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Semantics of lambda-I and of other substructure lambda calculi[1]

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The ordinary untyped λ-calculus (the main object of study in [3]) will be denoted here by λK. Church originally introduced the λI-calculus, which can be understood as the λK-calculus without weakening*: one cannot throw away variables. Similarly there is a* affine *calculus λA without* contraction*: there, one cannot duplicate variables. There is also a* linear *calculus λL in which one has neither weakening nor contraction. In λL variables occur precisely once.*

We give a systematic description of the semantics of these four calculi. It starts with two sorts of domain theoretic models: graph models and filter models (of intersection types) are constructed for each of these calculi. Later on, we describe an appropriate categorical way to capture such structures in terms of monoidal categories (with diagonals or projections).

1 Introduction

Logical systems in which the use of structural rules is restricted are called *substructure logics*. In the λI-calculus one cannot use weakening. In a similar way, there is a calculus λA without contraction and a *linear* calculus λL in which one has neither weakening nor contraction. Therefore, λI, λL and λA will be called substructure λ-calculi.

A bit more concretely, let M be a term from the ordinary λ-calculus (denoted by λK). Let $V(M)$ denote the union of the sets of free and bound variables in M; by convention, this union is a disjoint one. Intuitively,

$$\begin{array}{lcl} M \text{ is in } \lambda I & \Leftrightarrow & \text{every } x \in V(M) \text{ occurs \textit{at least} once} \\ M \text{ is in } \lambda A & \Leftrightarrow & \text{every } x \in V(M) \text{ occurs \textit{at most} once.} \\ M \text{ is in } \lambda L & \Leftrightarrow & \text{every } x \in V(M) \text{ occurs \textit{precisely} once} \end{array}$$

A precise syntax using sequents will be presented in the next section.

The subsequent third section deals with an order theoretic description. It will be shown that one can interprete

λK-terms	as	*continuous* functions	(preserving directed sups)
λI-terms	as	*strict* functions	(preserving directed sups and bottom)
λA-terms	as	*affine* functions	(preserving all non-empty sups)
λL-terms	as	*linear* functions	(preserving all sups)

For all four cases concrete models D will be described where the set of appropriate endofunctions $D \to D$ is a retract of D. The case of the ordinary λK-calculus is of course well-known.

[1]The research reported here was done during the academic year '91–'92 at the Pure Maths Dep., Univ. of Cambridge, UK. An early version is [5].

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Thus far, no categorical tools are used. Given the fact that in the λI-calculus one has "duplication" and in the λA-calculus one has "projections" we are lead to describe these notions of duplication and projection in monoidal categories. It turns out that the well-known smash product in the category of complete lattices with strict functions is a "tensor with diagonals". Similarly there is a "tensor with projections" in the category of complete lattices with affine functions.

In the subsequent section the notion of a (categorical) λK-algebra is recalled. A λL-algebra is described analogously by replacing the finite products in the definition of a λK-algebra by a monoidal structure. A λI-algebra (resp. λA-algebra) is then defined as a λL-algebra plus diagonals (resp. projections). Finally, a uniform construction is given to obtain such categories from set theoretic examples.

2 Syntax

The formation of terms in the substructure λ-calculi is most adequately described by sequent rules. Let *Var* be an infinite collection of (term) variables. Contexts Γ are finite sequences of such variables, in which multiple occurrences are allowed. A sequent $\Gamma \vdash M$ expresses that M is a well-formed term in context Γ.

In all systems λK, λI, λA and λL one has the following rules.

$$\frac{}{x \vdash x}\ (x \in \mathit{Var}) \qquad \frac{\Gamma \vdash M}{\Gamma' \vdash M}\ (\Gamma' \text{ is a permutation of } \Gamma) \qquad \frac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x.M}\ (x \text{ not in } \Gamma)$$

Only in λK and λA one has weakening: $\dfrac{\Gamma \vdash M}{\Gamma, x \vdash M}$

and only in λK and λI one has contraction: $\dfrac{\Gamma, x, x \vdash M}{\Gamma, x \vdash M.}$

Application in λL and λA is described by $\dfrac{\Gamma \vdash M \qquad \Delta \vdash N}{\Gamma, \Delta \vdash MN}$ $(\Gamma, \Delta$ disjoint$)$

and in λK and in λI by $\dfrac{\Gamma, \Theta \vdash M \qquad \Theta, \Delta \vdash N}{\Gamma, \Theta, \Delta \vdash MN}$ $(\Gamma, \Delta$ disjoint$)$

Notice that there are the following inclusions (of terms).

$$\begin{array}{ccc} \lambda L & \subset & \lambda I \\ \cap & & \cap \\ \lambda A & \subset & \lambda K \end{array}$$

In fact, λL can be understood as the intersection λI and λA, and λK as their union.

2.1. Example. The combinators $\boldsymbol{I} \equiv \lambda x.x$ (identity), $\boldsymbol{B} \equiv \lambda xyz.x(yz)$ (composition) and $\boldsymbol{C} \equiv \lambda xyz.xzy$ (exchange) can be formed in λL. One has $\boldsymbol{S} \equiv \lambda xyz.xz(yz)$ (duplication) in λI and $\boldsymbol{K} \equiv \lambda xy.x$ (projection) in λA.

Put for $n \in \mathbb{N}$, $\mathbf{1}_n \equiv \lambda z x_1 \ldots x_n.\ z x_1 \ldots x_n$. Then $\mathbf{1}_0 = \boldsymbol{I}$ and $\mathbf{1}_{n+1} = \boldsymbol{B}\mathbf{1}_n$; these $\mathbf{1}_n$ exist in λL (and hence in all systems).

3 Domain models

This section is organized as follows. We start with some general domain theoretic notions, which will be used subsequently for two specific constructions. The first one generalizes Engeler's graph models (see e.g. [3]) to all four of the present λ-calculi, and the second one generalizes the filter models from [4].

3.1. Definition. Let D, D' be complete lattices (i.e. posets with suprema of all subsets). A function $f\colon D \to D'$ is called

$$\begin{array}{rll} \textit{continuous} & \text{if} & f(\bigvee a) = \bigvee f(a) \text{ for all } \textit{directed } a \subseteq D, \\ & & \quad \text{(i.e. for non-empty } a \text{ satisfying } \forall x, y \in a.\ \exists z \in a.\ x \leq z \ \&\ y \leq z) \\ \textit{strict} & \text{if} & f(\bigvee a) = \bigvee f(a) \text{ for all directed or empty } a \subseteq D \\ \textit{affine} & \text{if} & f(\bigvee a) = \bigvee f(a) \text{ for all non-empty } a \subseteq D \\ \textit{linear} & \text{if} & f(\bigvee a) = \bigvee f(a) \text{ for all } a \subseteq D \end{array}$$

Thus a continuous function f is strict if and only if $f(\bot) = \bot$, where $\bot = \bigvee \emptyset$ is the bottom element. Similarly an affine function f is linear if and only if $f(\bot) = \bot$. Let's write $\mathbb{CL}_{cont}$, $\mathbb{CL}_{str}$, $\mathbb{CL}_{aff}$ and $\mathbb{CL}_{lin}$ for the categories of complete lattices with continuous, strict, affine and linear functions. There are inclusion functors

$$\begin{tikzcd} \mathbb{CL}_{lin} \arrow[r, hook] \arrow[d, hook] & \mathbb{CL}_{str} \arrow[d, hook] \\ \mathbb{CL}_{aff} \arrow[r, hook] & \mathbb{CL}_{cont} \end{tikzcd}$$

