Matrix Calculations: Determinants and Basis Transformation

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Outline

Last time

- Any linear map can be represented as a matrix:

$$f(\boldsymbol{v}) = \boldsymbol{A} \cdot \boldsymbol{v} \qquad\qquad g(\boldsymbol{v}) = \boldsymbol{B} \cdot \boldsymbol{v}$$

- Last time, we saw that composing linear maps could be done by multiplying their matrices:

$$f(g(\boldsymbol{v})) = \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{v}$$

- Matrix multiplication is pretty easy:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 & 1 \cdot (-1) + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot (-1) + 4 \cdot 4 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 3 & 13 \end{pmatrix}$$

  ...so if we can solve other stuff by matrix multiplication, we are pretty happy.

Last time

- For example, we can solve systems of linear equations:

$$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$$

...by finding the inverse of a matrix:

$$\boldsymbol{x} = \boldsymbol{A}^{-1} \cdot \boldsymbol{b}$$

- There is an easy shortcut formula for 2×2 matrices:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \Longrightarrow \qquad \boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

...as long as $ad - bc \neq 0$.

- We'll see today that "$ad - bc$" is an example of a special number we can compute for any square matrix (not just 2×2) called the determinant.

Determinants

What a determinant does

For an $n \times n$ matrix $\boldsymbol{A}$, the determinant $\det(\boldsymbol{A})$ is a number (in $\mathbb{R}$)

It satisfies:

$$\begin{aligned} \det(\boldsymbol{A}) = 0 &\iff \boldsymbol{A} \text{ is not invertible} \\ &\iff \boldsymbol{A}^{-1} \text{ does not exist} \\ &\iff \boldsymbol{A} \text{ has} < n \text{ pivots in its echolon form} \end{aligned}$$

Determinants have useful properties, but calculating determinants involves some work.

Determinant of a 2×2 matrix

- Assume $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- Recall that the inverse $\boldsymbol{A}^{-1}$ exists if and only if $ad - bc \neq 0$, and in that case is:

$$\boldsymbol{A}^{-1} = \tfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- In this 2×2-case we define:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Thus, indeed: $\det(\boldsymbol{A}) = 0 \iff \boldsymbol{A}^{-1}$ does not exist.

Determinant of a 2×2 matrix: example

- Example:

$$\boldsymbol{P} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} = \tfrac{1}{10} \begin{pmatrix} 8 & 1 \\ 2 & 9 \end{pmatrix}$$

- Then:

$$\begin{aligned} \det(\boldsymbol{P}) &= \tfrac{8}{10} \cdot \tfrac{9}{10} - \tfrac{1}{10} \cdot \tfrac{2}{10} \\ &= \tfrac{72}{100} - \tfrac{2}{100} \\ &= \tfrac{70}{100} = \tfrac{7}{10} \end{aligned}$$

- We have already seen that $\boldsymbol{P}^{-1}$ exists, so the determinant must be non-zero.

Determinant of a 3×3 matrix

- Assume $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
- Then one defines:

$$\det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= +a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

- Methodology:
  - take entries a_{i1} from first column, with alternating signs (+, -)
  - take determinant from square submatrix obtained by deleting the first column and the i-th row

Determinant of a 3×3 matrix, example

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ -2 & 0 & 1 \end{vmatrix} &= 1\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + -2\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} \\
&= \Big(3-0\Big) - 5\Big(2-0\Big) - 2\Big(8+3\Big) \\
&= 3 - 10 - 22 \\
&= -29
\end{aligned}$$

The general, $n \times n$ case

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = +a_{11} \cdot \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \dots & a_{nn} \end{vmatrix} - a_{21} \cdot \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$+ a_{31} \begin{vmatrix} \dots \\ \dots \\ \dots \end{vmatrix} \quad \cdots \quad \pm a_{n1} \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{(n-1)2} & \dots & a_{(n-1)n} \end{vmatrix}$$

(where the last sign $\pm$ is $+$ if n is odd and - if n is even)

Then, each of the smaller determinants is computed recursively.

