Bases as Coalgebras*

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Abstract. The free algebra adjunction, between the category of algebras of a monad and the underlying category, induces a comonad on the category of algebras. The coalgebras of this comonad are the topic of study in this paper (following earlier work). It is illustrated how such coalgebras-on-algebras can be understood as bases, decomposing each element x into primitives elements from which x can be reconstructed via the operations of the algebra. This holds in particular for the free vector space monad, but also for other monads. For instance, continuous dcpos or stably continuous frames, where each element is the join of the elements way below it, can be described as such coalgebras. Further, it is shown how these coalgebras-on-algebras give rise to a comonoid structure for copy and delete, and thus to diagonalisation of endomaps like in linear algebra.

1 Introduction

In general, algebras are used for composition and coalgebras for decomposition. An algebra $a\colon T(X) \to X$, for a functor or a monad T, can be used to produce elements in X from ingredients structured by T. Conversely, a coalgebra $c\colon X \to T(X)$ allows one to decompose an element in X into its ingredients with structure according to T. This is the fundamental difference between algebraic and coalgebraic data structures. In this paper we apply this view to the special situation where one has a coalgebra of a comonad on top of an algebra of a monad, where the comonad is canonically induced by the monad, namely as arising from the free algebra adjunction, see (1) below. Here it is proposed that such coalgebras can be seen as bases. In particular, it will be shown that the concept of basis in linear algebra gives rise to such a coalgebra $X \to \mathcal{M}(X)$ for the multiset monad $\mathcal{M}$; this coalgebra decomposes an element x of a vector space X into a formal sum $\sum_i x_i e_i \in \mathcal{M}(X)$ given by its coefficients x_i for a Hamel basis (e_i), see Theorem 1 for more details.

Other examples arise in an order-theoretic setting, formalised via the notion of monad of Kock-Zölberlein type (where $T(\eta_X) \leq \eta_{TX}$, see [14, 7]). We describe how they fit in the present setting (with continuous dcpos as coalgebras), and add a new result (Theorem 4) about algebras-on-coalgebras-on-algebras, see Section 4. This builds on rather old (little noticed) work of the author [10].

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In recent work [5] in the categorical foundations of quantum mechanics it is shown that orthonormal bases in finite-dimensional Hilbert spaces are equivalent to comonoids structures (in fact, Frobenius algebras). These comonoids are used for copying and deleting elements. In Section 5 it is shown how bases as coalgebras (capturing bases-as-decomposition) also give rise to such comonoids (capturing bases-as-copier-and-deleter). These comonoids can be used to formulate in general terms what it means for an endomap to be diagonalisable. This is illustrated for the Pauli functions.

2 Comonads on categories of algebras

In this section we investigate the situation of a monad and the induced comonad on its category of algebras. We shall see that coalgebras of this comonad capture the notion of basis, in a very general sense. This will be illustrated later in several situations see in particular Subsection 3.2.

For an arbitrary monad $T\colon \mathbf{A} \to \mathbf{A}$, with unit η and multiplication μ, there is a category $\mathit{Alg}(T)$ or (Eilenberg-Moore) algebras, together with a left adjoint F (for free algebra functor) to the forgetful functor $U\colon \mathit{Alg}(T) \to \mathbf{A}$. This adjunction $\mathit{Alg}(T) \leftrightarrows \mathbf{A}$ induces a comonad on the category $\mathit{Alg}(T)$, which we shall write as $\overline{T} = FU$ in:

$$
\begin{tikzcd}
\mathit{Alg}(T) \arrow[loop right, "\overline{T}=FU\ \text{comonad}"] \arrow[d, "U"] \arrow[d, phantom, shift right=2ex, "\dashv"] \\
\mathbf{A} \arrow[u, bend left=60, "F"] \arrow[loop right, "T=UF\ \text{monad}"]
\end{tikzcd}
\tag{1}
$$

For an algebra $(TX \xrightarrow{a} X) \in \mathit{Alg}(T)$ there are counit $\varepsilon\colon \overline{T} \Rightarrow \mathrm{id}$ and comultiplication $\delta\colon \overline{T} \Rightarrow \overline{\overline{T}}$ maps in $\mathit{Alg}(T)$ given by:

$$
\begin{pmatrix} TX \\ \downarrow a \\ X \end{pmatrix} \xleftarrow{\quad \varepsilon = a \quad} \begin{pmatrix} T^2X \\ \downarrow \mu_X \\ TX \end{pmatrix} \xrightarrow{\quad \delta = T(\eta_X) \quad} \begin{pmatrix} T^3X \\ \downarrow \mu_{TX} \\ T^2X \end{pmatrix}
\tag{2}
$$

Definition 1. *Consider a monad $T\colon \mathbf{A} \to \mathbf{A}$ together with the induced comonad $\overline{T}\colon \mathit{Alg}(T) \to \mathit{Alg}(T)$ as in (1). A* basis *for a T-algebra $(TX \xrightarrow{a} X) \in \mathit{Alg}(T)$ is a $\overline{T}$-coalgebra on this algebra, given by a map of algebras b of the form:*

$$
\begin{pmatrix} TX \\ \downarrow a \\ X \end{pmatrix} \xrightarrow{\qquad b \qquad} \overline{T} \begin{pmatrix} TX \\ \downarrow a \\ X \end{pmatrix} = FU \begin{pmatrix} TX \\ \downarrow a \\ X \end{pmatrix} = \begin{pmatrix} T^2X \\ \downarrow \mu_X \\ TX \end{pmatrix}
$$

Thus, a basis b is a map $X \xrightarrow{b} TX$ in $\mathbf{A}$ *satisfying $b \circ a = \mu_X \circ T(b)$ and $a \circ b = \mathrm{id}$ and $T(\eta_X) \circ b = T(b) \circ b$ in:*

$$\begin{tikzcd} T(X) \arrow[r, "T(b)"] \arrow[d, "a"'] & T^2(X) \arrow[d, "\mu_X"] \\ X \arrow[r, "b"'] & T(X) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "b"] \arrow[dr, equal] & T(X) \arrow[d, "\varepsilon = a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T(X) \arrow[r, "T(b)"] & T^2(X) \\ X \arrow[u, "b"] \arrow[r, "b"'] & T(X) \arrow[u, "\delta = T(\eta_X)"'] \end{tikzcd}$$

As we shall see a basis as described above may be understood as providing a decomposition of each element x into a collection $b(x)$ of basic elements that together form x. The actual basic elements $X_b \rightarrowtail X$ involved can be obtained as the indecomposable ones, via the following equaliser in the underlying category.

$$\begin{tikzcd} X_b \arrow[r, tail, "e"] & X \arrow[r, shift left, "b"] \arrow[r, shift right, "\eta"'] & TX \end{tikzcd} \tag{3}$$

One can then ask in which cases the map of algebras $T(X_b) \to X$, induced by the equaliser $e\colon X_b \to U(TX \to X)$, is an isomorphism. This is (almost always) the case for monads on **Sets**, see Proposition 1 below. But first we observe that free algebras always carry a basis.