The associated function spaces will be denoted as follows; here we write

$$\begin{array}{ll} [D \to_K D'] & \text{for the set of } \textit{continuous} \text{ functions } D \to D' \\ [D \to_I D'] & \text{for the set of } \textit{strict} \text{ functions } D \to D' \\ [D \to_A D'] & \text{for the set of } \textit{affine} \text{ functions } D \to D' \\ [D \to_L D'] & \text{for the set of } \textit{linear} \text{ functions } D \to D' \end{array}$$

These function spaces are complete lattices again under pointwise ordering. Finally, for $\diamondsuit \in \{K, I, L, A\}$ we call a complete lattice D *$\diamondsuit$-reflexive* if it comes equipped with two functions

$$\begin{array}{lll} F \in [\,D \to_\diamondsuit [\,D \to_\diamondsuit D]\,] & & \text{for application} \\ G \in [\,[\,D \to_\diamondsuit D\,] \to_\diamondsuit D\,] & & \text{for abstraction} \end{array}$$

satisfying $F \circ G = id$. In that case one has that $[\,D \to_\diamondsuit D]$ is a retract of D, which is often written as $[\,D \to_\diamondsuit D] \lhd D$.

We claim that for $\diamondsuit \in \{K, I, A, L\}$ one has that a $\diamondsuit$-reflexive complete lattice (with F, G as described above) forms a model for the $\lambda\diamondsuit$-calculus. This goes as follows. For a valuation (or environment) $\rho\colon Var \to D$ one defines the interpretation $[\![\, M \,]\!]_\rho$ of a $\lambda\diamondsuit$-term M in the standard way, i.e.

$$\begin{array}{rcl} [\![\, x \,]\!]_\rho & = & \rho(x) \\ [\![\, MN \,]\!]_\rho & = & F([\![\, M \,]\!]_\rho)([\![\, N \,]\!]_\rho) \\ [\![\, \lambda x. M \,]\!]_\rho & = & G(\boldsymbol{\lambda} d \in D. [\![\, M \,]\!]_{\rho(x:=d)}). \end{array}$$

3.2. Lemma. *The above interpretation is well-defined: for a $\lambda\diamondsuit$-term of the form $\lambda x.M$ one has that $f = \boldsymbol{\lambda} d \in D. [\![\, M \,]\!]_{\rho(x:=d)}$ is in $[D \to_\diamondsuit D]$.*

Proof. By induction on M. We shall do the application case $M \equiv M_1 M_2$ for $\diamondsuit = L, A, I$.

$\diamondsuit = L$ Assume without loss of generality that $x \in FV(M_1)$ and $x \notin FV(M_2)$. Then

$$\begin{array}{rcll} f(\bigvee_i d_i) & = & F([\![\, M_1 \,]\!]_{\rho(x:=\bigvee_i d_i)})([\![\, M_2 \,]\!]_\rho) & \\ & = & F(\bigvee_i [\![\, M_1 \,]\!]_{\rho(x:=d_i)})([\![\, M_2 \,]\!]_\rho) & \text{by induction hypothesis} \\ & = & \bigvee_i F([\![\, M_1 \,]\!]_{\rho(x:=d_i)})([\![\, M_2 \,]\!]_\rho) & F \text{ is linear (in both places)} \\ & = & \bigvee_i f(d_i). & \end{array}$$

$\diamondsuit = A$ The interesting remaining case is when $x \notin FV(M_1)$ and $x \notin FV(M_2)$. Then for a non-empty collection $\{d_i\}$,

$$\begin{array}{rcll} f(\bigvee_i d_i) & = & F([\![\, M_1 \,]\!]_\rho)([\![\, M_2 \,]\!]_\rho) & \\ & = & f(d_{i_0}) & \text{for some } i_0 \\ & = & \bigvee_i f(d_i). & \end{array}$$

$\diamondsuit = I$ Assume now (without loss of generality) that $x \in FV(M_1)$. The $\bot$-case is the most interesting:

$$\begin{array}{rcll} f(\bot) & = & F([\![\, M_1 \,]\!]_{\rho(x:=\bot)})([\![\, M_2 \,]\!]_{\rho(x:=\bot)}) & \\ & = & F(\bot)([\![\, M_2 \,]\!]_{\rho(x:=\bot)}) & \text{by induction hypothesis} \\ & = & \bot & F \text{ is strict (in both places).} \end{array}$$

□

In the rest of this section one finds examples of such $\diamondsuit$-reflexive complete lattices.

Graph models. Let's call a set X a *K-set* if it comes equipped with an injection $\langle -, - \rangle : \mathcal{P}_f(X) \times X \rightarrowtail X$, where $\mathcal{P}_f(X)$ is the set of finite subsets of X. As is well-known, the powerset $\mathcal{P}(X)$ of a K-set X yields a model for the λK-calculus with application and abstraction

$$\begin{array}{rcl} F(a)(b) & = & \{x \in X \mid \exists \beta \in \mathcal{P}_f(X).\ \beta \subseteq b \text{ and } \langle \beta, x \rangle \in a\} \\ G(f) & = & \{\langle \beta, x \rangle \mid \beta \in \mathcal{P}_f(X) \text{ and } x \in f(\beta)\} \end{array}$$

using that a continuous function on a powerset lattice is determined by its behaviour on finite sets. One obtains that $\mathcal{P}(X)$ is a K-reflexive complete lattice. The first example of such a K-set is the set ω of natural numbers (as noticed by Scott and Plotkin, see [3], 18.1.3). Later, Engeler constructed code-free variants of this model.

In order to code up a strict function $f\colon \mathcal{P}(X) \to \mathcal{P}(X)$ it suffices to code up its behaviour on *non-empty* finite sets — since $f(\emptyset) = \emptyset$. Hence we call X an *I-set* if it comes with an injection $\mathcal{P}_f^+(X) \times X \rightarrowtail X$, where $\mathcal{P}_f^+(X)$ is the set of non-empty finite subsets of X. Then one can define application and abstraction functions as above — with $\mathcal{P}_f$ replaced by $\mathcal{P}_f^+$ — and verify that the powerset $\mathcal{P}(X)$ becomes an I-reflexive complete lattice (and thus a model of the λI-calculus).

Next we notice that a linear function $f\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is determined by its behaviour on singletons. Hence we define an *L-set* to be a set X together with an injection $X \times X \rightarrowtail X$. The powerset of an L-set is an L-reflexive complete lattice. Specifically, it was noted by Abramsky and by Phoa that the pairing on ω makes $\mathcal{P}\omega$ a model of the linear λ-calculus.

