On Block Structures in Quantum Computation

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Abstract

A block is a language construct in programming that temporarily enlarges the state space. It is typically opened by initialising some local variables, and closed via a return statement. The "scope" of these local variables is then restricted to the block in which they occur. In quantum computation such temporary extensions of the state space also play an important role. This paper axiomatises "logical" blocks in a categorical manner. Opening a block may happen via a measurement, so that the block captures the various possibilities that result from the measurement. Following work of Coecke and Pavlović we show that von Neumann projective measurements can be described as an Eilenberg-Moore coalgebra of a comonad associated with a particular kind of block structure. Closing of a block involves a collapse of options. Such blocks are investigated in non-deterministic, probabilistic, and quantum computation. In the latter setting it is shown that there are two block structures in the category of C^*-algebras, via copowers and via matrices.

Keywords: Block structure, non-deterministic, probabilistic, quantum program semantics, effect logic

1 Introduction

In imperative programming languages one may find block structures of the form:

$$\texttt{\{int v = 0; ...; return\}} \tag{1}$$

Such a block is a temporary extension of the state space. It is "opened" by initialisation of some variables, and "closed" by a return statement. Although quantum programming is still in an embryonic state, it is clear, at least at the

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abstract level, that some sort of block structure is essential. For instance, in [9, Corollary 4.19] one finds that each completely positive map $S\colon \mathcal{DM}(H) \to \mathcal{DM}(H)$ between density matrices on a Hilbert space H — the interpretation of a quantum program — is of the form:

$$S(\rho) = tr_K\big(U(\rho \otimes \xi)U^{\dagger}\big),$$

where U is unitary operator on a state space $H \otimes K$ enlarging H with an "ancilla" space K, ξ is a pure state $|\,v\,\rangle\langle\,v\,|$ for some vector $|\,v\,\rangle \in K$, and tr_K is the partial trace operation. Essentially, this normal form result is based on Stinespring's Theorem (see *loc. cit.*). Here we see, similar to (1), extension of the state with K, opening of this block via the initial value ξ, and closing of the block via the partial trace tr_K.

In this paper we explore block structures at a more elementary level. They are defined as a collection of endofunctors $\mathcal{B}_n\colon \mathbf{A} \to \mathbf{A}$, for natural numbers $n > 0$, on a category $\mathbf{A}$, together with "in" and "out" maps for opening and closing a block. A "logical" block structures comes equipped with "characteristic" or "measurement" maps $X \to \mathcal{B}_n(X)$, induced by n-tests of predicates. Such maps can also open a block structure, via the various options that result from measurement. These logical block structures will be described in various categories, for non-deterministic, probabilistic (both discrete and continuous), and quantum computation. Interestingly, on Hilbert spaces with their standard logic of effects, there is no logical block structure, because there is no operation for closing blocks. This structure does exist on C^*-algebras. Hence, not directly on a Hilbert space H, but on the associated C^*-algebra $\mathcal{L}(H)$ of endomaps, we find the relevant logical block structure.

In the final section we use these logical block structures to give a diagrammatic description of two familiar quantum protocols, namely superdense coding and teleportation. The ultimate goal is to develop an appropriate logic for such protocols and to formalise the representation in a computer algebra tool, for simulation and verification. Thus, the paper follows earlier work on semantics of quantum programming languages, like, for instance [1,19,20,21] and [7].

Among the logical predicates that we use there is a subclass of "projections", with as typical property that iterated measurements give the same outcome. In [3] it was noticed that this property (of von Neumann projective measurements) is captured categorically by the "δ-law" for an Eilenberg-Moore coalgebra c, which requires $\delta \circ c = T(c) \circ c$, where T is the comonad involved; this corresponds to the requirement $P_i P_j = \delta_{ij} P_i$. The other equality that such a coalgebra must satisfy, namely $\varepsilon \circ c = \mathrm{id}$, corresponds to the condition $\sum_i P_i = 1$. The block structures that we use here allow us to generalise this approach in several directions, by showing that it also:

- occurs in simpler situations than quantum models, namely in non-deterministic and in probabilistic models, represented by the Kleisli categories of the powerset monad $\mathcal{P}$, and of the distribution and Giry monads $\mathcal{D}$ and $\mathcal{G}$;
- extends to C^*-algebras, but only in the commutative case, for one of the available block structures, namely the "copower" one that forms a comonad.

This paper unveils two block structures on C^*-algebras: one given by copowers and one by matrices. At this stage it fails to provide an answer to the question whether one of them is the right one, and in which sense? This will require more research.

2 Block structures

This section contains the basic definition of a block structure as a collection of endofunctors indexed by natural numbers, and also some examples. It starts with a very basic result describing some of the relevant endofunctors as (co)monads.

We shall write $+$ for a coproduct in a category, with coprojections $\kappa_i \colon X_i \to X_1 + X_2$ and cotupling $[f_1, f_2] \colon X_1 + X_2 \to Y$, for $f_i \colon X_i \to Y$. For maps $g_i \colon X_i \to Y_i$ there is the coproduct of maps $g_1 + g_2 = [\kappa_1 \circ g_1, \kappa_2 \circ g_2] \colon X_1 + X_2 \to Y_1 + Y_2$. Dually, we write products as $\times$, with projections $\pi_i \colon X_1 \times X_2 \to X_i$ and tupling $\langle f, g\rangle \colon Y \to X_1 \times X_2$.

Lemma 2.1 *Let $\boldsymbol{C}$ be a category with coproducts $+$. For each natural number $n > 0$, the n-fold copower functor $n \cdot (-) \colon \boldsymbol{C} \to \boldsymbol{C}$ is a comonad, where*

$$n \cdot X = \underbrace{X + \cdots + X}_{n \textit{ times}}$$

The counit $\varepsilon \colon n \cdot X \to X$ and comultiplication $\delta \colon n \cdot X \to n \cdot (n \cdot X)$ are given by:

$$\varepsilon = \nabla = [\mathrm{id}, \ldots, \mathrm{id}] \qquad \delta = \kappa_1 + \cdots + \kappa_n = [\kappa_i \circ \kappa_i]_{i \le n}.$$

Dually, in presence of products $\times$, the n-fold power functor $(-)^n$ is a monad on $\boldsymbol{C}$, with unit $\eta = \Delta = \langle \mathrm{id}, \ldots, \mathrm{id}\rangle \colon X \to X^n$ and multiplication $\mu = \langle \pi_i \circ \pi_i\rangle_{i \in n} \colon (X^n)^n \to X^n$. □

On an abstract level these (co)monad structures arise because the n-element set n carries a comonoid structure $1 \xleftarrow{!} n \xrightarrow{\langle \mathrm{id}, \mathrm{id}\rangle} n \times n$. But on a more concrete level, it is not hard to verify the comonad equations $\varepsilon \circ \delta = \mathrm{id} = (n \cdot \varepsilon) \circ \delta$ and $\delta \circ \delta = (n \cdot \delta) \circ \delta$.

Definition 2.2 A *block structure* on a category $\mathbf{A}$ consists of a collection of endofunctors $\mathcal{B}_n \colon \mathbf{A} \to \mathbf{A}$, for $n > 0$, with natural isomorphisms

$$\mathcal{B}_1(X) \cong X \qquad \text{and} \qquad \mathcal{B}_m(\mathcal{B}_n(X)) \cong \mathcal{B}_{m\times n}(X), \tag{2}$$

and with two collections of natural transformations $in_n \colon \mathrm{Id} \Rightarrow \mathcal{B}_n$ and $out_n \colon \mathcal{B}_n \Rightarrow \mathrm{Id}$ with $out_n \circ in_n = \mathrm{id}$, as in:

$$\begin{tikzcd}
X \arrow[r, "in_n"] \arrow[dr, equal] & \mathcal{B}_n(X) \arrow[d, "out_n"] \\
 & X
\end{tikzcd}$$

For the comonad $X \mapsto n \cdot X$ and monad $X \mapsto X^n$ from Lemma 2.1 there are obvious isomorphisms as in (2), namely:

$$1 \cdot X \cong X \qquad m \cdot (n \cdot X) \cong (m \times n) \cdot X \qquad X^1 \cong X \qquad (X^n)^m \cong X^{m\times n}.$$

One can turn the copower $n \cdot (-)$ into a block structure by choosing the first coprojection $\kappa_1 \colon X \to n \cdot X$ as “in”. This however, looks rather arbitrary. In the next example we see that a more natural option exists in a quantitative setting, as given by the Kleisli category of the distribution monad $\mathcal{D}$. We recall that $\mathcal{D}$ is the (finite discrete) distribution monad $\mathcal{D} \colon \mathbf{Sets} \to \mathbf{Sets}$, given by formal finite convex sums:

$$\mathcal{D}(X) = \{\varphi \colon X \to [0,1] \mid supp(\varphi) \text{ is finite, and } \textstyle\sum_x \varphi(x) = 1\}.$$

Such an element $\varphi \in \mathcal{D}(X)$ may be identified with a finite, formal convex sum $\sum_i r_i x_i$ with $x_i \in X$ and $r_i \in [0,1]$ satisfying $\sum_i r_i = 1$. The unit $\eta \colon X \to \mathcal{D}(X)$ and multiplication $\mu \colon \mathcal{D}^2(X) \to \mathcal{D}(X)$ of this monad are given by singleton/Dirac convex sum $\eta^{\mathcal{D}}(x) = 1x$ and by matrix multiplication: $\mu^{\mathcal{D}}(\Phi)(x) = \sum_\varphi \Phi(\varphi) \cdot \varphi(x)$.