Lemma 1. *Free algebras have a canonical basis: each $FX = \big(T^2X \xrightarrow{\mu} TX\big) \in \mathit{Alg}(T)$ carries a $\overline{T}$-coalgebra, namely given by $T(\eta_X)$. This gives a situation:*

$$\begin{tikzcd} \mathit{CoAlg}(\overline{T}) \arrow[r] & \mathit{Alg}(T) \arrow[loop right, "\overline{T}"] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] \\ & \mathbf{A} \arrow[ul, "F"] \arrow[u, bend left, "F"] \end{tikzcd}$$

Proof It is easy to check that $T(\eta_X)$ is a morphism in $\mathit{Alg}(T)$ and a $\overline{T}$-coalgebra:

$$F(X) = \begin{pmatrix} T^2X \\ \downarrow \mu_X \\ TX \end{pmatrix} \xrightarrow{\quad T(\eta_X) \quad} \begin{pmatrix} T^3X \\ \downarrow \mu_{TX} \\ T^2X \end{pmatrix} = \overline{T}(FX). \qquad \square$$

The object X_b of basic element, as in (3), in the situation of this lemma is the original set X in case the monad T satisfies the so-called *equaliser requirement* [16], which says precisely that $\eta_X\colon X \to TX$ is the equaliser of $T(\eta_X), \eta_{TX}\colon TX \rightrightarrows T^2X$.

The comonad $\overline{T}\colon \mathit{Alg}(T) \to \mathit{Alg}(T)$ from (1) gives rise to a category of coalgebras $\mathit{CoAlg}(\overline{T}) \to \mathit{Alg}(T)$, where this forgetful functor has a right adjoint, which maps an algebra $TY \to Y$ to the diagonal coalgebra $\delta\colon \mu_Y \to \mu_{TY}$ as in (2). Thus we obtain a monad on the category $\mathit{CoAlg}(\overline{T})$, written as $\overline{\overline{T}}$. On a basis $c\colon a \to \overline{T}(a)$, for an algebra $a\colon TX \to X$, there is a unit $\eta_c = c\colon c \to \delta$ and multiplication $\mu_c = T(c)\colon \delta \to \delta$ in $\mathit{CoAlg}(\overline{T})$.

By iterating this construction one obtains alternating monads and comonads. Such iterations are studied for instance in [3, 10, 14, 18]. In special cases it is known that the iterations stop after a number of cycles. This happens after 2 iterations for monads on sets, as we shall see next, and after 3 iterations for Kock-Zölberlein monads in Section 4.

3 Set-theoretic examples

It turns out that for monads on the category **Sets** only free algebras have bases. This result goes back to [3]. We repeat it in the present context, with a sketch of proof. Subsequently we describe the situation for the powerset monad (from [10]) and the free vector space monad.

Proposition 1. *For a monad T on* **Sets***, if an algebra $TX \xrightarrow{a} X$ has a basis $X \xrightarrow{b} TX$ with non-empty equaliser $X_b \rightarrowtail X \rightrightarrows TX$ as in (3), then the induced map $T(X_b) \to X$ is an isomorphism of algebras and coalgebras. In particular, in the set-theoretic case any algebra with a non-empty basis is free.*

Proof Let's consider the equaliser $X_b \rightarrowtail X$ of $b, \eta \colon X \rightrightarrows T(X)$ from (3) in **Sets**. It is a so-called coreflexive equaliser, because there is a map $TX \to X$, namely the algebra a, satisfying $a \circ b = \mathrm{id} = a \circ \eta$. It is well-known—see *e.g.* [15, Lemma 6.5] or the dual result in [4, Volume I, Example 2.10.3.a]—that if $X_b \neq \emptyset$ such coreflexive equalisers in **Sets** are split, and thus absolute. The latter means that they are preserved under any functor application. In particular, by applying T we obtain a new equaliser in **Sets**, of the form:

$$\begin{tikzcd} T(X_b) \arrow[r, tail, "T(e)"] & T(X) \arrow[r, shift left, "T(b)"] \arrow[r, shift right, "T(\eta)=\delta"'] & T^2(X) \\ X \arrow[u, dashed, "b'"] \arrow[ur, "b"'] & & \end{tikzcd} \tag{4}$$

The resulting b' is the inverse to the adjoint transpose $a \circ T(e) \colon T(X_b) \to X$, since:

- $a \circ T(e) \circ b' = a \circ b = \mathrm{id}$;
- the other equation follows because $T(e)$ is equaliser, and thus mono:

$$\begin{aligned} T(e) \circ b' \circ a \circ T(e) &= b \circ a \circ T(e) \\ &= \mu \circ T(b) \circ T(e) && \text{see Definition 1} \\ &= \mu \circ T(\eta) \circ T(e) && \text{since } e \text{ is equaliser} \\ &= T(e) \;=\; T(e) \circ \mathrm{id}. \end{aligned}$$

Hence the homomorphism of algebras $a \circ T(e)$, from $F(X_b) = \mu_{X_b}$ to a is an isomorphism. In particular, $b' \colon X \to T(X_b)$ in (4) is a map of algebras, as inverse of an isomorphism of algebras. It is not hard to see that it is also an isomorphism between the coalgebras $b \colon X \to T(X)$ and $T(\eta) \colon T(X_b) \to T^2(X_b)$, as in Lemma 1. □

3.1 Complete lattices

Consider the powerset monad $\mathcal{P}$ on **Sets**, with the category $\mathbf{CL} = Alg(\mathcal{P})$ of complete lattices and join-preserving maps as its category of algebras. The induced comonad $\overline{\mathcal{P}}\colon \mathbf{CL} \to \mathbf{CL}$ as in (1) sends a complete lattice $(L, \leq)$ to the lattice $(\mathcal{P}(L), \subseteq)$ of subsets, ignoring the original order $\leq$. The counit $\varepsilon\colon \overline{\mathcal{P}}(L) \to L$ sends a subset $U \in \mathcal{P}(L)$ to its join $\varepsilon(U) = \bigvee U$; the comultiplication $\delta\colon \overline{\mathcal{P}}(L) \to \overline{\mathcal{P}}^2(L)$ sends $U \in \mathcal{P}(L)$ to the subset of singletons $\delta(U) = \{\{x\} \mid x \in U\}$.