Finally, an affine function $f\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is determined by its behaviour on the set $\bot(X)$ of what are usually called *subsingletons* of X. It is given by

$$\begin{array}{rcl} \bot(X) & = & \{\alpha \in \mathcal{P}(X) \mid \forall x, y \in \alpha.\ x = y\} \\ & = & \{\{x\} \mid x \in X\} \cup \{\emptyset\}. \end{array}$$

Hence an *A-set* consists of a set X together with an injection $\bot(X) \times X \rightarrowtail X$. Its powerset $\mathcal{P}(X)$ is a A-reflexive complete lattice. We summarize

	K-set	I-set	A-set	L-set
definition	$\mathcal{P}_f(X) \times X \rightarrowtail X$	$\mathcal{P}_f^+(X) \times X \rightarrowtail X$	$\bot(X) \times X \rightarrowtail X$	$X \times X \rightarrowtail X$
$\mathcal{P}(X)$ is ..	K-reflexive	I-reflexive	A-reflexive	L-reflexive
.. and model of	λK-calculus	λI-calculus	λA-calculus	λL-calculus

Finally, we mention how to obtain such sets. Let A be a (non-empty) set of atoms. The least L-set containing A is $\bigcup_{n\in\mathbb{N}}.\, A_n$, where

$$A_0 \;=\; A, \qquad\qquad A_{n+1} \;=\; A_n \cup (A_n \times A_n).$$

A similar construction applies to the other cases. For example, to obtain an I-set one takes

$$A_{n+1} \;=\; A_n \cup (\mathcal{P}_f^+(A_n) \times A_n).$$

In such a way, one also obtains Engeler's D_A as the least K-set containing A, see e.g. [3].

Filter models. The intersection type discipline introduced by the Turin school captures finite forms of polymorphism via finite intersections of types $\sigma \wedge \tau$ and ω, where ω stands for the empty intersection, i.e. the top element. The intention is that $M : \sigma \wedge \tau$ if and only if both $M : \sigma$ and $M : \tau$ and that $M : \omega$ always holds. Thus one describes overloading of addition by

$$+ : (\mathsf{int} \to \mathsf{int} \to \mathsf{int}) \;\wedge\; (\mathsf{real} \to \mathsf{real} \to \mathsf{real})$$

The relevance for our present investigation lies in the fact that the finite intersections allow us to introduce types in the contraction and weakening rules. More specifically, $\wedge$ is used for contraction and ω for weakening:

$$\frac{\Gamma, x : \sigma, x : \tau \;\vdash\; M : \rho}{\Gamma, x : \sigma \wedge \tau \;\vdash\; M : \rho}\;(\mathrm{C}) \qquad\qquad \frac{\Gamma \;\vdash\; M : \rho}{\Gamma, x : \omega \;\vdash\; M : \rho}\;(\mathrm{W})$$

We are thus led to adapt the filter models of the λK-calculus described in [4] to the other λ-calculi by using

at least one intersection	for	contraction
at most one intersection	for	weakening

This will be worked out below. The 'K' case covers what was done in [4].

Let *TypeVar* be a non-empty set of type variables. The sets of types T_K, T_I, T_A and T_L are the least containing *TypeVar* and closed under $\to$ such that the following extra requirements are satisfied.

$$\begin{tikzcd}
\boxed{T_L : \quad -} \arrow[r, hook] \arrow[d, hook] & \boxed{T_I : \quad \sigma, \tau \in T_I \Rightarrow (\sigma \wedge \tau) \in T_I} \arrow[d, hook] \\
\boxed{T_A : \quad \omega \in T_A} \arrow[r, hook] & \boxed{T_K : \quad \omega \in T_K,\ \ \sigma, \tau \in T_K \Rightarrow (\sigma \wedge \tau) \in T_K}
\end{tikzcd}$$

where ω is a constant type (not in *TypeVar*). Further, consider the following conditions for an order relation $\leq$ on types.

$$(L)\begin{cases} \sigma \leq \sigma \\ \sigma \leq \tau,\, \tau \leq \rho \;\Rightarrow\; \sigma \leq \rho \\ \sigma' \leq \sigma,\, \tau \leq \tau' \;\Rightarrow\; \sigma \to \tau \leq \sigma' \to \sigma' \end{cases} \qquad (I)\begin{cases} \sigma \leq \sigma \wedge \sigma \\ \sigma \wedge \tau \leq \sigma \\ \sigma \wedge \tau \leq \tau \\ (\sigma \to \tau) \wedge (\sigma \to \rho) \leq \sigma \to (\tau \wedge \rho) \\ \sigma \leq \sigma',\, \tau \leq \tau' \;\Rightarrow\; \sigma \wedge \tau \leq \sigma' \wedge \tau' \end{cases}$$

$$(A)\begin{cases} \sigma \leq \omega \\ \omega \leq \omega \to \omega \end{cases}$$

We then define:

$\leq_L$ is the least relation on T_L satisfying (L)
$\leq_A$ is the least relation on T_A satisfying $(L) + (A)$
$\leq_I$ is the least relation on T_I satisfying $(L) + (I)$
$\leq_K$ is the least relation on T_K satisfying $(L) + (A) + (I)$

and write $\sigma \sim_\diamond \tau$ for $\sigma \leq_\diamond \tau$ and $\tau \leq_\diamond \sigma$ with $\diamond \in \{L, A, I, K\}$.

The following is essentially Lemma 2.4 in [4]; the proof is similar and left to the interested reader. It gives some technical properties of the preorders $\langle T_\diamond, \leq_\diamond \rangle$ which are needed later.

3.3. Lemma. (i) *Let $\diamond$ be A or K. Then $\sigma \to \tau \sim_\diamond \omega \Leftrightarrow \tau \sim_\diamond \omega$.*

(ii) *In $\langle T_L, \leq_L \rangle$ one has $\mu \to \nu \leq_L \sigma \to \tau \Rightarrow \sigma \leq_L \mu \;\&\; \nu \leq_L \tau$.*
In $\langle T_A, \leq_A \rangle$, one has $\mu \to \nu \leq_A \sigma \to \tau \;\&\; \tau \not\sim_A \omega \Rightarrow \sigma \leq_A \mu \;\&\; \nu \leq_A \tau$.
In $\langle T_I, \leq_I \rangle$, one has that $(\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_n \to \nu_n) \leq_I \sigma \to \tau$ implies
$\exists i_1, \ldots, i_m \in \{1, \ldots, n\}.\ \sigma \leq_I \mu_{i_1} \wedge \cdots \wedge \mu_{i_m} \;\&\; \nu_{i_1} \wedge \cdots \wedge \nu_{i_m} \leq_I \tau$.
In $\langle T_K, \leq_K \rangle$, one has that $(\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_n \to \nu_n) \leq_K \sigma \to \tau \;\&\; \tau \not\sim_K \omega$ implies
$\exists i_1, \ldots, i_m \in \{1, \ldots, n\}.\ \sigma \leq_K \mu_{i_1} \wedge \cdots \wedge \mu_{i_m} \;\&\; \nu_{i_1} \wedge \cdots \wedge \nu_{i_m} \leq_K \tau$. $\square$

In a preorder an *upset* is a subset a which is upwardly closed, i.e. satisfies $x \geq y \in a \Rightarrow x \in a$. We then define an *$L$-filter* to be an upset in the preorder $\langle T_L, \leq_L \rangle$.
Next, an *A-filter* is an upset a in $\langle T_A, \leq_A \rangle$ satisfying $\omega \in a$, i.e. A-filters are non-empty L-filters. Further, an *I-filter* is an upset a in $\langle T_I, \leq_I \rangle$ which satisfies $\sigma, \tau \in a \Rightarrow \sigma \wedge \tau \in a$ and a *K-filter* is an upset a in $\langle T_K, \leq_K \rangle$ satisfying both $\omega \in a$ and $\sigma, \tau \in a \Rightarrow \sigma \wedge \tau \in a$.