Example 2.3 The Kleisli category $\mathcal{K}\ell(\mathcal{D})$ of the distribution monad $\mathcal{D} \colon \mathbf{Sets} \to \mathbf{Sets}$ inherits coproducts $+$ from $\mathbf{Sets}$, so that $X \mapsto n \cdot X$ is a comonad, following Lemma 2.1. We can turn $n \cdot (-)$ into a block structure via an “in” map in $\mathcal{K}\ell(\mathcal{D})$, namely

$$X \xrightarrow{\;in_n\;} n \cdot X \qquad \text{given by} \qquad x \longmapsto \tfrac{1}{n}\kappa_1 x + \cdots + \tfrac{1}{n}\kappa_n x.$$

Thus, $in_n(x) \in \mathcal{D}(X)$ defines a uniform distribution over the various coprojections $\kappa_i x \in n \cdot X$. Taking the counit $\varepsilon = \nabla \colon n \cdot X \to X$ as “out” map we get a block structure. Writing Kleisli composition as $g \odot f = \mu^{\mathcal{D}} \circ \mathcal{D}(g) \circ f$, we have:

$$\begin{aligned}
\big(\nabla \odot \mathit{in}_n\big)(x) &= \big(\mu^{\mathcal{D}} \circ \mathcal{D}([\eta^{\mathcal{D}}, \ldots, \eta^{\mathcal{D}}])\big)\big(\textstyle\sum_i \frac{1}{n}\kappa_i x\big) \\
&= \mu^{\mathcal{D}}\big(\textstyle\sum_i \frac{1}{n}([\eta^{\mathcal{D}}, \ldots, \eta^{\mathcal{D}}] \circ \kappa_i)x\big) \\
&= \mu^{\mathcal{D}}\big(\textstyle\sum_i \frac{1}{n}1x\big) \;=\; \mu^{\mathcal{D}}\big(1(1x)\big) \;=\; 1x \;=\; \eta^{\mathcal{D}}(x) \;=\; \mathrm{id}(x).
\end{aligned}$$

The product case $X \mapsto X^n$ is a bit more subtle, because $\times$ is a tensor, not a cartesian product, on $\mathcal{K}\ell(\mathcal{D})$. But since $\mathcal{D}(1) = 1$, the tensor unit is the terminal object 1, so we have a tensor with projections. This allows us to define η and μ as in Lemma 2.1. An associated "out" map can be defined, again via a uniform distribution:

$$X^n \xrightarrow{\;\mathit{out}_n\;} X \qquad \text{namely} \qquad (x_1, \ldots, x_n) \longmapsto \tfrac{1}{n}x_1 + \cdots + \tfrac{1}{n}x_n$$

Then:

$$\begin{aligned}
\big(\mathit{out}_n \odot \eta\big)(x) &= \big(\mu^{\mathcal{D}} \circ \mathcal{D}(\mathit{out}_n)\big)\big(1(x, \ldots, x)\big) \\
&= \mu^{\mathcal{D}}\big(1\mathit{out}_n(x, \ldots, x)\big) \\
&= \mu^{\mathcal{D}}\big(1(\tfrac{1}{n}x + \cdots + \tfrac{1}{n}x)\big) \;=\; \mu^{\mathcal{D}}\big(1(1x)\big) \;=\; 1x.
\end{aligned}$$

Example 2.4 In the Kleisli category $\mathcal{K}\ell(\mathcal{P})$ of the powerset monad $\mathcal{P}\colon \mathbf{Sets} \to \mathbf{Sets}$ the coproducts $+$ are also products (and thus "biproducts"). This means that we have a particularly simple example of a block structure, namely:

$$X \xrightarrow{\;\mathit{in}=\eta=\Delta\;} n \cdot X \xrightarrow{\;\mathit{out}=\varepsilon=\nabla\;} X \tag{3}$$

where η and ε are the unit and counit from Lemma 2.1. Explicitly, $\mathit{in}(x) = \{\kappa_1 x, \ldots, \kappa_n x\}$ and $\mathit{out}(\kappa_i x) = \{x\}$.

Example 2.5 The category **Hilb** of Hilbert spaces (over the complex numbers) also has biproducts $\oplus$, given by direct sums. Hence we can form blocks $\mathcal{B}_n(H) = n \cdot H = H \oplus \cdots \oplus H$ as before, for a Hilbert space H. But the obvious maps $\mathit{in} = \Delta$ and $\mathit{out} = \nabla$ as in (3) do not work in this case. One has to compensate by appropriate division. This can be done on either side, as in:

$$\begin{tikzcd}
H \arrow[r, "\mathit{in}=\frac{1}{n}\Delta"] \arrow[dr, equal] & n \cdot H \arrow[d, "\mathit{out}=\nabla=\sum"] \\
& H
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
H \arrow[r, "\mathit{in}=\Delta"] \arrow[dr, equal] & n \cdot H \arrow[d, "\mathit{out}=\frac{1}{n}\nabla"] \\
& H
\end{tikzcd} \tag{4}$$

where $(\frac{1}{n}\Delta)(x) = (\frac{1}{n}x, \ldots, \frac{1}{n}x)$ and $(\frac{1}{n}\nabla)(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$. Alternatively, it can be done in a more symmetric manner:

$$\begin{tikzcd}
H \arrow[r, "in=\frac{1}{\sqrt{n}}\Delta"] \arrow[dr, equal] & n \cdot H \arrow[d, "out=\frac{1}{\sqrt{n}}\nabla"] \\
& H
\end{tikzcd} \tag{5}$$

In this symmetric case we have $in^{\dagger} = out$, where $(-)^{\dagger}$ is the conjugate transpose. The equation $in^{\dagger} \circ in = \mathrm{id}$ makes in a dagger mono — and out a dagger epi.

3 Blocks and predicates

This section describes how predicates may be related to block structures via certain "characteristic" or "measurement" maps, much like in [10]. We assume that the predicates, on an object in a base category, carry the structure of an *effect algebra*. Such effect algebras are generalisations of logical structures used in classical logic (esp. Boolean algebras), in probabilistic logic (fuzzy predicates), and in quantum logic (projections and effects). Briefly, an effect algebra is a partial commutative monoid, with partial binary operation $\ovee$ and zero 0, together with a unique orthocomplenent $x^{\perp}$, such that $x \ovee x^{\perp} = 1 = 0^{\perp}$, and such that $x \ovee 1$ is defined only for $x = 0$. The main example is the unit interval $[0,1]$, with $r \ovee s$ defined and equal to the sum $r+s$ if $r+s \leq 1$, and with $r^{\perp} = 1 - r$. In a pointwise manner this structure extends to fuzzy predicates $[0,1]^{X}$, see below. Each Boolean algebra also forms an effect algebra, with $x \ovee y$ defined and equal to the join $x \vee y$ if $x \wedge y = 0$. We shall use this below for powerset Boolean algebras $\mathcal{P}(X)$, where $\ovee$ is union of disjoint sets. For more information, see *e.g.* [5,4,12,13]. A morphism of effect algebras $f \colon E \to D$ is a function between the underlying sets satisfying $f(1) = 1$ and: if $x \perp y$, then $f(x) \perp f(y)$ and $f(x \ovee y) = f(x) \ovee f(y)$. This yields a category $\mathbf{EA}$.