An (Eilenberg-Moore) coalgebra of the comonad $\overline{\mathcal{P}}$ on **CL** is a map $b\colon L \to \overline{\mathcal{P}}(L)$ in **CL** satisfying $\varepsilon \circ b = \mathrm{id}$ and $\delta \circ b = \overline{\mathcal{P}}(b) \circ b$. More concretely, this says that $\bigvee b(x) = x$ and $\{\{y\} \mid y \in b(x)\} = \{b(y) \mid y \in b(x)\}$. It is then shown in [10] that a complete lattice L carries such a coalgebra structure b if and only if L is *atomic*, where $b(x) = \{a \in L \mid a \text{ is an atom with } a \leq x\}$. Thus, such a coalgebra of the comonad $\overline{\mathcal{P}}$, if it exists, is uniquely determined and gives a decomposition of lattice elements into the atoms below it. The atoms in the lattice thus form a basis.

(The complete lattice L is atomic when each element is the join of the atoms below it. And an atom $a \in L$ is a non-zero element with no non-zero elements below it, satisfying: $a \leq \bigvee U$ implies $a \leq x$ for some $x \in U$.)

The equaliser (3) for the basic elements in this situation, for an atomic complete lattice L, is the set of atoms:

$$X_b = \{x \in L \mid \{x\} = b(x)\} = \{x \in L \mid x \text{ is an atom}\}.$$

If $X_b \neq \emptyset$, the induced map $\mathcal{P}(X_b) \to L$ is an isomorphism, by Lemma 1.

3.2 Vector spaces

For a semiring S one can define the multiset monad $\mathcal{M}_S$ on **Sets** by $\mathcal{M}_S(X) = \{\varphi\colon X \to S \mid \mathrm{supp}(\varphi) \text{ is finite}\}$. Such an element φ can be identified with a formal finite sum $\sum_i s_i x_i$ with multiplicities $s_i \in S$ for elements $x_i \in X$. The category of algebras $Alg(\mathcal{M}_S)$ of the multiset monad $\mathcal{M}_S$ is the category of $\mathbf{Mod}_S$ of modules over S: commutative monoids with S-scalar multiplication, see *e.g.* [6] for more information. The induced comonad $\overline{\mathcal{M}_S}\colon \mathbf{Mod}_S \to \mathbf{Mod}_S$ from (1) sends such a module $X = (X, +, 0, \bullet)$ to the free module $\mathcal{M}_S(X)$ of finite multisets (formal sums) on the underlying set X, ignoring the existing module structure on X. The counit and comultiplication are given by:

$$\begin{array}{ccccc} X & \xleftarrow{\quad\varepsilon\quad} & \mathcal{M}_S(X) & \xrightarrow{\quad\delta\quad} & \mathcal{M}_S^2(X) \\ \big(\sum_j s_j \bullet x_j\big) & \longleftarrow\!\shortmid & \big(\sum_j s_j x_j\big) & \longmapsto & \big(\sum_j s_j (1x_j)\big). \end{array} \tag{5}$$

The formal sum (multiset) in the middle is mapped by the counit ε to an actual sum in X, namely to its interpretation. The comultiplication δ maps this formal sum to a multiset of multisets, with the inner multisets given by singletons $1x_j = \eta(x_j)$.

The following is a novel observation, motivating the view of coalgebras on algebras as bases.

Theorem 1. *Let X be a vector space, say over $S = \mathbb{R}$ or $S = \mathbb{C}$. Coalgebras $X \to \overline{\mathcal{M}_S}(X)$ correspond to (Hamel) bases on X.*

Proof Suppose we have a basis $B \subseteq X$. Then we can define a coalgebra $b\colon X \to \overline{\mathcal{M}_S}(X)$ via (finite) formal sums $b(x) = \sum_j s_j a_j$, where $s_j \in S$ is the j-th coefficient of x wrt $a_j \in B \subseteq X$. By construction we have $\varepsilon \circ b = \mathrm{id}$. The equation $\delta \circ b = \mathcal{M}_S(b) \circ b$ holds because $b(a) = 1a$, for basic elements $a \in B$.

Conversely, given a coalgebra $b\colon X \to \overline{\mathcal{M}_S}(X)$ take $X_b = \{a \in X \mid b(a) = 1a\}$ as in (3). Any finite subset of elements of X_b is linearly independent: if $\sum_j s_j \bullet a_j = 0$, for finitely many $a_j \in X_b$, then in $\mathcal{M}_S(X)$,

$$0 \;=\; b(0) \;=\; b(\textstyle\sum_j s_j \bullet a_j) \;=\; \textstyle\sum_j s_j b(a_j) \;=\; \textstyle\sum_j s_j(1a_j) \;=\; \textstyle\sum_j s_j a_j.$$

Hence $s_j = 0$, for each j. Next, since $\delta \circ b = \mathcal{M}_S(b) \circ b$, each a_j in $b(x) = \sum_j s_j a_j$ satisfies $b(a_j) = 1a_j$, so that $a_j \in X_b$. Because $\varepsilon \circ b = \mathrm{id}$, each element $x \in X$ can be expressed as sum of such basic elements. □

A basis for complete lattices in Subsection 3.1, if it exists, is uniquely determined. In the context of vector spaces bases are unique up to isomorphism.

4 Order-theoretic examples

Assume $\mathbf{C}$ is a poset-enriched category. This means that all homsets $\mathbf{C}(X,Y)$ are posets, and that pre- and post-composition are monotone. In this context maps $f\colon X \to Y$ and $g\colon Y \to X$ in opposite direction form an adjunction $f \dashv g$ (or Galois connection) if there inequalities $\mathrm{id}_X \leq g \circ f$ and $f \circ g \leq \mathrm{id}_Y$, corresponding to unit and counit of the adjunction. In such a situation the adjoints f, g determine each other.

A monad $T = (T, \eta, \mu)$ on such a poset-enriched category $\mathbf{C}$ is said to be of *Kock-Zölberlein type* or just a *Kock-Zölberlein monad* if $T\colon \mathbf{C}(X,Y) \to \mathbf{C}(TX,TY)$ is monotone and $T(\eta_X) \leq \eta_{TX}$ holds in the homset $\mathbf{C}\big(T(X), T^2(X)\big)$. This notion is introduced in [14] in proper 2-categorical form. Here we shall use the special 'poset' instance—like in [7] where the dual form occurs. The following result goes back to [14]; for convenience we include the proof.

Theorem 2. *Let T be a Kock-Zölberlein monad on a poset-enriched category $\mathbf{C}$. For a map $a\colon T(X) \to X$ in $\mathbf{C}$ the following statements are equivalent.*

1. *$a\colon T(X) \to X$ is an (Eilenberg-Moore) algebra of the monad T;*
2. *$a\colon T(X) \to X$ is a left-adjoint-left-inverse of the unit $\eta\colon X \to T(X)$; this means that $a \dashv \eta_X$ is a reflection.*

Proof First assume $a\colon T(X) \to X$ is an algebra, *i.e.* satisfies $a \circ \eta = \mathrm{id}$ and $a \circ \mu = a \circ T(a)$. It suffices to prove $\mathrm{id} \leq \eta \circ a$, corresponding to the unit of the reflection, since the equation $a \circ \eta = \mathrm{id}$ is the counit (isomorphism). This is easy, by naturality: $\eta \circ a = T(a) \circ \eta \geq T(a) \circ T(\eta) = \mathrm{id}$.