Applications

- Determinants detect when a matrix is invertible
- Though we showed an inefficient way to compute determinants, there is an efficient algorithm using, you guessed it...Gaussian elimination!
- Solutions to non-homogeneous systems can be expressed directly in terms of determinants using *Cramer's rule* (wiki it!)
- Most importantly: determinants will be used to calculate *eigenvalues* in the next lecture

Vectors in a basis

Recall: a basis for a vector space V is a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ in V such that:

1. They **uniquely** *span* V, i.e. for all $\boldsymbol{v} \in V$, there exist **unique** a_i such that:

$$\boldsymbol{v} = a_1\boldsymbol{v}_1 + \ldots + a_n\boldsymbol{v}_n$$

Because of this, we use a special notation for this linear combination:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_{\mathcal{B}} := a_1\boldsymbol{v}_1 + \ldots + a_n\boldsymbol{v}_n$$

Same vector, different outfits

The *same vector* can look different, depending on the choice of basis. Consider the standard basis: $\mathcal{S} = \{(1,0),(0,1)\}$ vs. another basis:

$$\mathcal{B} = \left\{ \begin{pmatrix} 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 1 \end{pmatrix} \right\}$$

Is this a basis? Yes...

- It's independent because: $\begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}$ has 2 pivots.
- It's spanning because... we can make every vector in $\mathcal{S}$ using linear combinations of vectors in $\mathcal{B}$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{100} \begin{pmatrix} 100 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 100 \\ 1 \end{pmatrix} - \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

...so we can also make any vector in $\mathbb{R}^2$.

Same vector, different outfits

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 1 \end{pmatrix} \right\}$$

Examples:

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}} \qquad \begin{pmatrix} 300 \\ 1 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathcal{B}}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} \frac{1}{100} \\ 0 \end{pmatrix}_{\mathcal{B}} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\mathcal{B}}$$

Why???

- Many find the idea of *multiple bases* confusing the first time around.
- $\mathcal{S} = \{(1,0),(0,1)\}$ is a perfectly good basis for $\mathbb{R}^2$. Why bother with others?
  1. Some vector spaces don't have one "obvious" choice of basis. Example: subspaces $S \subseteq \mathbb{R}^n$.
  2. Sometimes it is way more efficient to write a vector with respect to a different basis, e.g.:

$$\begin{pmatrix} 93718234 \\ -438203 \\ 110224 \\ -5423204980 \\ \vdots \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}_{\mathcal{B}}$$

  3. The choice of basis for *vectors* affects how we write *matrices* as well. Often this can be done cleverly. Example: JPEGs, MP3s, search engine rankings, ...

Transforming bases, part I

- **Problem:** given a vector written in $\mathcal{B} = \{(100,0),(100,1)\}$, how can we write it in the standard basis? Just use the definition:

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\mathcal{B}} = x \cdot \begin{pmatrix} 100 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 100 \\ 1 \end{pmatrix} = \begin{pmatrix} 100x + 100y \\ y \end{pmatrix}_{\mathcal{S}}$$

- Or, as matrix multiplication:

$$\underbrace{\begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}} \cdot \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{in basis } \mathcal{B}} = \underbrace{\begin{pmatrix} 100x + 100y \\ y \end{pmatrix}}_{\text{in basis } \mathcal{S}}$$

- Let $\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}$ be the matrix whose *columns* are the basis vectors $\mathcal{B}$. Then $\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}$ *transforms* a vector written in $\mathcal{B}$ into a vector written in $\mathcal{S}$.

Transforming bases, part II

- How do we transform back? Need $\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}}$ which undoes the matrix $\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}$.
- Solution: use the inverse! $\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} := (\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}})^{-1}$
- Example:

$$(\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}})^{-1} = \begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{100} & -1 \\ 0 & 1 \end{pmatrix}$$

- ...which indeed gives:

$$\begin{pmatrix} \frac{1}{100} & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a-100b}{100} \\ b \end{pmatrix}$$

Transforming bases, part IV

- How about two non-standard bases?

$$\mathcal{B} = \left\{ \begin{pmatrix} 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