An *n-test* in an effect algebra E is an n-tuple $e = (e_1, \ldots, e_n)$ of elements $e_i \in E$ which satisfy $e_1 \ovee \cdots \ovee e_n = 1$. In this setting we describe a "logic of effects" categorically as a functor (or "indexed category") $\mathit{Pred} \colon \mathbf{A} \to \mathbf{EA}^{\mathrm{op}}$. It maps an object $X \in \mathbf{A}$ to the effect algebra $\mathit{Pred}(X)$ of predicates on X. A map $f \colon X \to Y$ gives rise to a "substitution" functor $\mathit{Pred}(f) \colon \mathit{Pred}(Y) \to \mathit{Pred}(X)$. In categorical logic it is often written as f^{-1}.

Definition 3.1 Let $\mathbf{A}$ be a category with an indexed category $\mathit{Pred} \colon \mathbf{A} \to \mathbf{EA}^{\mathrm{op}}$ of effect algebras, and with a block structure $\mathcal{B}_n \colon \mathbf{A} \to \mathbf{A}$. We say this is a *logical block structure* if

(i) for each $X \in \mathbf{A}$ and $n > 0$ there is a "universal" n-test on $\mathcal{B}_n(X)$, written as $\Omega = (\Omega_1, \ldots, \Omega_n)$, with $\Omega_i \in \mathit{Pred}(\mathcal{B}_n(X))$ satisfying $\Omega_1 \ovee \cdots \ovee \Omega_n = 1$;

moreover, these Ω_i should be stable under substitution, in the sense that $\mathcal{B}_n(f)^{-1}(\Omega_i) = \Omega_i$, for each $f\colon X \to Y$ in $\mathbf{A}$;

(ii) for each $X \in \mathbf{A}$ and n-test $p = (p_1, \ldots, p_n)$ on X, where $p_i \in \mathit{Pred}(X)$ satisfy $p_1 \ovee \cdots \ovee p_n = 1$, there is a "characteristic" map $\mathit{char}_p\colon X \to \mathcal{B}_n(X)$ in $\mathbf{A}$ with $\mathit{char}_p^{-1}(\Omega_i) = p_i$, for each $i \in n$.

The characteristic map yields a block opening $\mathit{char}_p(x) \in \mathcal{B}_n(X)$ whose n different options are determined by the n predicates p_i in p.

Our first example clearly shows the importance of understanding powersets of predicates as effect algebras, because the *disjoint* union is crucial for having characteristic maps.

Example 3.2 On the Kleisli category $\mathcal{K}\ell(\mathcal{P})$ of the powerset monad $\mathcal{P}$ there is an indexed category $\mathit{Pred}\colon \mathcal{K}\ell(\mathcal{P}) \to \mathbf{EA}^{\mathrm{op}}$ given by ordinary predicates: $\mathit{Pred}(X) = \mathcal{P}(X)$. This set of predicates is a Boolean algebra, and thus an effect algebra, with sum $\ovee$ defined as union, but only for disjoint subsets. For a Kleisli map $f\colon X \to Y$ we have a substitution functor:

$$\mathcal{P}(Y) \xrightarrow{f^{-1} = \mathit{Pred}(f)} \mathcal{P}(X) \qquad \text{given by} \qquad V \longmapsto \{x \mid f(x) \subseteq V\}.$$

(This substitution f^{-1} is not the same as inverse image, which is often also written as f^{-1}.)

We show that the block structure $\mathcal{B}_n(X) = n \cdot X$ from Example 2.4 is a logical block structure. For each number $n > 0$ and set X there is an n-test $\Omega = (\Omega_1, \ldots, \Omega_n)$ on $\mathcal{B}_n(X) = n \cdot X$ given by subsets:

$$\Omega_i = \{\kappa_i x \mid x \in X\} \subseteq n \cdot X = X + \cdots + X = \mathcal{B}_n(X).$$

These subsets Ω_i are all disjoint, so their effect algebra sum $\Omega_1 \ovee \cdots \ovee \Omega_n$ exists and equals the maximal predicate $1 = n \cdot X \subseteq n \cdot X$ in $\mathit{Pred}(n \cdot X)$. It is easy to see that Ω is stable under composition: for $f\colon X \to Y$ in $\mathcal{K}\ell(\mathcal{P})$,

$$\mathcal{B}_n(f)^{-1}(\Omega_i) = \{z \in n \cdot X \mid (n \cdot f)(z) \subseteq \Omega_i\} = \{\kappa_i x \mid x \in X\} = \Omega_i.$$

For an arbitrary n-test $U = (U_1, \ldots, U_n)$ on X, where $U_1 \ovee \cdots \ovee U_n = 1$, there is a characteristic map in the Kleisli category $\mathcal{K}\ell(\mathcal{P})$:

$$X \xrightarrow{\mathit{char}_U} \mathcal{B}_n(X) \qquad \text{namely} \qquad x \longmapsto \{\kappa_i x\}, \quad \text{if } x \in U_i.$$

Since the predicates U_i are mutually disjoint with join X, this forms a well-defined map. The required substitution equation in Definition 3.1 (2) holds:

$$\mathit{char}_U^{-1}(\Omega_i) = \{x \mid \mathit{char}_U(x) \subseteq \Omega_i\} = \{x \mid x \in U_i\} = U_i.$$

It is not hard to verify that this map $\mathit{char}_U\colon X \to \mathcal{B}_n(X)$ is an Eilenberg-Moore coalgebra of the comonad $\mathcal{B}_n = n \cdot (-)$, *i.e.* that the equations $\mathit{out}_n \odot \mathit{char}_U = \mathrm{id}$ and $\delta \odot \mathit{char}_U = \mathcal{B}_n(\mathit{char}_U) \odot \mathit{char}_U$ hold, where δ is the comultiplication from Lemma 2.1. In fact one can prove that there is a bijective correspondence:

$$\begin{array}{c} \text{Boolean } n\text{-tests } U = (U_1, \ldots, U_n) \text{ in } \mathcal{P}(X) \\ \hline\hline \text{Eilenberg-Moore coalgebras } X \longrightarrow \mathcal{B}_n(X) \text{ in } \mathcal{K}\ell(\mathcal{P}) \end{array} \tag{6}$$

With intersection $\cap$ as multiplication operation, each of these predicates U_i is a projection, since $U_i^2 = U_i \cap U_i = U_i$.

Example 3.3 The Kleisli category $\mathcal{K}\ell(\mathcal{D})$ carries an indexed category $\mathit{Pred}\colon \mathcal{K}\ell(\mathcal{D}) \to \mathbf{EA}^{\mathrm{op}}$ given by fuzzy predicates: $\mathit{Pred}(X) = [0,1]^X$. The effect algebra structure on $[0,1]^X$ is inherited pointwise from $[0,1]$. In particular, for $p, q \in [0,1]^X$, if $p(x) + q(x) \leq 1$ for all $x \in X$, then $p \ovee q$ is defined and $(p \ovee q)(x) = p(x) + q(x)$. Each map $f\colon X \to Y$ in $\mathcal{K}\ell(\mathcal{D})$ yields a substitution functor:

$$[0,1]^Y \xrightarrow{f^{-1} = \mathit{Pred}(f)} [0,1]^X \qquad \text{by} \qquad q \longmapsto \lambda x.\, \textstyle\sum_y f(x)(y) \cdot q(y).$$

We show that the copower block structure $X \mapsto n \cdot X$ from Example 2.3 is logical.

(i) The "universal" n-test Ω consists of predicates $\Omega_i \in [0,1]^{n \cdot X}$, given by $\Omega_i(\kappa_j x)$ is 1 if $i = j$ and 0 otherwise. Then: $\Omega_1 \ovee \cdots \ovee \Omega_n = 1$; moreover these predicates Ω_i are stable under substitution.

