying equations, and computing the outcome.

Another example, to learn the mechanics

$$
\begin{pmatrix} 9 & 3 & 2 & 9 & 7 \\ 8 & 5 & 6 & 6 & 3 \\ 4 & 5 & 8 & 9 & 3 \\ 3 & 4 & 3 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 5 \\ 2 \\ 5 \\ 7 \end{pmatrix}
$$

$$
= \begin{pmatrix} 9\cdot 9 + 3\cdot 5 + 2\cdot 2 + 9\cdot 5 + 7\cdot 7 \\ 8\cdot 9 + 5\cdot 5 + 6\cdot 2 + 6\cdot 5 + 3\cdot 7 \\ 4\cdot 9 + 5\cdot 5 + 8\cdot 2 + 9\cdot 5 + 3\cdot 7 \\ 3\cdot 9 + 4\cdot 5 + 3\cdot 2 + 3\cdot 5 + 4\cdot 7 \end{pmatrix}
$$

$$
= \begin{pmatrix} 81 + 15 + 4 + 45 + 49 \\ 72 + 25 + 12 + 30 + 21 \\ 36 + 25 + 16 + 45 + 21 \\ 27 + 20 + 6 + 15 + 28 \end{pmatrix} = \begin{pmatrix} 194 \\ 160 \\ 143 \\ 96 \end{pmatrix}
$$

Linear map from matrix

- We have seen how a linear map can be described via a matrix
- One can also read each matrix as a linear map

Example

- Consider the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & -3 \end{pmatrix}$
- It has 3 columns/inputs and two rows/outputs. Hence it describes a map $f \colon \mathbb{R}^3 \to \mathbb{R}^2$
- Namely: $f((x_1, x_2, x_3)) = (2x_1 - x_3,\, 5x_1 + x_2 - 3x_3)$.

Examples of linear maps and matrices I

Projections are linear maps that send higher-dimensional vectors to lower ones. Consider $f : \mathbb{R}^3 \to \mathbb{R}^2$

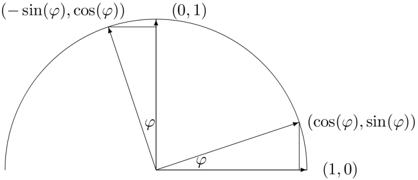
$$f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} x \\ y \end{pmatrix}.$$

f maps $3d$ space to the the $2d$ plane.
The matrix of f is the following 2×3 matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Examples of linear maps and matrices II

We have already seen: Rotation over an angle φ is a linear map



This rotation is described by $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f((x,y)) = \big(\, x\cos(\varphi) - y\sin(\varphi),\; x\sin(\varphi) + y\cos(\varphi)\,\big)$$

The matrix that describes f is

$$\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}.$$

Example: systems of equations

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ & \vdots & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{array} \quad \Rightarrow \quad \begin{array}{c} \boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b} \\ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \end{array}$$

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} \boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{0} \\ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \end{array}$$

Matrix summary

- Take the standard bases: $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\} \subset \mathbb{R}^n$ and $\{\boldsymbol{e}'_1, \ldots, \boldsymbol{e}'_m\} \subset \mathbb{R}^m$
- Every linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by a matrix, and every matrix represents a linear map:

$$f(\boldsymbol{v}) = \boldsymbol{A} \cdot \boldsymbol{v}$$

- The i-th column of $\boldsymbol{A}$ is $f(\boldsymbol{e}_i)$, written in terms of the standard basis $\boldsymbol{e}'_1, \ldots, \boldsymbol{e}'_m$ of $\mathbb{R}^m$.
- (Next time, we'll see the matrix of f depends on the choice of basis: for different bases, a different matrix is obtained)