It is then easy to verify that for $\diamond \in \{L, A, I, K\}$ one has that the set $\mathcal{F}_\diamond$ of $\diamond$-filters is closed under arbitrary intersections. For an arbitrary $b \subseteq T_\diamond$ put

$$\uparrow_\diamond b \;=\; \bigcap \{a \in \mathcal{F}_\diamond \mid a \supseteq b\}$$

for the least $\diamond$-filter containing b. As usual, one writes $\uparrow_\diamond \sigma$ for $\uparrow_\diamond \{\sigma\}$. One then easily verifies that

$$\begin{aligned}
\uparrow_L b &= \{\sigma \in T_L \mid \exists \tau \in b.\ \tau \leq_L \sigma\} \\
\uparrow_A b &= \{\sigma \in T_A \mid \exists \tau \in b.\ \tau \leq_A \sigma\} \;\cup\; \{\omega\} \\
\uparrow_I b &= \{\sigma \in T_I \mid \exists n \geq 1.\ \exists \tau_1, \ldots, \tau_n \in b.\ \tau_1 \wedge \cdots \wedge \tau_n \leq_I \sigma\} \\
\uparrow_K b &= \{\sigma \in T_K \mid \exists n \geq 0.\ \exists \tau_1, \ldots, \tau_n \in b.\ \tau_1 \wedge \cdots \wedge \tau_n \leq_K \sigma\}
\end{aligned}$$

whereby the empty intersection in the latter case (for $n = 0$) is understood to be ω. Thus one has that $\uparrow_A b = \uparrow_L b \cup \{\omega\}$ and $\uparrow_K b = \uparrow_I b \cup \{\omega\}$. Hence by defining joins of filters as $\bigvee_i a_i = \uparrow_\diamond (\bigcup_i a_i)$ one obtains that $\mathcal{F}_\diamond$ is a complete lattice.

Finally we are in a position to define application and abstraction functions which make the sets of filters $\mathcal{F}_\diamond$ into appropriate reflexive complete lattices. For $a, b \in \mathcal{F}_\diamond$ put

$$F(a)(b) \;=\; \{\tau \in T_\diamond \mid \exists \sigma \in b.\ \sigma \to \tau \in a\}$$

and for a function $f : \mathcal{F}_\diamond \to \mathcal{F}_\diamond$ put

$$G(f) \;=\; \updownarrow_{\diamond}\{\sigma \to \tau \mid \sigma \in T_{\diamond} \text{ and } \tau \in f(\updownarrow_{\diamond}\sigma)\}.$$

It is then not difficult to verify that indeed F is in $[\mathcal{F}_{\diamond} \to_{\diamond} [\mathcal{F}_{\diamond} \to_{\diamond} \mathcal{F}_{\diamond}]]$ and that G is in $[[\mathcal{F}_{\diamond} \to_{\diamond} \mathcal{F}_{\diamond}] \to_{\diamond} \mathcal{F}_{\diamond}]$ and with a little more effort that $F \circ G = id$, using the above Lemma and characterization of $\updownarrow_{\diamond} b$. Thus $\mathcal{F}_{\diamond}$ is a model of the $\lambda\diamondsuit$-calculus.

In [2] one finds various (other) subsystems of the intersection type discipline. It was already noticed there that a filter model of intersection types without ω yields a model of the λI-calculus. In [8] the intersection $\wedge$ is generalized to second order quantification $\forall$. The resulting filters model only the affine λA-calculus, because one cannot type the contraction rule with $\forall$.

4 Towards a categorical understanding

A *monoidal category* is a category $\mathbb{B}$ provided with a *tensor product* $\otimes$ with a neutral element I. A bit more formally, $\otimes$ is a functor $\mathbb{B}\times\mathbb{B} \to \mathbb{B}$ which comes equipped with natural isomorpisms

$$X \otimes (Y \otimes Z) \xrightarrow[\sim]{\alpha} (X \otimes Y) \otimes Z \qquad\qquad I \otimes X \xrightarrow[\sim]{\lambda} X \xleftarrow[\sim]{\rho} X \otimes I$$

satisfying some equations, see e.g. [9]. The monoidal category is called *symmetric* if there are additional isomorphism

$$X \otimes Y \xrightarrow[\sim]{\gamma} Y \otimes X$$

satisfying some additional equations. A monoidal category is called *closed* if it comes equipped with "function spaces" or "internal homs". That is, if for each pair of objects X, Y, there is an object $X \multimap Y$ together with an evaluation map $ev : (X \multimap Y) \otimes X \to Y$ such that for every $f\colon Z \otimes X \to Y$ there is a unique $\Lambda(f) : Z \to (X \multimap Y)$ satisfying $ev \circ \Lambda(f) \otimes id = f$.

We shall abbreviate 'symmetric monoidal category' as SMC and 'symmetric monoidal closed category' as SMCC. The category of abelian groups is the standard example of an SMCC.

Suppose now Ω is an object in an SMC $\mathbb{B}$. The idea is to interpret a sequent $x_1, \ldots, x_n \vdash M$ as a morphism in $\mathbb{B}$, $\Omega^n \to \Omega$, where $\Omega^0 = I$ and $\Omega^{i+1} = \Omega^i \otimes \Omega$. Because the tensor is associative, the precise bracketing in Ω^n can be ignored. Since we have symmetry as well, the components in Ω^n can be exhanged. This gives the validity of the context permutation rule. An elementary but crucial observation is that one needs *diagonals* $X \to (X \otimes X)$ for the validity of the contraction rule and *projections* $X \leftarrow (X \otimes Y) \to Y$ for the weakening rule. This can be seen as follows. Suppose we have a term $\Gamma, x, x \vdash M$ interpreted as a morphism $f : \Omega^n \otimes \Omega \otimes \Omega \to \Omega$ then by using $\delta\colon \Omega \to \Omega \otimes \Omega$ one can form the arrow which corresponds to the outcome $\Gamma, x \vdash M$ of the contraction rule by $f \circ id_{\Omega^n} \otimes \delta : \Omega^n \otimes \Omega \to \Omega^n \otimes \Omega \otimes \Omega \to \Omega$. For weakening, suppose we have $\Gamma \vdash M$ interpreted as $g : \Omega^n \to \Omega$, then one can use the (first) projection $\pi\colon (\Omega^n) \otimes \Omega \to \Omega^n$ to interprete $\Gamma, x \vdash M$ as $g \circ \pi : \Omega^n \otimes \Omega \to \Omega^n \to \Omega$.

These observations lead to the following Definition.

4.1. Definition. Let $(\mathbb{B}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ be an SMC as described above. We say that $\mathbb{B}$ has

(i) *diagonals* if there is a natural transformation $\delta\colon Id \stackrel{\cdot}{\longrightarrow} \Delta$, where the latter functor is described by $\Delta(X) = X \otimes X$. In addition, commutation of the following diagrams is required.

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, equal] & X \otimes X \arrow[r, "id \otimes \delta"] & X \otimes (X \otimes X) \arrow[d, "\alpha"] \\
X \arrow[r, "\delta"'] & X \otimes X \arrow[r, "\delta \otimes id"'] & (X \otimes X) \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[d, "\delta"'] \arrow[dr, equal] & \\
I \otimes I \arrow[r, "\rho = \lambda"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "\delta"'] \arrow[dr, "\delta"] & \\
X \otimes X \arrow[r, "\gamma"'] & X \otimes X
\end{tikzcd}$$

(ii) *projections* if the neutral element I is terminal.