(ii) For an arbitrary n-test p on X, given by $p_i \in [0,1]^X$ with $p_1 \ovee \cdots \ovee p_n = 1$, we define a characteristic map $\mathit{char}_p\colon X \to n \cdot X$ in $\mathcal{K}\ell(\mathcal{D})$ via the convex sums:

$$\mathit{char}_p(x) = p_1(x)\kappa_1 x + \cdots + p_n(x)\kappa_n x,$$

using that $p_1(x) + \cdots + p_n(x) = 1$. Then:

$$\begin{aligned} \mathit{char}_p^{-1}(\Omega_i)(x) &= \textstyle\sum_{z \in n \cdot X} \mathit{char}_p(x)(z) \cdot \Omega_i(z) \\ &= \textstyle\sum_{y \in X} \mathit{char}_p(x)(\kappa_i y) \\ &= p_i(x). \end{aligned}$$

In general the map $\mathit{char}_p\colon X \to \mathcal{B}_n(X)$ does not form an Eilenberg-Moore coalgebra of the comonad $\mathcal{B}_n$: we do have $\mathit{out}_n \odot \mathit{char}_p = \mathrm{id}$, but the δ-law may fail. However, the law holds for n-tests given by fuzzy projections $p_i \in [0,1]^X$, satisfying $p_i^2 = p_i$. Automatically, $p_i p_j = 0$, for $j \neq i$, since the p_i add up to 1. It is not hard to see that these projections correspond to "Boolean" fuzzy tests, determined by indicator functions $\mathbf{1}_{U_i}\colon X \to [0,1]$ with

$\mathbf{1}_{U_i}(x) = 1$ if $x \in U_i$ and $\mathbf{1}_{U_i}(x) = 0$ otherwise, for disjoint subsets $U_i \subseteq X$ with $U_1 \ovee \cdots \ovee U_n = 1$. Thus we have a bijective correspondence like in (6):

$$\begin{array}{c} \text{Boolean } n\text{-tests } U = (U_1, \ldots, U_n) \text{ in } \mathcal{P}(X) \\ \hline\hline n\text{-tests of projections } p = (p_1, \ldots, p_n) \text{ in } [0,1]^X \text{ with } p_i^2 = p_i \\ \hline\hline \text{Eilenberg-Moore coalgebras } X \longrightarrow \mathcal{B}_n(X) \text{ in } \mathcal{K}\ell(\mathcal{D}) \end{array} \qquad (7)$$

The fuzzy predicates in $[0,1]^X$ form not only an effect algebra but an *effect module* (see [13] for details): they come with scalar multiplication, with scalars r from $[0,1]$, via $r \cdot p = \lambda x.\, r \cdot p(x)$. In the subcategory $\mathbf{EMod} \hookrightarrow \mathbf{EA}$ of such effect modules maps preserve the scalar multiplication. In this setting there are alternative characterisations of n-tests in $[0,1]^X$, in the style of [10], which we express via bijective correspondences:

$$\begin{array}{c} \text{fuzzy } n\text{-tests } p = (p_1, \ldots, p_n) \text{ in } [0,1]^X \\ \hline\hline \text{effect module maps } [0,1]^n \longrightarrow [0,1]^X \\ \hline\hline \text{Kleisli maps } X \longrightarrow \mathcal{B}_n(1) \text{ in } \mathcal{K}\ell(\mathcal{D}) \\ \hline\hline \text{Kleisli maps } X \xrightarrow[f]{} \mathcal{B}_n(X) \text{ with } \mathit{out}_n \odot f = \mathrm{id} \end{array} \qquad (8)$$

where $1 = \{*\}$ is the singleton set. The first correspondence is standard. An n-test p corresponds to a Kleisli map $g \colon X \to \mathcal{B}_n(1)$ via $g(x) = \sum_i p_i(x)\kappa_i *$, and to a map $f \colon X \to \mathcal{B}_n(X)$ via $f = \mathit{char}_p$. Such a map f gives rise to an n-test with predicates $p_i = \lambda x.\, f(x)(\kappa_i x)$.

Example 3.4 The distribution monad $\mathcal{D}$ is used in a categorical approach to *discrete* probability. For the continuous case one uses the Giry monad [8]. It is defined as monad $\mathcal{G} \colon \mathbf{Meas} \to \mathbf{Meas}$ on the category of measurable spaces, where $\mathcal{G}(X)$ contains the probability measures $\Sigma_X \to [0,1]$, defined on the measurable subsets $\Sigma_X \subseteq \mathcal{P}(X)$. We briefly illustrate how it carries a logical block structure, in line with [11]. We follow the constructions and notation used there.

The logic is given by a functor $\mathit{Pred} \colon \mathcal{K}\ell(\mathcal{G}) \to \mathbf{EMod}^{\mathrm{op}}$ that sends a measurable space X to the homset $\mathit{Pred}(X) = \mathbf{Meas}(X, [0,1])$ of measurable maps to $[0,1]$, with pointwise effect module structure. For a Kleisli map $f \colon X \to \mathcal{G}(Y)$ and predicate $q \colon Y \to [0,1]$ one defines substitution by integration:

$$\mathit{Pred}(f)(q) = f^{-1}(q) = \lambda x.\, \int q \, \mathrm{d}f(x).$$

There is a (comonad) block structure $\mathcal{B}_n(X) = n \cdot X$ defined via copowers on $\mathcal{K}\ell(\mathcal{G})$, with the $\mathit{in} \colon X \to \mathcal{G}(n \cdot X)$ and $\mathit{out} \colon n \cdot X \to \mathcal{G}(X)$ maps given by:

$$\mathit{in}(x) = \lambda M \in \Sigma_{n \cdot X}.\, \tfrac{1}{n} \textstyle\sum_i \mathbf{1}_M(\kappa_i x) \qquad \mathit{out}(\kappa_i x) = \lambda N \in \Sigma_X.\, \mathbf{1}_N(x).$$

The predicates $\Omega_i \colon n \cdot X \to [0,1]$ are defined, as in Example 3.3, as $\Omega_i = \mathbf{1}_{\kappa_i X}$, *i.e.* as $\Omega_i(\kappa_j x) = 1$ if $i = j$ and $\Omega_i(\kappa_j x) = 0$ otherwise. And for an n-test $p = (p_1, \ldots, p_n)$ of predicates $p_i \in \mathit{Pred}(X)$ with $p_1 \ovee \cdots \ovee p_n = 1$, we can define a characteristic map $\mathit{char}_p \colon X \to \mathcal{B}_n(X)$ in $\mathcal{K}\ell(\mathcal{G})$ by:

$$\mathit{char}_p(x) = \lambda M \in \Sigma_{n \cdot X}.\ \textstyle\sum_i p_i(x) \cdot \mathbf{1}_M(\kappa_i x).$$

Also in this case the n-tests $p_i \colon X \to [0,1]$ that consist of projections, *i.e.* that satisfy $p_i^2 = p_i$, can be characterised as Eilenberg-Moore coalgebras, like in (7). They also correspond to indicator functions $\mathbf{1}_{M_i}$, for $M_i \in \Sigma_X$ pairwise disjoint. Since the measurable subsets Σ_X form an effect algebra, with $\ovee$ given by disjoint union, they form n-tests in Σ_X. Thus we get bijective correspondences:

$$\begin{array}{c}
n\text{-tests } M = (M_1, \ldots, M_n) \text{ in } \Sigma_X \\
\hline\hline
n\text{-tests of projections } p = (p_1, \ldots, p_n) \text{ in } \mathbf{Meas}(X, [0,1]) \text{ with } p_i^2 = p_i \\
\hline\hline
\text{Eilenberg-Moore coalgebras } X \longrightarrow \mathcal{B}_n(X) \text{ in } \mathcal{K}\ell(\mathcal{G})
\end{array} \qquad (9)$$

Example 3.5 In the context of Hilbert spaces, several of the ingredients encountered above are present, but we do *not* find a logical block structure, for the standard logic of effects. We briefly describe the situation, building on Example 2.5.

We start with the logic. We write $\mathbf{Hilb}_{\text{isom}} \hookrightarrow \mathbf{Hilb}$ for the subcategory of Hilbert spaces with isometries between them. Such an isometry f is bounded linear function that is a "dagger mono", *i.e.* satisfies $f^\dagger \circ f = \mathrm{id}$. There is an "effect" predicate functor $\mathcal{E}f \colon \mathbf{Hilb}_{\text{isom}} \to \mathbf{EMod}^{\mathrm{op}}$ that sends a Hilbert space H to the set of effects:

$$\mathcal{E}f(H) = \{A \colon H \to H \mid 0 \le A \le \mathrm{id}\}.$$

These effects are the quantum fuzzy/unsharp predicates, see *e.g.* [16,15,5]. An effect $A \ovee B$ is defined and equal to $A + B$ if $A + B \le \mathrm{id}$. The orthocomplement is given by $A^\perp = \mathrm{id} - A$. Scalar multiplication rA, for $r \in [0,1]$ is done in a pointwise manner. Hence this $\mathcal{E}f(H)$ is an effect module.

For a dagger monic map $f \colon H \rightarrowtail K$ one defines $f^{-1} = \mathcal{E}f(f) = f^\dagger \circ (-) \circ f \colon \mathcal{E}f(K) \to \mathcal{E}f(H)$. This substitution functor f^{-1} preserves the effect module structure because f is a dagger mono.

Let $\mathcal{B}_n(H) = n \cdot H = H \oplus \cdots \oplus H$ be the block structure on $\mathbf{Hilb}$ from Example 2.5. There is an n-test $\Omega = (\Omega_1, \ldots, \Omega_n)$ of effects $\Omega_i = \kappa_i \circ \pi_i \in \mathcal{E}f(\mathcal{B}_n(H))$. More explicitly, $\Omega_i(x_1, \ldots, x_n) = (0, \ldots, 0, x_i, 0, \ldots, 0)$. These Ω_i's are stable under substitution.