- **Problem**: translate a vector from $\begin{pmatrix} a \\ b \end{pmatrix}_{\mathcal{B}}$ to $\begin{pmatrix} a' \\ b' \end{pmatrix}_{\mathcal{C}}$
- **Solution**: do this in two steps:

$$\underbrace{\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \cdot \boldsymbol{v}}_{\text{first translate from } \mathcal{B} \text{ to } \mathcal{S}\ldots}$$

$$\underbrace{\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{C}} \cdot \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \cdot \boldsymbol{v}}_{\ldots\text{then translate from } \mathcal{S} \text{ to } \mathcal{C}} \quad = \quad \left(\boldsymbol{T}_{\mathcal{C}\Rightarrow\mathcal{S}}\right)^{-1} \cdot \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \cdot \boldsymbol{v}$$

Transforming bases, example

- For bases:

$$\mathcal{B} = \{\begin{pmatrix}100\\0\end{pmatrix}, \begin{pmatrix}100\\1\end{pmatrix}\} \qquad \mathcal{C} = \{\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\}$$

- ...we need to find a' and b' such that

$$\begin{pmatrix}a'\\b'\end{pmatrix}_{\mathcal{C}} = \begin{pmatrix}a\\b\end{pmatrix}_{\mathcal{B}}$$

- Translating both sides to the standard basis gives:

$$\begin{pmatrix}-1 & 1\\2 & 2\end{pmatrix} \cdot \begin{pmatrix}a'\\b'\end{pmatrix} = \begin{pmatrix}100 & 100\\0 & 1\end{pmatrix} \cdot \begin{pmatrix}a\\b\end{pmatrix}$$

- This we can solve using the matrix-inverse:

$$\begin{pmatrix}a'\\b'\end{pmatrix} = \begin{pmatrix}-1 & 1\\2 & 2\end{pmatrix}^{-1} \cdot \begin{pmatrix}100 & 100\\0 & 1\end{pmatrix} \cdot \begin{pmatrix}a\\b\end{pmatrix}$$

Transforming bases, example

For:

$$\underbrace{\begin{pmatrix} a' \\ b' \end{pmatrix}}_{\text{in basis } \mathcal{C}} = \underbrace{\begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}^{-1}}_{\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{C}}} \cdot \underbrace{\begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}} \cdot \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\text{in basis } \mathcal{B}}$$

we compute

$$\begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -200 & -199 \\ 200 & 201 \end{pmatrix}$$

which gives:

$$\underbrace{\begin{pmatrix} a' \\ b' \end{pmatrix}}_{\text{in basis } \mathcal{C}} = \underbrace{\frac{1}{4} \begin{pmatrix} -200 & -199 \\ 200 & 201 \end{pmatrix}}_{\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{C}}} \cdot \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\text{in basis } \mathcal{B}}$$

Basis transformation theorem

Theorem

Let $\mathcal{S}$ be the standard basis for $\mathbb{R}^n$ and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be other bases.

1. *Then there is an invertible $n \times n$ basis transformation matrix $\mathbf{T}_{\mathcal{B}\Rightarrow\mathcal{C}}$ such that:*

$$\begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} = \mathbf{T}_{\mathcal{B}\Rightarrow\mathcal{C}} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix}_{\mathcal{C}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_{\mathcal{B}}$$

2. *$\mathbf{T}_{\mathcal{B}\Rightarrow\mathcal{S}}$ is the matrix which has the vectors in $\mathcal{B}$ as columns, and*

$$\mathbf{T}_{\mathcal{B}\Rightarrow\mathcal{C}} := (\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}})^{-1} \cdot \mathbf{T}_{\mathcal{B}\Rightarrow\mathcal{S}}$$

3. $\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{B}} = (\mathbf{T}_{\mathcal{B}\Rightarrow\mathcal{C}})^{-1}$

Matrices in other bases

- Since *vectors* can be written with respect to different bases, so too can *matrices*.
- For example, let g be the linear map defined by:

$$g(\begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{S}}) = \begin{pmatrix}0\\1\end{pmatrix}_{\mathcal{S}} \qquad\qquad g(\begin{pmatrix}0\\1\end{pmatrix}_{\mathcal{S}}) = \begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{S}}$$

- Then, naturally, we would represent g using the matrix:

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}_{\mathcal{S}}$$

- Because indeed:

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

(the columns say where each of the vectors in $\mathcal{S}$ go, written in the basis $\mathcal{S}$)

On the other hand...