The word 'projections' used in the second point of the above Definition is appropriate because in case the neutral element is terminal, one obtains two natural transformations $\pi : \otimes \stackrel{\cdot}{\longrightarrow} Fst$ and $\pi' : \otimes \stackrel{\cdot}{\longrightarrow} Snd$ with components

$$\begin{array}{rcllllll}
\pi & = & \rho^{-1} & \circ \; id \otimes ! & : & X \otimes Y & \to & X \otimes I \xrightarrow{\sim} X \\
\pi' & = & \lambda^{-1} & \circ \; ! \otimes id & : & X \otimes Y & \to & I \otimes Y \xrightarrow{\sim} Y
\end{array}$$

These projections satisfy

$$\begin{array}{rcl}
\pi \circ \pi \circ \alpha & = & \pi \\
\pi' \circ \alpha & = & \pi' \circ \pi' \\
\pi' \circ \pi \circ \alpha & = & \pi \circ \pi'
\end{array}
\qquad\qquad
\begin{array}{rcl}
\pi \circ \gamma & = & \pi' \\
\pi' \circ \gamma & = & \pi
\end{array}$$

Notice that if a (symmetric) monoidal category has both diagonals and projections as described above and additionally the following eqnations hold,

$$\pi \circ \delta \;=\; id \qquad\qquad \pi \circ \delta \;=\; id \qquad\qquad \delta \circ (\pi \otimes \pi') \;=\; id$$

then $\otimes$ becomes a cartesian product with pairing $\langle f, g\rangle = (f \otimes g) \circ \delta$. Moreover, in that case one has $\alpha = \langle\langle \pi, \pi \circ \pi'\rangle, \pi' \circ \pi'\rangle$ and $\gamma = \langle \pi', \pi\rangle$.

In the rest of this section it will be shown that the categories of complete lattices introduced in Definition 3.1 give examples of the above monoidal structures. Much of the next result is standard; maybe points (ii) and (iii) are not so well-known.

4.2. Proposition. *The category of complete lattices with*

(i) *linear functions (i.e.* $\mathbb{CL}_{lin}$*) is an* SMCC*;*

(ii) *affine functions (i.e.* $\mathbb{CL}_{\mathsf{aff}}$*) is an* SMCC *with projections;*

(iii) *strict functions (i.e.* $\mathbb{CL}_{str}$*) is an* SMCC *with diagonals;*

(iv) *continuous functions (i.e.* $\mathbb{CL}_{cont}$*) is a* CCC *(cartesian closed category; it thus has both projections and diagonals).*

Proof. (i) For complete lattices D_1 and D_2 one defines a linear tensor

$$D_1 \otimes D_2 \;=\; \{A \in \mathcal{P}(D_1 \times D_2) \mid \forall a \in \mathcal{P}(D_1).\, \forall b \in \mathcal{P}(D_2).\; a \times b \subseteq A \text{ iff } (\bigvee a, \bigvee b) \in A\}$$

It comes together with a map $\mathsf{lin}\colon D_1 \times D_2 \to D_1 \otimes D_2$ given by

$$(x, y) \;\mapsto\; \downarrow(x, y) \;=\; \{(x', y') \mid x' \leq x \text{ and } y' \leq y\}$$

This map is *bi-linear*, i.e. linear in each coordinate separately. Even more, it is a universal bi-linear function with domain $D_1 \times D_2$: for every bi-linear function $f\colon D_1 \times D_2 \to D_3$ there is a unique linear function $\overline{f}\colon D_1 \otimes D_2 \to D_3$ with $\overline{f} \circ \mathsf{lin} = f$. This shows that the category $\mathbb{CL}_{lin}$ is closed (i.e. has internal homs). The neutral element for the linear tensor is the two-element complete lattice.

(ii) The affine tensor of D_1 and D_2 is

$$D_1 \otimes D_2 \;=\; \{A \in \mathcal{P}^{+}(D_1 \times D_2) \mid \forall a \in \mathcal{P}^{+}(D_1).\, \forall b \in \mathcal{P}^{+}(D_2).\; a \times b \subseteq A \text{ iff } (\bigvee a, \bigvee b) \in A\}$$

where $\mathcal{P}^{+}$ denotes the non-empty powerset operation. One now has a universal *bi-affine* function $\mathsf{aff}\colon D_1 \times D_2 \to D_1 \otimes D_2$. The neutral element for the affine tensor is the one-element complete lattice — which is terminal. Thus one has projections $D_1 \leftarrow D_1 \otimes D_2 \to D_2$. These exist because the cartesian projections $D_1 \leftarrow D_1 \times D_2 \to D_2$ are bi-affine functions (but bot bi-linear!).

(iii) For complete lattices D_1 and D_2 there is the *smash product*

$$D_1 \otimes D_2 \quad = \quad \{\langle x, y\rangle \in D_1 \times D_2 \mid x = \bot \text{ iff } y = \bot\}$$

which comes equipped with a universal *bi-strict* function $\mathsf{str}: D_1 \times D_2 \to D_1 \otimes D_2$, given by

$$(x, y) \quad \mapsto \quad \text{if } x \neq \bot \,\wedge\, y \neq \bot \text{ then } \langle x, y\rangle \text{ else } \langle \bot, \bot\rangle.$$

The 2-element complete lattice is neutral element for $\otimes$. There are obvious diagonals $\delta: D \to D \otimes D$ by $x \mapsto \langle x, x\rangle$.

(iv) Well-known. □

5 Appropriate categorical notions

Following earlier work by Obtulowicz, it is argued in [6] that a categorical structure for the λK-calculus looks as follows.

5.1. Definition. (i) A *(categorical) λK-algebra* is given by a category $\mathbb{B}$ with finite products, containing a distinguished non-empty object Ω (i.e. the homset $\mathbb{B}(1,\ \Omega) \neq \emptyset$ where 1 is the terminal object) provided with

- an application map $app: \Omega \times \Omega \to \Omega$;
- abstraction operations $\lambda(-) : \mathbb{B}(X \times \Omega,\ \Omega) \to \mathbb{B}(X,\ \Omega)$,

such that

$$app \circ \lambda(f) \times id \;=\; f \qquad\qquad \lambda(f \circ g \times id) \;=\; \lambda(f) \circ g.$$

(ii) A *$\lambda\eta K$-algebra* is a λK-algebra in which $\lambda(app) = id$.

The second equation $\lambda(f \circ g \times id) = \lambda(f) \circ g$ above is a naturality condition which provides the proper distribution of substitution over abstraction. The first equation is of course β; and the equation $\lambda(app) = id$ in (ii) is η.

5.2. Examples. (i) Suppose $\langle D, \cdot\rangle$ is a set theoretic λ-algebra as in [3]. For $n \in \mathbb{N}$, put

$$(D^n \to D) \quad = \quad \{a \in D \mid \mathbf{1}_n \cdot a = a\}.$$

Define a category $\mathbb{D}$ with $n \in \mathbb{N}$ as objects. Morphisms $n \to m$ in $\mathbb{D}$ are m-tuples of elements from $(D^n \to D)$. The object 0 is then terminal and $n + m$ is the cartesian product of n and m. As distinguished object ('Ω') one takes $1 \in \mathbb{D}$. One has $app = \lambda xy.xy: 1 + 1 \to 1$ and for $a: n + 1 \to 1$ one has $\lambda(a) = a: n \to 1$.