For an n-test $A = (A_1, \ldots, A_n)$ of effects $A_i \in \mathcal{E}f(H)$ we can define a characteristic map $\mathit{char}_A \colon H \to \mathcal{B}_n(H)$ in $\mathbf{Hilb}_{\text{isom}}$ as n-tuple of square roots of (positive) maps:

$$\mathit{char}_A = \langle \sqrt{A_1}, \ldots, \sqrt{A_n} \rangle \colon H \longrightarrow H \oplus \cdots \oplus H = \mathcal{B}_n(H).$$

This characteristic map is a dagger mono, since, as shown in [10]:

$$\big(\mathit{char}_A\big)^\dagger \circ \mathit{char}_A = [\sqrt{A_1}, \ldots, \sqrt{A_n}] \circ \langle \sqrt{A_1}, \ldots, \sqrt{A_n} \rangle = A_1 + \cdots + A_n = \mathrm{id}.$$

Clearly, we have:

$$\begin{aligned} (\mathit{char}_A)^{-1}(\Omega_i) &= \big(\mathit{char}_A\big)^\dagger \circ \Omega_i \circ \mathit{char}_A \\ &= [\sqrt{A_1}, \ldots, \sqrt{A_n}] \circ \kappa_1 \circ \pi_i \circ \langle \sqrt{A_1}, \ldots, \sqrt{A_n} \rangle \\ &= \sqrt{A_i} \circ \sqrt{A_i} \\ &= A_i. \end{aligned}$$

The map $\mathit{in} = \frac{1}{\sqrt{n}}\Delta \colon H \to \mathcal{B}_n(H)$ in (5) arises in this manner as characteristic map of the n-test $(\frac{1}{n}\mathrm{id}, \ldots, \frac{1}{n}\mathrm{id})$. However, the corresponding "out" map in (5), $\mathit{out} = \mathit{in}^\dagger = \frac{1}{\sqrt{n}}\nabla$, is *not* a morphism in the category $\mathbf{Hilb}_{\mathrm{isom}}$, since it is not a dagger mono (but a dagger epi). Thus this does *not* give us a logical block structure in $\mathbf{Hilb}_{\mathrm{isom}}$.

Using $\oplus$ as coproduct we have a comonad structure (ε, δ) on $\mathcal{B}_n = n\cdot(-)$, as in Lemma 2.1. Following [3] we call a map $c \colon H \to \mathcal{B}_n(H)$ in **Hilb** *self-adjoint* if the following diagram commutes.

$$\begin{tikzcd} H \arrow[r, "c"] \arrow[d, "\eta=\Delta"'] & \mathcal{B}_n(H) \\ \mathcal{B}_n(H) \arrow[r, "\delta=\kappa_1\oplus\cdots\oplus\kappa_n"'] & \mathcal{B}_n(\mathcal{B}_n(H)) \arrow[u, "\mathcal{B}_n(c^\dagger)"'] \end{tikzcd}$$

This means that each component $c_i = \pi_i \circ c \colon H \to H$ is self-adjoint, *i.e.* satisfies $c_i^\dagger = c_i$.

The subset of projections (or sharp predicates) $\mathcal{P}r(H) \hookrightarrow \mathcal{E}f(H)$ contains those $p \colon H \to H$ with $p \circ p = p = p^\dagger$. An n-test $A = (A_1, \ldots, A_n)$ in $\mathcal{E}f(H)$ is called a *von Neumann* test if each A_i satisfies $A_i \circ A_i = A_i$ and $A_i \circ A_j = 0$ for each $j \neq i$. Such an A_i is then a projection. One of the main results of [3] (specifically: Thm. 16.6) says that there is a bijective correspondence:

$$\frac{\text{von Neumann } n\text{-tests } p = (p_1, \ldots, p_n) \text{ in } \mathcal{E}f(H)}{\text{self-adjoint Eilenberg-Moore coalgebras } H \to \mathcal{B}_n(H)} \tag{10}$$

A test p corresponds to its characteristic map $\mathit{char}_p = \langle \sqrt{p_1}, \ldots, \sqrt{p_n} \rangle = \langle p_1, \ldots, p_n \rangle$.

4 Copower block structure on C^*-algebras

In the present context all C^*-algebras have a unit. The maps $f\colon A \to B$ between C^*-algebras that we consider are linear functions which are unital (preserve the unit) and positive (preserve positive elements: for each $x \in A$ there is an $y \in B$ with $f(x^*x) = y^*y$). We often refer to these morphisms as 'PU-maps'. We shall write $\mathbf{Cstar}_{\mathrm{PU}}$ for the category of C^*-algebras with such unital positive maps, and $\mathbf{CCstar}_{\mathrm{PU}} \hookrightarrow \mathbf{Cstar}_{\mathrm{PU}}$ for the full subcategory of C^*-algebras with commutative multiplication. These categories of C^*-algebras are most naturally used in opposite form — as $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$ and $(\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ — just like the category $\mathbf{cHA}$ of complete Heyting algebras typically occurs in opposite form, as category of locales $\mathbf{Loc} = \mathbf{cHA}^{\mathrm{op}}$, see *e.g.* [14].

In the literature on C^*-algebras it is most common to use *-homomorphism as maps. These preserve multiplication (M), involution (I) and are unital (U). In [6] these *-homomorphisms are called MIU-maps, in order to distinguish them from the PU-maps which are used here. MIU-maps are very restrictive, which is useful for Gelfand duality. But the PU-maps are the appropriate notion in a probabilistic or quantum context.

Let's write $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \hookrightarrow \mathcal{K}\ell(\mathcal{D})$ for the full subcategory with natural numbers $n \in \mathbb{N}$ as objects, considered as n-element set. There is a full and faithful functor $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \to (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$, which sends an object n to $\mathbb{C}^n = \mathbb{C}\times\cdots\times\mathbb{C}$, the n-fold power of the complex numbers $\mathbb{C}$; it sends a Kleisli map $f\colon n \to m$ to the PU-map $\mathbb{C}^m \to \mathbb{C}^n$ given by $v \mapsto \lambda i \in n.\, \sum_{j\in m} f(i)(j)\cdot v(j)$. This functor restricts to an equivalence between $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D})$ and the subcategory of finite dimensional commutative C^*-algebras, see [6]. In fact, in [6] it is shown that there is an equivalence between $(\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ and the Kleisli category of the "Radon" monad on the compact Hausdorff spaces. The point we are trying to make is that the category $(\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ of *commutative* C^*-algebras is a natural universe for probabilistic (monadic) computation.

In general, the multiplication term ab, for two positive elements a, b in a C^*-algebra, need not be positive. The following easy observations will be useful.

Lemma 4.1 *Let a be a positive element in an arbitrary C^*-algebra. Then:*

(i) *x^*ax is positive, for each element x;*

(ii) *xax is positive, for each* positive *x;*

Proof Write $a = y^*y$; then $x^*ax = x^*y^*yx = (yx)^*(yx)$ is clearly positive. If x is positive itself, then $x^* = x$, so the second point follows from the first one. □

For two C^*-algebras A, B we write $A \oplus B$ for the C^*-algebra with product $A \times B$ as underlying set, with componentwise operations, and with maximum of the norms. Together with the usual projection and pairing operations this $\oplus$ forms a product in $\mathbf{Cstar}_{\mathrm{PU}}$ and $\mathbf{CCstar}_{\mathrm{PU}}$, and thus a coproduct in their dual categories. By Lemma 2.1 the mapping

$$A \longmapsto \mathcal{B}_n(A) \stackrel{\text{def}}{=} n \cdot A = A \oplus \cdots \oplus A$$

is a comonad on $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$ and $(\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$. We show that it extends to a block structure, both on $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$ and $(\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$, namely:

$$A \xrightarrow{\;in_n\;} \mathcal{B}_n(A) \xrightarrow{\;out_n\;} A, \tag{11}$$

where out_n is the diagonal (counit) map $A \to A^n$ given by $out_n(a) = (a, \ldots, a)$. The map $in_n \colon A^n \to A$ takes the average: $in_n(a_1, \ldots, a_n) = \frac{a_1 + \cdots + a_n}{n}$. Keeping the 'opposite' in mind we see that the required block structure equation holds:

$$\big(out_n \circ^{\mathrm{op}} in_n\big)(a) = in_n\big(out_n(a)\big) = \frac{a + \cdots + a}{n} = a.$$

We further notice that in_n is a PU-map, but out_n is a MIU-map.