- Lets look at what g does to another basis:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

- First $(1,1) \in \mathcal{B}$:

$$g(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}}) = g(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = g(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \ldots$$

- Then, by linearity:

$$\ldots = g(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + g(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

On the other hand...

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

- Similarly $(1, -1) \in \mathcal{B}$:

$$g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{B}} \right) = g\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \ldots$$

- Then, by linearity:

$$\ldots = g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) - g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{\mathcal{B}}$$

A new matrix

- From this:

$$g(\begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{B}}) = \begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{B}} \qquad g(\begin{pmatrix}0\\1\end{pmatrix}_{\mathcal{B}}) = \begin{pmatrix}0\\-1\end{pmatrix}_{\mathcal{B}}$$

- It follows that we should instead use *this* matrix to represent g:

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}_{\mathcal{B}}$$

- Because indeed:

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

(the columns say where each of the vectors in $\mathcal{B}$ go, written in the basis $\mathcal{B}$)

A new matrix

- So on different bases, g acts in a totally different way!

$$g(\begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{S}}) = \begin{pmatrix}0\\1\end{pmatrix}_{\mathcal{S}} \qquad g(\begin{pmatrix}0\\1\end{pmatrix}_{\mathcal{S}}) = \begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{S}}$$

$$g(\begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{B}}) = \begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{B}} \qquad g(\begin{pmatrix}0\\1\end{pmatrix}_{\mathcal{B}}) = \begin{pmatrix}0\\-1\end{pmatrix}_{\mathcal{B}}$$

- ...and hence gets a totally different matrix:

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}_{\mathcal{S}} \quad \text{vs.} \quad \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}_{\mathcal{B}}$$

Transforming bases, part II

Theorem

Assume again we have two bases $\mathcal{B}, \mathcal{C}$ for $\mathbb{R}^n$.

If a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ has matrix $\boldsymbol{A}$ w.r.t. to basis $\mathcal{B}$, then, w.r.t. to basis $\mathcal{C}$, f has matrix $\boldsymbol{A}'$:

$$\boldsymbol{A}' = \boldsymbol{T}_{\mathcal{B} \Rightarrow \mathcal{C}} \cdot \boldsymbol{A} \cdot \boldsymbol{T}_{\mathcal{C} \Rightarrow \mathcal{B}}$$

Thus, via $\boldsymbol{T}_{\mathcal{B} \Rightarrow \mathcal{C}}$ and $\boldsymbol{T}_{\mathcal{C} \Rightarrow \mathcal{B}}$ one tranforms $\mathcal{B}$-matrices into $\mathcal{C}$-matrices. In particular, a matrix can be translated from the standard basis to basis $\mathcal{B}$ via:

$$\boldsymbol{A}' = \boldsymbol{T}_{\mathcal{S} \Rightarrow \mathcal{B}} \cdot \boldsymbol{A} \cdot \boldsymbol{T}_{\mathcal{B} \Rightarrow \mathcal{S}}$$

Example basis transformation, part I

- Consider the standard basis $\mathcal{S} = \{(1,0),(0,1)\}$ for $\mathbb{R}^2$, and as alternative basis $\mathcal{B} = \{(-1,1),(0,2)\}$
- Let the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, w.r.t. the standard basis $\mathcal{S}$, be given by the matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

- What is the representation $\boldsymbol{A}'$ of f w.r.t. basis $\mathcal{B}$?
- Since $\mathcal{S}$ is the standard basis, $\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$ contains the $\mathcal{B}$-vectors as its columns

Example basis transformation, part II

- The basis transformation matrix $\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}}$ in the other direction is obtained as matrix inverse:

$$\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} = \left(\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}\right)^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{-2-0}\begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$$

- Hence:

$$\begin{aligned} \boldsymbol{A}' &= \boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} \cdot \boldsymbol{A} \cdot \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \\ &= \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -2 & 2 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 4 & 4 \\ -1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ -\frac{1}{2} & 2 \end{pmatrix} \end{aligned}$$