(ii) D. Scott first constructed a complete lattice D_∞ for which the complete lattice $[D_\infty \to_K D_\infty]$ of continuous endofunctions is isomorphic to D_∞ itself (say via $F: D_\infty \to [D_\infty \to_K D_\infty]$ and $G: [D_\infty \to_K D_\infty] \to D_\infty$). A category $\mathbb{B}$ is formed with $n \in \mathbb{N}$ as objects. A morphism $n \to m$ is an m-tuple $(f_1, \ldots, f_m)$ of continuous functions $f_i: (D_\infty)^n \to D_\infty$. Again 0 is terminal, $n + m$ is cartesian product and 1 serves as distinguished object. Then

$$app \quad = \quad \boldsymbol{\lambda}(x, y).\, F(x)(y) \quad : \quad 1 + 1 \quad \to \quad 1$$

and for $f: n + 1 \to 1$ one has $\lambda(f): n \to 1$ described by

$$\lambda(f) \quad = \quad \boldsymbol{\lambda}(x_1, \ldots, x_n).\, G\big(\boldsymbol{\lambda}y.\, f(x_1, \ldots, x_n, y)\big)$$

where the boldface $\boldsymbol{\lambda}$ is used for meta abstraction. One obtains a $\lambda\eta K$-algebra.

D. Scott used in [10] a *reflexive object* in a cartesian closed category (CCC) to describe the semantics of the untyped λ-calculus (like in Definition 3.1). This notion can be described in an arbitrary symmetric monoidal closed category (SMCC). Actually, symmetry is not needed.

5.3. Definition. Let $\mathbb{B}$ be an SMCC. An object $\Omega \in \mathbb{B}$ will be called *reflexive* if it comes equipped with two morphisms

$$F \;:\; \Omega \;\to\; (\Omega \multimap \Omega) \qquad\qquad G \;:\; (\Omega \multimap \Omega) \;\to\; \Omega$$

satisfying $F \circ G = id$. It will be called *extensional* if additionally $G \circ F = id$.

The complete lattice D_∞ in the above Example (ii) is an example of an extensional reflexive object in the cartesian closed category $\mathbb{CL}_{cont}$ of complete lattices and continuous functions. The following is now an easy result.

5.4. Lemma. *Suppose $\mathbb{B}$ is a CCC with a reflexive object Ω. One obtains a λK-algebra by putting*

$$app \;=\; ev \;\circ\; F \times id \qquad\qquad \textit{and} \qquad\qquad \lambda(f) \;=\; G \;\circ\; \Lambda(f).$$

One obtains a $\lambda\eta K$-algebra if and only if the reflexive object Ω is extensional. □

Notice that a categorical description of untyped λ-calculi in terms of reflexive objects requires an ambient category with *all* exponents, whereas for λK-algebras (in Definition 5.1) we use a weaker ambient category with essentially only one exponent. This more economical formulation gives the straightforward descriptions in Examples 5.2 above. Below one finds how every λK-algebra can be fully embedded into a CCC with a reflexive object.

The following is a straightforward generalization of Definition 5.1.

5.5. Definition. (i) A *λL-algebra* is given by a symmetric monoidal category (SMC) $\mathbb{B}$ containing a distinguished non-empty object Ω provided with a map $app: \Omega \otimes \Omega \to \Omega$ and operations $\lambda(-) : \mathbb{B}(X \otimes \Omega,\ \Omega) \to \mathbb{B}(X,\ \Omega)$ such that $app \circ \lambda(f) \otimes id = f$ and $\lambda(f \circ g \otimes id) = \lambda(f) \circ g$.

(ii) A *$\lambda\eta L$-algebra* is a λL-algebra in which $\lambda(app) = id$.

The next step is of course to add diagonals and projections.

5.6. Definition. (i) A *$\lambda(\eta)I$-algebra* is a $\lambda(\eta)L$-algebra with diagonals.

(ii) A *$\lambda(\eta)A$-algebra* is a $\lambda(\eta)L$-algebra with projections.

5.7. Interpretation. Let $(\mathbb{B}, \Omega)$ be a λL-algebra and let Γ be a context of length n; put $[\![\Gamma]\!] = \Omega^n$, where $\Omega^0 = I$ and $\Omega^{n+1} = \Omega^n \otimes \Omega$. A term $\Gamma \vdash M$ can be interpreted as a morphism $[\![M]\!]_\Gamma: [\![\Gamma]\!] \to \Omega$. We only describe abstraction and application. How to interprete the contraction and weakening rules may be found before Definition 4.1.

Assume $\Gamma, x \vdash M$ interpreted as $[\![M]\!]_{\Gamma,x}: [\![\Gamma]\!] \otimes \Omega \to \Omega$. One takes

$$[\![\lambda x.M]\!]_\Gamma \quad = \quad \lambda([\![M]\!]_{\Gamma,x}) \quad : \quad [\![\Gamma]\!] \quad \to \quad \Omega$$

Let's turn to application. Given terms $\Gamma \vdash M$ and $\Delta \vdash N$ in λL or λA with Γ, Δ disjoint, one puts

$$[\![MN]\!]_{\Gamma,\Delta} \quad = \quad app \;\circ\; [\![M]\!]_\Gamma \otimes [\![N]\!]_\Delta \quad : \quad [\![\Gamma]\!] \otimes [\![\Delta]\!] \quad \to \quad \Omega \otimes \Omega \quad \to \quad \Omega.$$

For the interpretation of application in λK or λI one needs diagonals: given terms $\Gamma, \Theta \vdash M$ and $\Theta, \Delta \vdash N$ with Γ, Δ disjoint, one takes

$$\begin{aligned} [\![MN]\!]_{\Gamma,\Theta,\Delta} \quad = \quad & app \;\circ\; [\![M]\!]_{\Gamma,\Theta} \otimes [\![N]\!]_{\Theta,\Delta} \;\circ\; id_{[\![\Gamma]\!]} \otimes \delta_{[\![\Theta]\!]} \otimes id_{[\![\Delta]\!]} \;: \\ & [\![\Gamma]\!] \otimes [\![\Theta]\!] \otimes [\![\Delta]\!] \;\to\; [\![\Gamma]\!] \otimes [\![\Theta]\!] \otimes [\![\Theta]\!] \otimes [\![\Delta]\!] \;\to\; \Omega \otimes \Omega \;\to\; \Omega. \end{aligned}$$

5.8. Examples. In analogy with Lemma 5.4 one obtains that a reflexive object in an SMCC (with diagonals/projections) yields an example of a λL-algebra (λI-/λA-algebra). Thus Proposition 4.2 provides the setting for the following examples.