For a C^*-algebra A we write $[0,1]_A = \{x \in A \mid 0 \leq x \leq 1\}$ for the "effects" in A, that is, for the positive elements below the unit. These form an effect algebra, with $x \ovee y$ defined and equal to $x + y$ if $x + y \leq 1$. The orthocomplement of $x \in [0,1]_A$ is $1 - x$. In fact, $[0,1]_A$ is not just an effect algebra but an effect module, since scalar multiplication rx, where r is in the unit interval $[0,1] \subseteq \mathbb{R}$ and $x \in [0,1]_A$, yields an effect $rx \in [0,1]_A$. Each PU-map $f \colon A \to B$ restricts to a map of effect algebras $[0,1]_A \to [0,1]_B$. In [6] it is shown that the mapping $A \mapsto [0,1]_A$ yields a full and faithful functor $\mathbf{Cstar}_{\mathrm{PU}} \to \mathbf{EMod}$. We shall use it as $\mathit{Pred} \colon (\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}} \to \mathbf{EMod}^{\mathrm{op}}$, where $\mathit{Pred}(A) = [0,1]_A$. The substitution functor $\mathit{Pred}(f) = f^{-1} \colon [0,1]_B \to [0,1]_A$ associated with $f \colon A \to B$ in $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$ is obtained simply by restriction.

In this situation, like in (8), tests can be characterised in various ways.

Lemma 4.2 *For a C^*-algebra A, there are bijective correspondences between:*

$$\begin{array}{c} n\text{-}tests\ p = (p_1, \ldots, p_n)\ \ in\ [0,1]_A \\ \hline\hline effect\ module\ maps\ [0,1]^n \longrightarrow [0,1]_A \\ \hline\hline maps\ A \longrightarrow \mathcal{B}_n(\mathbb{C})\ \ in\ (\boldsymbol{Cstar}_{PU})^{op} \end{array} \tag{12}$$

Proof Given an n-test $e = (e_1, \ldots, e_n)$, define $h \colon A \to \mathcal{B}_n(\mathbb{C})$ in $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$, that is $h \colon \mathbb{C}^n \to A$ in $\mathbf{Cstar}_{\mathrm{PU}}$, by $h(z_1, \ldots, z_n) = \sum_i z_i e_i$. This h is clearly positive, and unital since $h(1, \ldots, 1) = \sum_i e_i = 1$. Conversely, a PU-map $f \colon \mathbb{C}^n \to A$ is determined by the values $f(|i\rangle)$, where $|i\rangle$ is the standard base vector $(0, \ldots, 1, \ldots, 0) \in \mathbb{C}^n$. Since $0 \leq |i\rangle \leq 1$ one has $f(|i\rangle) \in [0,1]_A$. □

Proposition 4.3 *Let $\overrightarrow{a_i} = (a_1, \ldots, a_n) \in A^n = \mathcal{B}_n(A)$ be an n-tuple in a C^*-algebra A.*

(i) *If $\sum_i a_i^* a_i = 1$ there is a PU-map:*

$$\mathcal{B}_n(A) \xrightarrow{meas(\overrightarrow{a_i})} A \quad \text{given by} \quad \overrightarrow{b_i} \longmapsto \textstyle\sum_i a_i^* b_i a_i$$

(ii) *If $a_i^* a_i = 1$ for each i, then there is a PU-map:*

$$\mathcal{B}_n(A) \xrightarrow{map(\overrightarrow{a_i}) = \prod_i a_i^*(-)a_i} \mathcal{B}_n(A) \quad \text{given by} \quad \overrightarrow{b_i} \longmapsto (a_1^* b_1 a_1, \ldots, a_n^* b_n a_n)$$

Proof The conditions $\sum_i a_i^* a_i = 1$ and $\forall i.\, a_i^* a_i = 1$ ensure that the functions $meas(\overrightarrow{a_i})$ and $map(\overrightarrow{a_i})$ are unital. Positivity is trivial, by Lemma 4.1. □

Example 4.4 Let $\psi = (z_1, \ldots, z_n) \in \mathbb{C}^n$ be a state, so that $\|\psi\| = 1$. This means that $\langle \psi \mid \psi \rangle = \sum_i \overline{z_i} z_i = \sum_i |z_i|^2 = 1$, where $\overline{\cdot}$ is conjugation of complex numbers. Hence in each C^*-algebra A this ψ gives rise to an n-tuple $z_i 1$ with $\sum_i (z_i 1)^*(z_i 1) = 1$. These elements $z_i 1 \in A$ arise via the unique map $\mathbb{C} \to A$, using initiality of $\mathbb{C}$ among C^*-algebras. The "measure" map from Proposition 4.3 then gives a PU-map $\mathcal{B}_n(A) \to A$, namely

$$(b_1, \ldots, b_n) \longmapsto \textstyle\sum_i (z_i 1)^* b_i (z_i 1) = \sum_i (\overline{z_i} z_i) b_i = \sum_i |z_i|^2 b_i.$$

In the opposite category this operation forms a map $A \to \mathcal{B}_n(A)$ which describes how a context is opened and initialised by the state $\psi \in \mathbb{C}^n$, via a probabilistic mixture determined by $|z_i|^2 \in [0,1]$, corresponding to the Born rule.

It turns out that the "copower" definition $\mathcal{B}_n = n \cdot (-)$ yields a logical block structure, also for C^*-algebras, with predicate logic given by their effects: $Pred(A) = [0,1]_A$. In the next section we show that there is another block structure.

Proposition 4.5 *The assignment $A \mapsto \mathcal{B}_n(A) = A \oplus \cdots \oplus A$, with maps (11), is a logical block structure, both on $(\boldsymbol{Cstar}_{PU})^{op}$ and on $(\boldsymbol{CCstar}_{PU})^{op}$.*

(i) *The universal n-test $\Omega = (\Omega_1, \ldots, \Omega_n)$ consists of $\Omega_i \in [0,1]_{\mathcal{B}_n(A)} = ([0,1]_A)^n$ given by the n-tuple of effects $(0, \ldots, 1, \ldots, 0)$, with 1 only at the i-th position.*

(ii) *For an n-test $e = (e_1, \ldots, e_n)$ one can define a characteristic maps $char_e \colon A \to \mathcal{B}_n(A)$ as:*

$$char_e(a_1, \ldots, a_n) = \sqrt{e_1} a_1 \sqrt{e_1} + \cdots + \sqrt{e_n} a_n \sqrt{e_n}.$$

If A is commutative, we get $char_e(a_1, \ldots, a_n) = \sum_i e_i a_i$.

Proof It is clear that the predicates Ω_i are stable under substitution. Further, $\mathit{char}_e^{-1}(\Omega_i) = \mathit{char}_e(0,\ldots,1,\ldots,0) = \sqrt{e_i}\sqrt{e_i} = e_i$. □

The following result gives a C^*-algebraic version of the correspondences (6), (7), and (10). It only applies in the commutative case.

Generalising Example 3.5 we call an n-test $e = (e_1,\ldots,e_n)$ in a C^*-algebra a *von Neumann* n-test if each e_i is a projection, *i.e.* satisfies $e_i^2 = e_i$, and satisfies $e_ie_j = 0$ for each $j \neq i$.

Theorem 4.6 *In a C^*-algebra A there are bijective correspondences:*

$$\begin{array}{c}
\begin{array}{c}
\textit{von Neumann } n\textit{-tests } e = (e_1,\ldots,e_n) \textit{ in } [0,1]_A \\
\hline\hline
\textit{maps } A \longrightarrow \mathcal{B}_n(\mathbb{C}) \textit{ in } (\boldsymbol{Cstar}_{MIU})^{op}
\end{array} \\
\hline\hline
\textit{Eilenberg-Moore coalgebras } A \longrightarrow \mathcal{B}_n(A) \textit{ in } (\boldsymbol{Cstar}_{PU})^{op}
\end{array} \; (*) \tag{13}$$

where the second correspondence, marked with $(*)$, only works if the C^*-algebra A is commutative.