In the other direction, assume $a\colon T(X) \to X$ is left-adjoint-left-inverse of the unit $\eta\colon X \to T(X)$, so that $a \circ \eta = \mathrm{id}$ and $\mathrm{id} \leq \eta \circ a$. We have to prove $a \circ \mu = a \circ T(a)$. In one direction, we have:

$$\mu \;\leq\; T(a), \tag{6}$$

since $\mu \leq \mu \circ T(\eta \circ a) = T(a)$, and thus $a \circ \mu \leq a \circ T(a)$. For the reverse inequality we use:

$$\begin{array}{rcl}
a \circ T(a) \;=\; a \circ T(a) \circ T(\mathrm{id}) & = & a \circ T(a) \circ T(\mu) \circ T(\eta) \\
& \leq & a \circ T(a) \circ T(\mu) \circ \eta \qquad\qquad \text{since } T(\eta) \leq \eta \\
& = & a \circ \eta \circ a \circ \mu \qquad\qquad\qquad\;\; \text{by naturality} \\
& = & a \circ \mu.
\end{array}$$

□

In a next step we consider the induced comonad $\overline{T}$ on the category $\mathit{Alg}(T)$ of algebra of a Kock-Zöberlein monad T. A first, trivial but important, observation is that the category $\mathit{Alg}(T)$ is also poset enriched. It is not hard to see that the comonad $\overline{T}$ is also of Kock-Zöberlein type, in the sense that for each algebra $(TX \overset{a}{\to} X)$ we have:

$$\varepsilon_{\overline{T}(a)} = \mu \leq T(a) = \overline{T}(\varepsilon_a)$$

by (6). Thus one may expect a result similar to Theorem 2 for coalgebras of this comonad $\overline{T}$. It is formulated in [14, Thm. 4.2] (and attributed to the present author). We repeat the poset version in the current context.

Theorem 3. *Let T be a Kock-Zölberlein monad on a poset-enriched category $\mathbf{C}$, with induced comonad $\overline{T}$ on the category of algebras $\mathrm{Alg}(T)$. Assume an algebra $a\colon T(X) \to X$. For a map $c\colon X \to T(X)$, forming a map of algebras in,*

$$\begin{pmatrix} TX \\ \downarrow{\scriptstyle a} \\ X \end{pmatrix} \xrightarrow{\quad c \quad} \overline{T}\begin{pmatrix} TX \\ \downarrow{\scriptstyle a} \\ X \end{pmatrix} = \begin{pmatrix} T^2X \\ \downarrow{\scriptstyle \mu_X} \\ TX \end{pmatrix} \tag{7}$$

the following statements are equivalent.

1. *$c\colon a \to \overline{T}(a)$ is an (Eilenberg-Moore) coalgebra of the comonad $\overline{T}$;*
2. *$c\colon a \to \overline{T}(a)$ is a left-adjoint-right-inverse of the counit $a\colon \overline{T}(a) \to a$; this means that $c \dashv a$ is a coreflection.*

Proof Assume c is a $\overline{T}$-coalgebra, *i.e.* $c \circ a = \mu \circ T(c)$, $a \circ c = \mathrm{id}$ and $T(\eta) \circ c = T(c) \circ c$. We have to prove $c \circ a \leq \mathrm{id}$, which is obtained in:

$$c \circ a = \mu \circ T(c) \overset{(6)}{\leq} T(a) \circ T(c) = \mathrm{id}.$$

Conversely, assume a coreflection $c \dashv a$, so that $a \circ c = \mathrm{id}$ and $c \circ a \leq \mathrm{id}$. We have to prove $T(\eta) \circ c = T(c) \circ c$. In one direction we have $T(c) \leq T(\eta \circ a) \circ T(c) = T(\eta)$, and thus $T(c) \circ c \leq T(\eta) \circ c$. In the other direction, we use:

$$\begin{array}{rcl} T(c) \circ c \;=\; T^2(\mathrm{id}) \circ T(c) \circ c & = & T^2(a \circ \eta) \circ T(c) \circ c \\ & \leq & T^2(a) \circ T(\eta) \circ T(c) \circ c \qquad \text{since } T(\eta) \leq \eta \\ & = & T^2(a) \circ T^2(c) \circ T(\eta) \circ c \qquad \text{by naturality} \\ & = & T(\eta) \circ c. \end{array}$$

□

As mentioned, one can iterate the $\overline{(-)}$ construction. Below we show that for Kock-Zölberlein monads the iteration stops after 3 steps. First we need another characterisation. The proof is as before.

Lemma 2. *Let T be a Kock-Zölberlein monad on a poset-enriched category $\mathbf{C}$, giving rise to comonad $\overline{T}$ on $Alg(T)$ and monad $\overline{\overline{T}}$ on $CoAlg(\overline{T})$. Assume:*

- *an algebra $a\colon T(X) \to X$ in $Alg(T)$;*
- *a coalgebra $c\colon X \to T(X)$ on a in $CoAlg(\overline{T})$;*
- *an algebra $b\colon T(X) \to X$ on c in $Alg(\overline{\overline{T}})$, where:*
  - *$b \circ c = \mathrm{id}$ and $b \circ T(b) = b \circ T(a)$, since b is a $\overline{\overline{T}}$-algebra;*
  - *$a \circ T(b) = b \circ \mu$, since b is a map of algebras $a \to \overline{T}(a) = \mu$;*
  - *$c \circ b = T(b) \circ T(\eta)$, since b is a map of algebras $\delta = \overline{\overline{c}} \to c$.*

The following statements are then equivalent.

1. *$b\colon \overline{\overline{T}}(c) \to c$ is an algebra of the monad $\overline{\overline{T}}$;*
2. *$b\colon \overline{\overline{T}}(c) \to c$ is a left-adjoint-left-inverse of the unit $c\colon c \to \overline{\overline{T}}(c)$.* □

The next result shows how such series of adjunctions can arise.

Lemma 3. *Assume an algebra $a\colon T(X) \to X$ of a Kock-Zöberlein monad. The free algebra $T(X)$ then carries multiple (co)reflections (algebras and coalgebras) in a situation:*

$$\begin{tikzcd} T^2(X) \arrow[d, bend right=80, "T(a)"'] \arrow[d, bend right=20, "\mu"'] \\ T(X) \arrow[u, bend right=45, "T(\eta)"'] \arrow[u, "\eta"'] \end{tikzcd} \qquad T(a) \dashv T(\eta) \dashv \mu \dashv \eta \tag{8}$$

This yields a functor $T\colon Alg(T) \to Alg(\overline{\overline{T}})$ between categories of algebras.