Examples	$\mathcal{P}(X)$, X L-set $\mathcal{F}_L$	$\mathcal{P}(X)$, X A-set $\mathcal{F}_A$ $\mathcal{F}_\forall$	$\mathcal{P}(X)$, X I-set $\mathcal{F}_I$	$\mathcal{P}(X)$, X K-set $\mathcal{F}_K$ D_∞
reflexive object in	$\mathbb{CL}_{lin}$	$\mathbb{CL}_{aff}$	$\mathbb{CL}_{str}$	$\mathbb{CL}_{cont}$
model of	λL-calculus	λA-calculus	λI-calculus	λK-calculus

where $\mathcal{F}_\forall$ is the filter model of $\forall$-types from [8].

The following result occurs in [6], 4.9 (iii). It says that the notion of a λK-algebra can be described via a reflexive object in a richer environment, namely in a topos of presheaves. Later, we give similar results for the other calculi. We assume the CCC structure of a category of presheaves to be known, but will provide the SMCC definitions.

5.9. Proposition. *Let $\mathbb{B}$ be a category with finite products and Ω a non-empty object in $\mathbb{B}$. Then*

$$(\mathbb{B}, \Omega) \text{ is a } \lambda K\text{-algebra} \quad \Leftrightarrow \quad \mathbb{B}(\_, \Omega) \text{ is a reflexive object in } \widehat{\mathbb{B}} = \mathcal{S}ets^{\mathbb{B}^{op}}$$

Proof. Let's write $H = \mathbb{B}(\_, \Omega) \colon \mathbb{B}^{op} \to \mathcal{S}ets$. The exponent $H \Rightarrow H$ in $\widehat{\mathbb{B}}$ is described by

$$\begin{array}{rcl} (H \Rightarrow H)(X) & = & Nat\big(\mathbb{B}(\_, X) \times H,\ H\big) \\ & \cong & Nat\big(\mathbb{B}(\_, X \times \Omega),\ \mathbb{B}(\_, \Omega)\big) \\ & \cong & \mathbb{B}(X \times \Omega,\ \Omega) \end{array}$$

The latter by Yoneda. Thus $H = \mathbb{B}(\_, \Omega)$ is a reflexive object in $\widehat{\mathbb{B}}$ if and only if there are natural transformations F, G with components

$$\mathbb{B}(X \times \Omega,\ \Omega) \underset{F_X}{\overset{G_X}{\rightleftarrows}} \mathbb{B}(X,\ \Omega)$$

such that $F \circ G = id$.

Now assume that $(\mathbb{B}, \Omega)$ is a λK-algebra with app and $\lambda(\_)$. Put

$$\begin{array}{rcl} F_X(f \colon X \to \Omega) & = & app \ \circ \ f \times id \\ G_X(g \colon X \times \Omega \to \Omega) & = & \lambda(f) \end{array}$$

Conversely, given natural transformations F, G as above, take

$$\begin{array}{rcl} app & = & F_\Omega(id_\Omega) \\ \lambda(f \colon X \times \Omega \to \Omega) & = & G_X(f) \end{array}$$ □

One easily extends this result to a correspondence between $\lambda\eta K$-algebras and extensional reflexive objects. For the other calculi, we need the following basic result, due to Day. A clear exposition may be found in [1], 4.1.3. The second point is added here.

5.10. Lemma. *Let $\mathbb{B}$ be an SMC.*

(i) *The category of presheaves $\widehat{\mathbb{B}} = \mathcal{S}ets^{\mathbb{B}^{op}}$ is then an SMCC and the Yoneda embedding $\mathcal{Y} \colon \mathbb{B} \to \widehat{\mathbb{B}}$ preserves the SMC structure.*

(ii) *If $\mathbb{B}$ has diagonals/projections, then so has $\widehat{\mathbb{B}}$.*

Proof. (i) The tensor of presheaves $A, B \colon \mathbb{B}^{op} \to \mathcal{S}ets$ is given (at X) as coend in $\mathcal{S}ets$

$$(A \otimes B)(X) \quad = \quad \int^{(Y,Z)} A(Y) \times B(Z) \times \mathbb{B}(X,\ Y \otimes Z)$$

and the exponent by the end

$$\begin{array}{rcl} (A \multimap B)(X) & = & \int_Z B(Z \otimes X)^{A(X)} \\ & \cong & Nat\big(A,\ B(X \otimes \_)\big) \end{array}$$

Finally the presheaf $\mathcal{Y}(I) = \mathbb{B}(\_, I)$ is neutral for this tensor on $\widehat{\mathbb{B}}$.

(ii) If I is terminal in $\mathbb{B}$, then so is $\mathcal{Y}(I)$ in $\widehat{\mathbb{B}}$. This settles the case of projections. For diagonals, notice that the above coend comes together with a universal wedge

$$A(Y) \times B(Z) \times \mathbb{B}(X,\ Y \otimes Z) \xrightarrow{\omega(X)_{(Z,Y)}} (A \otimes B)(X)$$

which enables us to form a diagonal in $\widehat{\mathbb{B}}$ with components

$$\boldsymbol{\lambda} a.\ \omega(X)_{(X,X)}(a, a, \delta_X) \quad : \quad A(X) \quad \to \quad (A \otimes A)(X).$$ □

5.11. Proposition. *Let $\mathbb{B}$ be an SMC and $\Omega \in \mathbb{B}$ a non-empty object. Reflexive objects below are with respect to the above implication $\multimap$ on $\widehat{\mathbb{B}} = \mathcal{S}ets^{\mathbb{B}^{op}}$.*

(i) *One has that*

$$(\mathbb{B}, \Omega) \text{ is a } \lambda L\text{-algebra} \quad \Leftrightarrow \quad \mathbb{B}(\_, \Omega) \text{ is a reflexive object in } \widehat{\mathbb{B}}$$

(ii) *The same holds for λA- and λI-algebras.*

Proof. (i) Note that $\mathbb{B}(\_, \Omega)$ is reflexive in $\widehat{\mathbb{B}}$ if and only if there are natural transformations F, G with components

$$Nat\big(\mathbb{B}(\_, \Omega),\ \mathbb{B}(X \otimes \_, \Omega)\big) \underset{F_X}{\overset{G_X}{\rightleftarrows}} \mathbb{B}(X,\ \Omega)$$

satisfying $F \circ G = id$. These can be obtained from app and $\lambda(\_)$ by

$$\begin{array}{rcl} \big(F_X(f)\big)_Y(g) & = & app \ \circ \ f \otimes g \\ G_X(\sigma) & = & \lambda\big(\sigma_\Omega(id_\Omega)\big) \end{array}$$

Conversely, given F and G one obtains app and $\lambda(\_)$ by

$$\begin{array}{rcl} app & = & \big(F_\Omega(id_\Omega)\big)_\Omega(id_\Omega) \\ \lambda(f \colon X \otimes \Omega \to \Omega) & = & G_X(\boldsymbol{\lambda} g.\ f \circ id \otimes g) \end{array}$$

(ii) By (ii) in the previous Lemma and (i) above using that the implication $\multimap$ remains the same. □

In case the SMC structure on $\mathbb{B}$ is cartesian, the induced SMCC structure on $\widehat{\mathbb{B}}$ described in Lemma 5.10 becomes the familiar CCC structure used in Proposition 5.9. Thus the results 5.9 and 5.11 are completely uniform.

6 Set theoretic versus categorical

The more set theoretic oriented reader may want to know what *set theoretic* λI-, λA- or λL-algebras are. One expects definitions similar to Definition 5.2.2 in [3] of (set theoretic) λK-algebras. Since the latter is aestethically so unappealing, we don't bother about this and contend ourselves with the clear and concise categorical definitions of λI-, λA- and λL-algebras.