Proof We first do the first correspondence. Given a von Neumann n-test $e = (e_1,\ldots,e_n)$ we can define a MIU-map $f\colon \mathbb{C}^n \to A$ as sum of scalar multiplications: $f(z_1,\ldots,z_n) = \sum_i z_ie_i$, as in the proof of Lemma 4.2. It now preserves multiplication:

$$\begin{aligned}
f(\overrightarrow{z_i}) \cdot f(\overrightarrow{w_i}) \;=\; \big(\sum_i z_ie_i\big) \cdot \big(\sum_i w_ie_i\big) &= \sum_{i,j} z_ie_i \cdot w_je_j \\
&= \sum_{i,j} (z_i \cdot w_j)(e_i \cdot e_j) \\
&= \sum_i (z_i \cdot w_i)e_i \;=\; f(\overrightarrow{(z\cdot w)_i}).
\end{aligned}$$

In the other direction, given such a MIU-map $f\colon \mathbb{C}^n \to A$ we obtain an n-test of effects $e_i = f(|\,i\,\rangle)$, like before. Now we have:

$$e_ie_j = f(|\,i\,\rangle)f(|\,j\,\rangle) = f(|\,i\,\rangle|\,j\,\rangle) = \begin{cases} f(|\,i\,\rangle) = e_i & \text{if } i = j \\ f(0) = 0 & \text{otherwise.} \end{cases}$$

Hence the e_i form mutually orthogonal projections, and thus a von Neumann test.

For the second correspondence, assume A is commutative. Let $e = (e_1,\ldots,e_n)$ be a von Neumann n-test. The corresponding characteristic PU-map $\mathcal{B}_n(A) \to A$ from Proposition 4.5, is given by $\mathit{char}_e(\overrightarrow{a}) = \sum_i \sqrt{e_i}a_i\sqrt{e_i} = \sum_i e_ia_i$. The latter simple form, resulting from commutativity, is crucial for proving the ε-equation for a coalgebra, in the opposite category $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$.

$$\begin{aligned}
\big(\varepsilon \circ^{\mathrm{op}} \mathit{char}_e\big)(a) &= \big(\mathit{char}_e \circ \mathit{out}\big)(a) \\
&= \mathit{char}_e(a, \ldots, a) \\
&= \textstyle\sum_i e_i a \\
&= (\textstyle\sum_i e_i)a \\
&= 1a \\
&= a. \\
\big(\mathcal{B}_n(\mathit{char}_e) \circ^{\mathrm{op}} \mathit{char}_e\big)(\overrightarrow{t_i}) &= \mathit{char}_e\big(\mathit{char}_e(t_1), \ldots, \mathit{char}_e(t_n)\big) \quad \text{for } t_i \in A^n \\
&= \textstyle\sum_i e_i(\sum_j e_j t_{ij}) \\
&= \textstyle\sum_{i,j} e_i e_j t_{ij} \\
&= \textstyle\sum_i e_i t_{ii} \\
&= \mathit{char}_e\big(\delta(\overrightarrow{t_i})\big) \\
&= \big(\delta \circ^{\mathrm{op}} \mathit{char}_e\big)(\overrightarrow{t_i}).
\end{aligned}$$

Finally, assuming a coalgebra $f\colon A \to \mathcal{B}_n(A)$ in $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$, we define effects $e_i = f(|\,i\,\rangle_A)$, where $|\,i\,\rangle_A = (0, \ldots, 1, \ldots, 0) \in ([0,1]_A)^n = [0,1]_{A^n} = [0,1]_{\mathcal{B}_n(A)}$. Clearly, $e_1 \ovee \cdots \ovee e_1 = f(1) = 1$. The equation $\varepsilon \circ^{\mathrm{op}} f = f \circ \mathit{out} = \mathrm{id}$ yields that f is a "map of bimodules":

$$b \cdot f(a_1, \ldots, a_n) \cdot c = f\big(\mathit{out}(b) \cdot (a_1, \ldots, a_n) \cdot \mathit{out}(c)\big) \tag{14}$$

This follows from [22, Thm. 1], because f is a PU-map and out a MIU-map, and will be used without further ado.

We can now prove that the e_i are mutually orthogonal projections. Consider the "matrix" $t = |\,j\,\rangle\langle\,i\,| \in \mathcal{B}_n(\mathcal{B}_n(A))$, so that $t(x)(y)$ is 1 if $x = i$ and $y = j$, and 0 otherwise. Then:

$$\begin{aligned}
e_i \cdot e_j \;=\; f(|\,i\,\rangle) \cdot e_j &= f\big(|\,i\,\rangle \cdot \mathit{out}(e_j)\big) && \text{by (14)} \\
&= f\big(0, \ldots, 0, e_j, 0, \ldots, 0\big) && \text{with } e_j \text{ at position } i \\
&= f\big(f(0), \ldots, f(|\,j\,\rangle), \ldots, f(0)\big) \\
&= \big(\mathcal{B}_n(f) \circ^{\mathrm{op}} f\big)(t) \\
&= \big(\delta \circ^{\mathrm{op}} f\big)(t) \\
&= f(\lambda x.\, t(x)(x)) \\
&= \begin{cases} f(|\,i\,\rangle) = e_i & \text{if } i = j \\ f(0) = 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

□

We conclude this section with some basic observations. First, the opening of a the block via $in_n \colon A \to \mathcal{B}_n(A)$ as in (11) can be described via the characteristic maps $A \to \mathcal{B}_n(A)$, for the "uniform" n-test $(\frac{1}{n}1, \ldots, \frac{1}{n}1)$. Alternatively, it may be understood as initialisation like in Example 4.4, given by the state $(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \in \mathbb{C}^n$.

Second, the functor $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \to (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ preserves block structures, since for $m \in \mathcal{K}\ell_{\mathbb{N}}(\mathcal{D})$ we have:

$$\mathcal{B}_n(\mathbb{C}^m) = (\mathbb{C}^m)^n \cong \mathbb{C}^{n\times m} = \mathbb{C}^{\mathcal{B}_n(m)}. \tag{15}$$

An n-test $p_1, \ldots, p_n \in [0,1]^m$ for $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \to \mathbf{EMod}^{\mathrm{op}}$ is at the same time an n-test for $(\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}} \to \mathbf{EMod}^{\mathrm{op}}$, since the effects $[0,1]_{\mathbb{C}^m} = [0,1]^m$ of $\mathbb{C}^m \in (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ are the same as the effects on $m \in \mathcal{K}\ell_{\mathbb{N}}(\mathcal{D})$, so that the diagram on the left commutes.

$$\begin{tikzcd}
& \mathbf{EMod}^{\mathrm{op}} & \\
\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \arrow[ur] \arrow[rr, "\mathbb{C}^{(-)}"'] & & (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}} \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{C}^m & \\
\mathbb{C}^{\mathcal{B}_n(m)} \arrow[ur, "\mathbb{C}^{char_p}"] \arrow[r, phantom, "\cong"] & (\mathbb{C}^m)^n \arrow[r, equal] & \mathcal{B}_n(\mathbb{C}^m) \arrow[ul, "char_p"']
\end{tikzcd}$$

The triangle on the right shows that the characteristic maps are also preserved via (15) in $\mathbf{CCstar}_{\mathrm{PU}}$, where $char_p$ on the left is in $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D})$, see Example 3.3, and $char_p$ on the right is in the category of C^*-algebras, see Proposition 4.5, using that $\mathbb{C}^m$ is commutative.

5 Matrix block structure on C^*-algebras

For a C^*-algebra A and number $n \in \mathbb{N}$ let $\mathcal{M}_n(A) = A^{n\times n}$ be the vector space of $n \times n$-matrices with entries from A. It is again a C^*-algebra with matrix multiplication, unit and conjugate transpose $(-)^{\dagger}$. Clearly $\mathcal{M}_1(A) \cong A$, but also $\mathcal{M}_k(\mathcal{M}_n(A)) \cong \mathcal{M}_{k\times n}(A)$. Hence these matrices behave like a block structure.

It turns out that $\mathcal{M}_n$ is not a functor $\mathbf{Cstar}_{\mathrm{PU}} \to \mathbf{Cstar}_{\mathrm{PU}}$, since $\mathcal{M}_n(f)$ need not be positive when f is positive. One therefore calls f *completely positive* when $\mathcal{M}_n(f)$ is positive, for each n. We write $\mathbf{Cstar}_{\mathrm{cPU}} \hookrightarrow \mathbf{Cstar}_{\mathrm{PU}}$ for the (non-full) subcategory of C^*-algebras with completely positive maps between them.

Each MIU-map is completely positive. When $f \colon A \to B$ is a PU-map, where either A or B is commutative, then f is completely positive. One thus requires complete positivity only in the non-commutative PU-case, that is, in a proper quantum setting. The following is the analogue of Proposition 4.3 for matrices.