Proof We check all (co)reflections from right to left.

- In the first case the counit is the identity since $\mu \circ \eta = \mathrm{id}$; because $T(\eta) \leq \eta$ for a Kock-Zöberlein monad, we get a unit $\eta \circ \mu = T(\mu) \circ \eta \geq T(\mu) \circ T(\eta) = \mathrm{id}$. (This follows already from Theorem 2.)
- In the next case we have a coreflection $T(\eta) \dashv \mu$ since the unit is the identity $\mu \circ T(\eta) = \mathrm{id}$, and: $T(\eta) \circ \mu = \mu \circ T^2(\eta) \leq \mu \circ T(\eta) = \mathrm{id}$.

– Finally one gets a reflection $T(a) \dashv T(\eta)$ from the reflection $a \dashv \eta$ from Theorem 2: $T(a) \circ T(\eta) = T(a \circ \eta) = \mathrm{id}$ and $T(\eta) \circ T(a) = T(\eta \circ a) \geq T(\mathrm{id}) = \mathrm{id}$. □

This lemma describes the only form that such structures can have. This is the main (new) result of this section.

Theorem 4. *If we have a reflection-coreflection-reflection chain $b \dashv c \dashv a \dashv \eta_X$ on an object X, like in Lemma 2, then X is a free algebra.*

Thus: for a Kock-Zöberlein monad T, the functor $T\colon \mathit{Alg}(T) \to \mathit{Alg}(\overline{\overline{T}})$ is an equivalence of categories.

Proof Assume $b \dashv c \dashv a \dashv \eta_X$ on X, and consider the equaliser (3) in:

$$\begin{tikzcd}
X_c \arrow[rr, tail, "e"] & & X \arrow[r, shift left, "c"] \arrow[r, shift right, "\eta"'] & T(X) \\
& T(X) \arrow[ur, "b"'] & & \\
X \arrow[uu, dashed, "k"] \arrow[ur, "\eta"'] & & &
\end{tikzcd} \tag{9}$$

We use the letter 'k' because the elements in X_c will turn out to be compact elements, in the examples later on. The first thing we note is:

$$k \circ e \;=\; \mathrm{id}_{X_c}. \tag{10}$$

This follows since e is a mono, and:

$$\begin{array}{rcll}
e \circ k \circ e & = & b \circ \eta \circ e & \text{by construction of } k \\
& = & b \circ c \circ e & \text{since } e \text{ is equaliser} \\
& = & e & \text{since } b \text{ is a } \overline{\overline{T}}\text{-algebra and } c \text{ is unit.}
\end{array}$$

Next we observe that the object X_c carries a T-algebra structure a_c inherited from $a\colon T(X) \to X$, as in:

$$a_c \stackrel{\text{def}}{=} \Big(T(X_c) \xrightarrow{T(e)} T(X) \xrightarrow{\;a\;} X \xrightarrow{\;k\;} X_c\Big)$$

It is an algebra indeed, since:

$$a_c \circ \eta = k \circ a \circ T(e) \circ \eta = k \circ a \circ \eta \circ e = k \circ e \stackrel{(10)}{=} \mathrm{id}.$$

The other algebra equation is left to the reader.

Next we show that the transpose $a \circ T(e)\colon T(X_c) \to X$ of the equaliser $e\colon X_c \rightarrowtail X$ is an isomorphism of algebras $\mu_{X_c} \cong a$. The inverse is $T(k) \circ$

$c\colon X \to T(X) \to T(X_c)$, since:

$$
\begin{array}{lll}
\multicolumn{2}{l}{\big(a \circ T(e)\big) \circ \big(T(k) \circ c\big)} & \\
= & a \circ T(b \circ \eta) \circ c & \text{by (9)} \\
= & a \circ \mu \circ T(\eta) \circ c & \text{see the assumptions in Lemma 2} \\
= & a \circ c & \\
= & \mathrm{id} & \text{see Theorem 3} \\
\multicolumn{2}{l}{\big(T(k) \circ c\big) \circ \big(a \circ T(e)\big)} & \\
= & T(k) \circ \mu \circ T(c) \circ T(e) & \text{since } c \text{ is a map } a \to \overline{T}(a) = \mu \\
= & T(k) \circ \mu \circ T(\eta) \circ T(e) & \text{since } e \text{ is equaliser of } c \text{ and } \eta \\
= & T(k) \circ T(e) & \\
= & \mathrm{id} & \text{by (10).}
\end{array}
$$

We continue to check that the assumed chain of adjunctions $b \dashv c \dashv a \dashv \eta_X$ is related to the chain $T(a_c) \dashv T(\eta) \dashv \mu \dashv \eta$ in (8) via these isomorphisms. In particular we still need to check that the following two square commute.

$$
\begin{tikzcd}
T^2(X_c) \arrow[r, "T(a\circ T(e))", "\cong"'] & T(X) \\
T(X_c) \arrow[u, "T(\eta)"] \arrow[r, "\cong", "a\circ T(e)"'] & X \arrow[u, "c"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T^2(X_c) \arrow[r, "T(a\circ T(e))", "\cong"'] \arrow[d, "T(a_c)"'] & T(X) \arrow[d, "b"] \\
T(X_c) \arrow[r, "\cong", "a\circ T(e)"'] & X
\end{tikzcd}
$$

These square commute since:

$$
\begin{array}{rcll}
T(a \circ T(e)) \circ T(\eta) & = & T(a) \circ T(\eta) \circ T(e) & \text{by naturality} \\
& = & T(e) & \\
& = & c \circ a \circ T(e) & \text{see Theorem 3} \\
a \circ T(e) \circ T(a_c) & = & a \circ T(e) \circ T(k \circ a \circ T(e)) & \\
& = & a \circ T(b \circ \eta) \circ T(a \circ T(e)) & \text{by (9)} \\
& = & b \circ \mu \circ T(\eta) \circ T(a \circ T(e)) & \text{see in Lemma 2} \\
& = & b \circ T(a \circ T(e)). &
\end{array}
$$

We still have to check that the functor $T\colon \mathit{Alg}(T) \to \mathit{Alg}(\overline{\overline{T}})$ is an equivalence. In the reverse direction, given a coalgebra $c\colon \overline{\overline{T}}(b) \to b$ on X, we take the induced algebra $T(X_c) \to X_c$ on the equaliser (9). Then $T(X_c) \xrightarrow{\cong} X$ is an isomorphism of $\overline{\overline{T}}$-algebras, as we have seen.