In order to massage away remaining feelings of (set theoretic) discomfort we intend to do the following. Let's assume that we do understand what a set theoretic λI- , λL- or λA-algebra is. We suppose it is an applicative structure $\langle D, \cdot\rangle$ with appropriate combinators ($\boldsymbol{I}, \boldsymbol{B}, \boldsymbol{C}$ for λL; additionally $\boldsymbol{S}$ for λI and $\boldsymbol{K}$ for λA) in which one can interprete the terms of the corresponding calculi. As usual, we reason directly in D. The following construction produces categorical λI-, λA- and λL-algebras from such data; it generalizes Example 5.2 (i).

6.1. Definition. Let $\diamondsuit \in \{L, A, I, K\}$ and $\langle D, \cdot\rangle$ be a "set theoretic $\lambda\diamondsuit$-algebra". We form a category $\mathbb{B}_\diamondsuit$ with $n \in \mathbb{N}$ as objects. A morphism $n \to m$ in $\mathbb{B}_\diamondsuit$ consists of an element a of D of the form

$$a \;=\; \lambda z x_1 \ldots x_n.\, z\big(a_1(\vec{x})_1\big) \ldots \big(a_m(\vec{x})_m\big) \qquad (*)$$

where $a_1, \ldots a_m$ are elements of D called *factors* and $(\vec{x})_1 \ldots (\vec{x})_m$ are subsequences of $x_1, \ldots, x_n$ in such a way that the term $(*)$ exists in the $\lambda\diamondsuit$-calculus. Composition of $a{:}\, n \to m$ and $b{:}\, m \to k$ is given by $b \circ a = \boldsymbol{B}ab = \lambda z.\, a(bz)$. It is thus associative. The identity on n is $\mathbf{1}_n = \lambda z x_1 \ldots x_n.\, z x_1 \ldots x_n$ (with factors $\boldsymbol{I}$).

6.2. Lemma. (i) *Each category $\mathbb{B}_\diamondsuit$ has a (strict) monoidal structure with $n \otimes m = n + m$.*

(ii) *The monoidal structure has projections in $\mathbb{B}_A$, has diagonals in $\mathbb{B}_I$ and is cartesian in $\mathbb{B}_K$.*

Proof. (i) For

$$\begin{array}{rcll} a & = & \lambda z x_1 \ldots x_n.\, z\big(a_1(\vec{x})_1\big) \ldots \big(a_l(\vec{x})_l\big) & : \; n \;\to\; l \\ b & = & \lambda z y_1 \ldots y_m.\, z\big(b_1(\vec{y})_1\big) \ldots \big(b_k(\vec{y})_k\big) & : \; m \;\to\; k \end{array}$$

define $a \otimes b : n \otimes m \to l \otimes k$ by

$$a \otimes b \;=\; \lambda z x_1 \ldots x_n y_1 \ldots y_m.\, z\big(a_1(\vec{x})_1\big) \ldots \big(a_l(\vec{x})_l\big)\big(b_1(\vec{y})_1\big) \ldots \big(b_k(\vec{y})_k\big)$$

i.e. $a \otimes b = \lambda z \vec{x} \vec{y}.\, b(az\vec{x})\vec{y}$. The object 0 is neutral element for this tensor. Further, symmetry is given by

$$\lambda z x_1 \ldots x_n y_1 \ldots y_m.\, z y_1 \ldots y_m x_1 \ldots x_n \;:\; n \otimes m \;\to\; m \otimes n.$$

(ii) We need to check that 0 is terminal in $\mathbb{B}_A$. The unique arrow $n \to 0$ in $\lambda z x_1 \ldots x_n.\, z$. Notice that weakening is used in an essential way to form this morphism. The resulting projections $\pi{:}\, n \otimes m \to n$ and $\pi'{:}\, n \otimes m \to m$ are

$$\pi \;=\; \lambda z x_1 \ldots x_n y_1 \ldots y_m.\, z x_1 \ldots x_n \qquad \pi' \;=\; \lambda z x_1 \ldots x_n y_1 \ldots y_m.\, z y_1 \ldots y_m.$$

Further, the category $\mathbb{B}_I$ has diagonals because one can use contraction:

$$\lambda z x_1 \ldots x_n.\, z x_1 \ldots x_n x_1 \ldots x_n \;:\; n \;\to\; n \otimes n.$$

In the category $\mathbb{B}_K$ these projections and diagonals both exist and yield cartesian products. □

6.3. Proposition. *For $\diamondsuit \in \{L, A, I, K\}$ one has that the pair $(\mathbb{B}_\diamondsuit, 1)$ is a $\lambda\diamondsuit$-algebra.*

Proof. In all four cases one has

$$app \;=\; \boldsymbol{B} \;=\; \lambda z x y.\, z(xy) \;:\; 1 \otimes 1 \;\to\; 1$$

with factor $\mathbf{1}_1 = \lambda xy.\, xy$. For $a{:}\, n \otimes 1 \to 1$ of the form $a = \lambda z x_1 \ldots x_n x_{n+1}.\, z\big(a_1(\vec{x})_1\big)$ one obtains an abstraction morphism

$$\lambda(a) \;=\; \lambda z x_1 \ldots x_n.z\big(\lambda x_{n+1}.\, a_1(\vec{x})_1\big) \;:\; n \;\to\; 1.$$ □

One easily establishes that the above category $\mathbb{B}_K$ is isomorphic to the category $\mathbb{D}$ described in Example 5.2 (i).

7 Final Remarks

Soon after investigating the matters reported here, I realized that the phenomena involved have a wider significance and give rise to a refinement of some aspect of Girard's linear logic, see [7]. The crucial "shriek" operation ! of linear logic reinstates weakening and contraction in annotated form. One can read $!A$ as "A, as many times as you like". This ! can be split into two operations: $\underset{w}{!}$ for weakening and $\underset{c}{!}$ for contraction. These can be read as

$\underset{w}{!}A$ means A, at *most* once $\qquad\qquad$ $\underset{c}{!}A$ means A, at *least* once

This distinction at least once/at most once underlies all models given above. In a linear setting where one has a linear implication $\multimap$, one can describe a model Ω for one of our untyped substructure λ-calculi as a retract of the following form.

$$\begin{array}{llcll} \lambda K & (!\,\Omega \multimap \Omega) \lhd \Omega & \qquad & \lambda A & (\underset{w}{!}\,\Omega \multimap \Omega) \lhd \Omega \\ \lambda I & (\underset{c}{!}\,\Omega \multimap \Omega) \lhd \Omega & & \lambda L & (\Omega \multimap \Omega) \lhd \Omega \end{array}$$

For more details on $\underset{w}{!}$ and $\underset{c}{!}$, see [7].

Interesting operational aspects crop up at this point. Under the above reading, a function $f{:}\, \underset{w}{!}A \to B$ uses its argument at *most* once. Hence in computing an application $f\,a$ one better first fully evaluates f—instead of a—because a may be thrown away. Similarly, a function $g{:}\, \underset{c}{!}A \to B$ uses its argument at *least* once. So in computing $g\,a$ one better computes a first. Such a type system involving $\underset{w}{!}$ and $\underset{c}{!}$ contains information about the reduction strategy to be followed.

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