For the isomorphism in the other direction, assume we start from an algebra $a\colon T(X) \to X$, obtain the $\overline{\overline{T}}$-algebra $T(a)$ described in the chain $T(a) \dashv T(\eta) \dashv \mu \dashv \eta$ in (8), and then form the equaliser (9); it now looks as follows.

$$
\begin{tikzcd}
X \arrow[r, tail, "\eta_X"] & T(X) \arrow[r, shift left, "T(\eta)"] \arrow[r, shift right, "\eta"'] & T^2(X)
\end{tikzcd}
$$

This is the equaliser requirement [16], which holds since X carries an algebra structure. Clearly, $\eta \circ \eta = T(\eta) \circ \eta$ by naturality. And if a map $f\colon Y \to T(X)$ satisfies $\eta \circ f = T(\eta) \circ f$, then f factors through $\eta\colon X \to T(X)$ via $f' = a \circ f$, since

$$\eta \circ f' = \eta \circ a \circ f = T(a) \circ \eta \circ f = T(a) \circ T(\eta) \circ f = f.$$

This f' is unique with this property, since if $g\colon Y \to X$ also satisfies $\eta \circ g = f$, then $f' = a \circ f = a \circ \eta \circ g = g$. $\square$

In the remainder of this section we review some examples.

4.1 Dcpos over Posets

The main example from [10] involves the ideal monad *Idl* on the category **PoSets** of partially ordered sets with monotone functions between them. In the light of Theorems 2 and 3 we briefly review the essentials.

For a poset $X = (X, \leq)$ let $\mathit{Idl}(X)$ be the set of directed downsets in X, ordered by inclusion. This *Idl* is in fact a monad on **PoSets** with unit $X \to \mathit{Idl}(X)$ given by principal downset $x \mapsto {\downarrow}x$ and multiplication $\mathit{Idl}^2(X) \to \mathit{Idl}(X)$ by union. This monad is of Kock-Zölberlein type since for $U \in \mathit{Idl}(X)$ we have:

$$\begin{aligned}
\mathit{Idl}({\downarrow})(U) \;=\; {\downarrow}\{{\downarrow}x \mid x \in U\} \;&=\; \{V \in \mathit{Idl}(X) \mid \exists x \in U.\, V \subseteq {\downarrow}x\} \\
&\subseteq\; \{V \in \mathit{Idl}(X) \mid V \subseteq U\} \qquad \text{since } U \text{ is a downset} \\
&=\; {\downarrow}U.
\end{aligned}$$

Applying Theorem 2 to the ideal monad yields the (folklore) equivalence of the following points.

1. X is a directed complete partial order (dcpo): each directed subset $U \subseteq X$ has a join $\bigvee U$ in X;
2. The unit ${\downarrow}\colon X \to \mathit{Idl}(X)$ has a left adjoint—which is the join;
3. X carries a (necessarily unique) algebra structure $\mathit{Idl}(X) \to X$, which is also the join.

Additionally, algebra maps are precisely the continuous functions. Thus we may use as category $\mathbf{Dcpo} = \mathit{Alg}(\mathit{Idl})$.

The monad *Idl* on **PoSets** induces a comonad on **Dcpo**, written $\overline{\mathit{Idl}}$, with counit $\varepsilon = \bigvee\colon \mathit{Idl}(X) \to X$ and comultiplication $\delta = \mathit{Idl}({\downarrow})\colon \mathit{Idl}(X) \to \mathit{Idl}^2(X)$, so that $\delta(U) = {\downarrow}\{{\downarrow}x \mid x \in U\}$. In order to characterise coalgebras of this comonad $\overline{\mathit{Idl}}$ we need the following. In a dcpo X, the way below relation $\ll$ is defined as: for $x, y \in X$,

$$x \ll y \iff \text{for each directed } U \subseteq X, \text{ if } y \leq \bigvee U \text{ then } \exists z \in U.\, x \leq z.$$

A *continuous poset* is then a dcpo in which for each element $x \in X$ the set $\twoheaddownarrow x = \{y \in X \mid y \ll x\}$ is directed and has x as join. These elements way-below x may be seen as a (local) basis.

The following equivalence formed the basis for [14, Thm. 4.2] (of which Theorem 3 is a special case). The equivalence of points (1) and (2) is known from the literature, see e.g. [11, VII, Proposition 2.1], [9, Proposition 2.3], or [8, Theorem I-1.10]. The equivalence of points (2) and (3) is given by Theorem 3.

For a dcpo X, the following statements are equivalent.

1. X is a continuous poset;
2. The counit $\bigvee : \mathit{Idl}(X) \to X$ of the comonad $\overline{\mathit{Idl}}$ on **Dcpo** has a left adjoint (in **Dcpo**); it is $x \mapsto \twoheaddownarrow x$.
3. X carries a (necessarily unique) $\overline{\mathit{Idl}}$-coalgebra structure $X \to \mathit{Idl}(X)$, which is also $\twoheaddownarrow(-)$.

Theorem 4 says that another iteration $\overline{\overline{\mathit{Idl}}}$ yields nothing new.

4.2 Frames over semi-lattices

For a poset X, the set $\mathit{Dwn}(X) = \{U \subseteq X \mid U \text{ is downclosed}\}$ of downsets of X is a frame (or complete Heyting algebra, or locale), see [11]. If the poset X has finite meets $\top, \wedge$, then the downset map $\downarrow : X \to \mathit{Dwn}(X)$ preserves meets: $\downarrow\top = X$ and $\downarrow(x \wedge y) = \downarrow x \cap \downarrow y$. Hence it is a morphism in the category **MSL** of meet semi-lattices. It is not hard to see that Dwn is a monad on **MSL** that is of Kock-Zöberlein type. For a (meet) semi-lattice $X = (X, \top, \wedge)$ the following are equivalent.

1. X is a frame: X has arbitrary joins and its finite meets distribute over these joins: $x \wedge \left(\bigvee_i y_i\right) = \bigvee_i (x \wedge y_i)$;
2. The unit $\downarrow : X \to \mathit{Dwn}(X)$ has a left adjoint in **MSL**—which is the join;
3. X carries a (necessarily unique) algebra structure $\mathit{Dwn}(X) \to X$ in **MSL**, which is also the join.

Moreover, the algebra maps are precisely the frame maps, preserving arbitrary joins and finite meets; thus $\mathbf{Frm} = \mathit{Alg}(\mathit{Dwn})$.

In a next step, for a frame X, the following statements are equivalent.

1. X is a stably continuous frame, *i.e.* a frame that is continuous as a dcpo, in which $\top \ll \top$, and also $x \ll y$ and $x \ll z$ implies $x \ll y \wedge z$;
2. The counit $\bigvee : \mathit{Dwn}(X) \to X$ of the comonad $\overline{\mathit{Dwn}}$ on **Frm** has a left adjoint in **Frm**; it is $x \mapsto \twoheaddownarrow x$.
3. X carries a (necessarily unique) $\overline{\mathit{Dwn}}$-coalgebra structure $X \to \mathit{Dwn}(X)$, which is also $\twoheaddownarrow(-)$. $\square$

One can show that coalgebra homomorphisms are the *proper* frame homomorphisms (from [2]) that preserve $\ll$. We recall from [11, VII, 4.5] that for a sober topological space X, its opens $\Omega(X)$ form a continuous lattice iff X is a locally compact space. Further, the stably continuous frames are precisely the retracts of frames of the form $\mathit{Dwn}(X)$, for X a meet semi-lattice—here via the coreflection $\twoheaddownarrow \dashv \bigvee$.

5 Comonoids from bases

A recent insight, see [5], is that orthonormal bases in finite-dimensional Hilbert spaces can be described via so-called Frobenius algebras. In general, such an algebra consists of an object carrying both a monoid and a comonoid structure that interact appropriately. In the self-dual category of Hilbert spaces, it suffices to have either a monoid or a comonoid, since the dual is induced by the dagger / adjoint transpose $(-)^{\dagger}$. In this section we show that the kind of coalgebras (on algebras) considered in this paper also give rise to comonoids, assuming that the category of algebras has monoidal (tensor) structure.

In a (symmetric) monoidal category $\mathbf{A}$ a comonoid is the dual of a monoid, given by maps $I \stackrel{u}{\leftarrow} X \stackrel{d}{\rightarrow} X \otimes X$ satisfying the duals of the monoid equations. Such comonoids are used for copying and deletion, in linear and quantum logic. If $\otimes$ is cartesian product $\times$, each object carries a unique comonoid structure $1 \stackrel{!}{\leftarrow} X \stackrel{\Delta}{\rightarrow} X \times X$. The no-cloning theorem in quantum mechanics says that copying arbitrary states is impossible. But copying wrt a basis is allowed, see [5, 17].

If a monad T on a symmetric monoidal category $\mathbf{A}$ is a commutative (aka. symmetric monoidal) monad, and the category $\mathit{Alg}(T)$ has enough coequalisers, then it is also symmetric monoidal, and the free functor $F\colon \mathbf{A} \to \mathit{Alg}(T)$ preserves this monoidal structure. This classical result goes back to [13, 12]. We shall use it for the special case where the monoidal structure on the base category $\mathbf{A}$ is cartesian.

Proposition 2. *In the setting described above, assume the category of algebras $\mathit{Alg}(T)$ is symmetric monoidal, for a monad T on a cartesian category $\mathbf{A}$. Each $\overline{T}$-coalgebra / basis $b\colon X \to F(X)$, say on algebra $a\colon T(X) \to X$, gives rise to a commutative comonoid in $\mathit{Alg}(T)$ by:*

$$\begin{aligned} d_b &= \Big(X \xrightarrow{b} FX \xrightarrow{T(\Delta)} F(X \times X) \xrightarrow[\cong]{\xi^{-1}} FX \otimes FX \xrightarrow{a \otimes a} X \otimes X\Big) \\ u_b &= \Big(X \xrightarrow{b} F(X) \xrightarrow{T(!)} F(1) = I\Big), \end{aligned} \tag{11}$$

where we use the underlying comonoid structure $1 \stackrel{!}{\leftarrow} X \stackrel{\Delta}{\rightarrow} X \times X$ on X in the underlying category $\mathbf{A}$.

Proof It is not hard to see that these d_b and u_b are maps of algebras. For instance,

$$\mu_1 \circ T(u_b) = \mu_1 \circ T^2(!) \circ T(b) = T(!) \circ \mu_X \circ T(b) = T(!) \circ b \circ a = u_b \circ a.$$

The verification of the comonoid properties involves lengthy calculations, which are basically straightforward. We just show that u is neutral element for d, using the equations from Definition 1.

$$
\begin{aligned}
&(u_b \otimes \mathrm{id}) \circ d_d \\
&\quad = (T(!) \otimes \mathrm{id}) \circ (b \otimes \mathrm{id}) \circ (a \otimes a) \circ \xi^{-1} \circ T(\Delta) \circ b \\
&\quad = (T(!) \otimes \mathrm{id}) \circ (\mu \otimes \mathrm{id}) \circ (T(b) \otimes a) \circ \xi^{-1} \circ T(\Delta) \circ b \\
&\quad = (T(!) \otimes a) \circ (\mu \otimes T(a)) \circ (T(b) \otimes T(b)) \circ \xi^{-1} \circ T(\Delta) \circ b \\
&\quad = (T(!) \otimes a) \circ (\mu \otimes \mu) \circ \xi^{-1} \circ T(b \otimes b) \circ T(\Delta) \circ b \\
&\quad = (T(!) \otimes a) \circ (\mu \otimes \mu) \circ \xi^{-1} \circ T(\Delta) \circ T(b) \circ b \\
&\quad = (T(!) \otimes a) \circ (\mu \otimes \mu) \circ \xi^{-1} \circ T(\Delta) \circ T(\eta) \circ b \\
&\quad = (T(!) \otimes a) \circ (\mu \otimes \mu) \circ \xi^{-1} \circ T(\eta \times \eta) \circ T(\Delta) \circ b \\
&\quad = (T(!) \otimes a) \circ (\mu \otimes \mu) \circ (T(\eta) \times T(\eta)) \circ \xi^{-1} \circ T(\Delta) \circ b \\
&\quad = (T(!) \otimes a) \circ \xi^{-1} \circ T(\Delta) \circ b \\
&\quad = (\mathrm{id} \otimes a) \circ (T(!) \otimes \mathrm{id}) \circ \xi^{-1} \circ T(\Delta) \circ b \\
&\quad = (\mathrm{id} \otimes a) \circ \xi^{-1} \circ T(! \times \mathrm{id}) \circ T(\Delta) \circ b \\
&\quad = (\mathrm{id} \otimes a) \circ \xi^{-1} \circ T(\lambda^{-1}) \circ b \qquad \text{where } \lambda\colon 1 \times X \xrightarrow{\cong} X \\
&\quad = (\mathrm{id} \otimes a) \circ \lambda^{-1} \circ b \qquad \text{since } \xi \text{ is monoidal, where } \lambda\colon I \otimes X \xrightarrow{\cong} X \\
&\quad = \lambda^{-1} \circ a \circ b \\
&\quad = \lambda^{-1}\colon X \xrightarrow{\cong} I \otimes X. \qquad \square
\end{aligned}
$$

Example 1. To make the comonoid construction (11) more concrete, let V be a vector space, say over the complex numbers $\mathbb{C}$, with a basis, described as a coalgebra $b\colon V \to \mathcal{M}_{\mathbb{C}}(V)$ like in Theorem 1, with basic elements (e_j), satisfying $b(e_j) = 1e_j$. The counit $u_b = \mathcal{M}_{\mathbb{C}}(!) \circ b\colon V \to \mathbb{C}$ from (11) is:

$$v \longmapsto \textstyle\sum_j v_j e_j \longmapsto \sum_j v_j.$$

Similarly, the comultiplication $d_b\colon V \to V \otimes V$ as in (11) is the composite:

$$v \longmapsto \textstyle\sum_j v_j e_j \longmapsto \sum_j v_j (e_j \otimes e_j),$$

like in [5]. (For Hilbert spaces one uses orthonormal bases instead of Hamel bases; the counit u of the comonoid then exists only in the finite-dimensional case. The comultiplication d seems more relevant, see also below, and may thus also be studied on its own, like in [1], without finiteness restriction.)

In general, given a comonoid $I \xleftarrow{u} X \xrightarrow{d} X \otimes X$, an endomap $f\colon X \to X$ may be called *diagonalisable*—wrt. this comonoid, or actually, comultiplication d—if there is a "map of eigenvalues" $v\colon X \to I$ such that f equals the composite:

$$X \xrightarrow{d} X \otimes X \xrightarrow{v \otimes \mathrm{id}} I \otimes X \xrightarrow[\cong]{\lambda} X. \tag{12}$$

In the special case where the comonoid comes from a coalgebra (basis) $b\colon X \to T(X)$, like in (11), an endomap of algebras $f\colon X \to X$, say on $a\colon T(X) \to X$, is diagonalisable if there is a map of algebras $v\colon X \to I = T(1)$ such that f is:

$$X \xrightarrow{b} T(X) \xrightarrow{T(\langle v,\mathrm{id}\rangle)} T(T(1)\times X) \xrightarrow{T(\mathsf{st})} T^2(1\times X) \xrightarrow{\cong} T^2(X) \xrightarrow{\mu} T(X) \xrightarrow{a} X,$$

where st is a strength map of the form $T(X)\times Y \to T(X\times Y)$, which exists because the monad T is assumed to be commutative.

We illustrate what this means for Pauli matrices.

Example 2. We consider the set $\mathbb{C}^2$ as vector space over $\mathbb{C}$, and thus as algebra of the (commutative) multiset monad $\mathcal{M}_{\mathbb{C}}\colon \mathbf{Sets} \to \mathbf{Sets}$ via the map $\mathcal{M}_{\mathbb{C}}(\mathbb{C}^2) \xrightarrow{a} \mathbb{C}^2$ that sends a formal sum $s_1(z_1,w_1)+\cdots+s_n(z_n,w_n)$ of pairs in $\mathbb{C}^2$ to the pair of sums $(s_1\cdot z_1+\cdots+s_n\cdot z_n, s_1\cdot w_1+\cdots+s_n\cdot w_n)\in\mathbb{C}^2$.

The familiar Pauli spin functions $\sigma_{\mathsf{x}},\sigma_{\mathsf{y}},\sigma_{\mathsf{z}}\colon \mathbb{C}^2\to\mathbb{C}^2$ are given by:

$$\sigma_{\mathsf{x}}(z,w) = (w,z) \qquad \sigma_{\mathsf{y}}(z,w) = (-iw, iz) \qquad \sigma_{\mathsf{z}}(z,w) = (z,-w).$$

We concentrate on σ_{x}; it satisfies $\sigma_{\mathsf{x}}(1,1) = (1,1)$ and $\sigma_{\mathsf{x}}(1,-1) = -(1,-1)$. These eigenvectors $(1,1)$ and $(1,-1)$ are organised in a basis $b_{\mathsf{x}}\colon \mathbb{C}^2 \to \mathcal{M}_{\mathbb{C}}(\mathbb{C}^2)$, as in Definition 1, via the following formal sum.

$$b_{\mathsf{x}}(z,w) \;=\; \tfrac{z+w}{2}(1,1) + \tfrac{z-w}{2}(1,-1).$$

It expresses an arbitrary element of $\mathbb{C}^2$ in terms of this basis of eigenvectors. It is not hard to see that b_{x} is a $\overline{\mathcal{M}_{\mathbb{C}}}$-coalgebra; for instance:

$$\big(a\circ b_{\mathsf{x}}\big)(z,w) = a\big(\tfrac{z+w}{2}(1,1)+\tfrac{z-w}{2}(1,-1)\big) = \big(\tfrac{z+w}{2}+\tfrac{z-w}{2}, \tfrac{z+w}{2}-\tfrac{z-w}{2}\big) = (z,w).$$

The comonoid structure $\mathbb{C} \xleftarrow{u_{\mathsf{x}}} \mathbb{C}^2 \xrightarrow{d_{\mathsf{x}}} \mathbb{C}^2\otimes\mathbb{C}^2$ induced by b_{x} as in (11) is given by $u_{\mathsf{x}}(z,w) = z$ and $d_{\mathsf{x}}(z,w) = \frac{z+w}{2}\big((1,1)\otimes(1,1)\big) + \frac{z-w}{2}\big((1,-1)\otimes(1,-1)\big)$. The eigenvalue map $v_{\mathsf{x}}\colon \mathbb{C}^2\to\mathbb{C}$ is given by $v_{\mathsf{x}}(z,w) = w$. The eigenvalues $1,-1$ appear by application to the basic elements: $v_{\mathsf{x}}(1,1) = 1$ and $v_{\mathsf{x}}(1,-1) = -1$. Further, the Pauli function σ_{x} is diagonalised as in (12) via these $d_{\mathsf{x}}, v_{\mathsf{x}}$, since:

$$\begin{array}{rl}
\multicolumn{2}{l}{\big(\lambda\circ(v_{\mathsf{x}}\otimes\mathrm{id})\circ d_{\mathsf{x}}\big)(z,w)} \\
= & \big(\lambda\circ(v_{\mathsf{x}}\otimes\mathrm{id})\big)\Big(\tfrac{z+w}{2}\big((1,1)\otimes(1,1)\big)+\tfrac{z-w}{2}\big((1,-1)\otimes(1,-1)\big)\Big) \\
= & \lambda\Big(\tfrac{z+w}{2}\big(1\otimes(1,1)\big)+\tfrac{z-w}{2}\big(-1\otimes(1,-1)\big)\Big) \\
= & \tfrac{z+w}{2}(1,1)-\tfrac{z-w}{2}(1,-1) \\
= & (w,z) \;=\; \sigma_{\mathsf{x}}(z,w).
\end{array}$$

In a similar way one defines for the other Pauli functions σ_{y} and σ_{z}:

$$\begin{array}{rclcrcl}
b_{\mathsf{y}}(z,w) & = & \tfrac{iz+w}{2}(-i,1)+\tfrac{iz-w}{2}(i,1) & \qquad & v_{\mathsf{y}}(z,w) & = & iz \\
b_{\mathsf{z}}(z,w) & = & z(1,0)-w(0,1) & & v_{\mathsf{z}}(z,w) & = & z-w.
\end{array}$$

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