Proposition 5.1 *Let $\overrightarrow{a_i} = (a_1, \ldots, a_n) \in A^n$ be an n-tuple in a C^*-algebra A. The tuple can be used to form "measurement" and "map" functions.*

(i) *If $\sum_i a_i^* a_i = 1$ there is a completely positive map:*

$$\mathcal{M}_n(A) \xrightarrow{meas(\overrightarrow{a_i})} M \quad \textit{given by} \quad M \longmapsto (a_1^* \ldots a_n^*) M \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

(ii) *If $a_i^* a_i = 1$ for each i, then there is a completely positive map:*

$$\mathcal{M}_n(A) \xrightarrow{map(\overrightarrow{a_i})} \mathcal{M}_n(A) \quad \textit{given by} \quad M \longmapsto \mathrm{diag}(\overrightarrow{a_i^*}) M \,\mathrm{diag}(\overrightarrow{a_i})$$

where $\mathrm{diag}(\overrightarrow{a_i})$ is the diagonal matrix $\begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix}$. □

We present an example later on in Example 5.5.

Lemma 5.2 *Taking $n \times n$-matrices yields a functor $\mathcal{M}_n \colon \boldsymbol{Cstar}_{cPU} \to \boldsymbol{Cstar}_{cPU}$, for each $n > 0$. It forms a block structure via "in" and "out" natural transformations in a commuting triangle in $(\boldsymbol{Cstar}_{cPU})^{op}$*

$$\begin{tikzcd}
A \arrow[r, "in_n"] \arrow[dr, equal] & \mathcal{M}_n(A) \arrow[d, "out_n"] \\
 & A
\end{tikzcd}$$

These natural transformations are given by:

$$in_n(M) = \tfrac{1}{n}\mathrm{tr}(M) = \tfrac{1}{n}\textstyle\sum_{i \le n} M_{ii} \qquad out_n(a) = aI_n = \begin{pmatrix} a & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a \end{pmatrix},$$

where $I_n \in \mathcal{M}_n(A)$ is the unit/identity matrix. Here, out_n is a MIU-map.

Moreover, the diagonal map $\mathrm{diag} \colon \mathcal{B}_n(A) \to \mathcal{M}_n(A)$ is a natural transformation that commutes with the in's and out's.

Proof By definition there is a functor $\mathcal{M}_n \colon \mathbf{Cstar}_{\mathrm{cPU}} \to \mathbf{Cstar}_{\mathrm{PU}}$. We have to prove that $\mathcal{M}_n(f)$ is completely positive, for a completely positive map f. Hence for each k, the map $\mathcal{M}_k(\mathcal{M}_n(f))$ must be positive. But the latter can also be described as $\mathcal{M}_{k \times n}(f)$, via the isomorphism $\mathcal{M}_k \circ \mathcal{M}_n \cong \mathcal{M}_{k \times n}$, which is positive because f is completely positive.

It is a basic fact that the trace map tr is completely positive. Hence so is $in_n = \frac{1}{n} tr$. The map $out_n \colon A \to \mathcal{M}_n(A)$ preserves multiplication and is thus completely positive. Clearly,

$$\big(out \circ^{\mathrm{op}} in\big)(a) = in\big(out(a)\big) = in \begin{pmatrix} a & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a \end{pmatrix} = \tfrac{1}{n} tr \begin{pmatrix} a & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a \end{pmatrix} = \tfrac{1}{n} na = a.$$

It is easy to see that *diag* is natural, *i.e.* that the equation $\mathcal{M}_n(f) \circ \mathit{diag} = \mathit{diag} \circ \mathcal{B}_n(f)$ holds. Moreover, *diag* commutes with the $\mathcal{B}$ and $\mathcal{M}$ maps:

$$\begin{array}{l}
\big(\mathit{diag} \circ^{\mathrm{op}} \mathit{in}^{\mathcal{M}}\big)(\overrightarrow{a_i}) \,=\, \mathit{in}^{\mathcal{M}} \left(\begin{smallmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{smallmatrix}\right) \,=\, \frac{1}{n}\sum_i a_i \,=\, \mathit{in}^{\mathcal{B}}(\overrightarrow{a_i}) \\
\big(\mathit{out}^{\mathcal{B}} \circ^{\mathrm{op}} \mathit{diag}\big)(a) \,=\, \mathit{diag}(a,\ldots,a) \,=\, \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a \end{smallmatrix}\right) \,=\, a \cdot I_n \,=\, \mathit{out}^{\mathcal{M}}(a). \qquad \square
\end{array}$$

The next step is to show that matrices form a logical block structure.

Proposition 5.3 *The matrix block structure $\mathcal{M}_n$ on $(\boldsymbol{Cstar}_{cPU})^{op}$ is logical, with:*

(i) *the universal n-test consisting of positive matrices $\Omega_i = |\,i\,\rangle\langle\,i\,| \in \mathcal{M}_n(A)$, clearly with $\ovee_i \Omega_i = I_n$;*

(ii) *for an arbitrary n-test $e_i \in [0,1]_A$ a characteristic map $\mathrm{char}_e \colon A \to \mathcal{M}_n(A)$ in $(\boldsymbol{Cstar}_{cPU})^{op}$ given by:*

$$\mathrm{char}_e(M) = (\sqrt{e_1}\ldots\sqrt{e_n})M \begin{pmatrix} \sqrt{e_1} \\ \vdots \\ \sqrt{e_n} \end{pmatrix}$$

The characteristic maps for the copower and matrix block structures $\mathcal{B}_n$ and $\mathcal{M}_n$ are related via the diagonal: $\mathrm{diag} \circ^{op} \mathrm{char}_e^{\mathcal{M}} = \mathrm{char}_e^{\mathcal{B}}$.

Proof Clearly $\mathit{char}_e^{-1}(\Omega_i) = \mathit{char}_e(|\,i\,\rangle\langle\,i\,|) = \sqrt{e_i}\sqrt{e_i} = e_i$. And:

$$\begin{aligned}
\big(\mathit{diag} \circ^{\mathrm{op}} \mathit{char}_e^{\mathcal{M}}\big)(a_1,\ldots,a_n) &= \mathit{char}_e^{\mathcal{M}} \left(\begin{smallmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{smallmatrix}\right) \\
&= (\sqrt{e_1}\ldots\sqrt{e_n}) \left(\begin{smallmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{smallmatrix}\right) \left(\begin{smallmatrix} \sqrt{e_1} \\ \vdots \\ \sqrt{e_n} \end{smallmatrix}\right) \\
&= \sqrt{e_1}a_1\sqrt{e_1} + \cdots + \sqrt{e_n}a_n\sqrt{e_n} \\
&= \mathit{char}_e^{\mathcal{B}}(a_1,\ldots,a_n). \qquad \square
\end{aligned}$$

The following result collects some standard facts.

Proposition 5.4 *Let $\mathcal{L}(H)$ be the set of bounded linear maps $H \to H$, where H is a Hilbert space. The mapping $H \mapsto \mathcal{L}(H)$ forms a functor*

$$\boldsymbol{Hilb}_{isom} \xrightarrow{\;\mathcal{L}\;} (\boldsymbol{Cstar}_{cPU})^{op}, \tag{16}$$

where $\boldsymbol{Hilb}_{isom}$ is the category of Hilbert spaces with isometries (dagger monos) between them. Each such a dagger mono $f \colon H \rightarrowtail K$ gives a completely positive map $\mathcal{L}(f) = f^\dagger \circ (-) \circ f \colon \mathcal{L}(K) \to \mathcal{L}(H)$. In this situation we have:

$$\mathcal{M}_n(\mathcal{L}(H)) \cong \mathcal{L}(H^n) \cong \mathcal{L}(H \otimes \mathbb{C}^n) \quad \textit{where} \quad H^n = H \oplus \cdots \oplus H.$$

*Thus $\mathcal{L}$ maps the copower block structure $n\cdot(-)$ on **Hilb**, from Example 2.5, to the matrix block structure $\mathcal{M}_n$ on $(\boldsymbol{Cstar}_{PU})^{op}$. This functor $\mathcal{L}$ also preserves effects and characteristic maps.*

Proof A matrix in $\mathcal{M}_n(\mathcal{L}(H))$ consists of $n \times n$ bounded maps $H \to H$. Since direct sum $\oplus$ is a biproduct for Hilbert spaces, these maps correspond to a single map $H^n \to H^n$, *i.e.* an element of $\mathcal{L}(H^n)$. Next we use that $\mathbb{C}$ is the tensor unit in **Hilb** and that $\otimes$ distributes over $\oplus$ in:

$$\begin{aligned} H^n \;=\; H \oplus \cdots \oplus H &\cong (H \otimes \mathbb{C}) \oplus \cdots \oplus (H \otimes \mathbb{C}) \\ &\cong H \otimes (\mathbb{C} \oplus \cdots \oplus \mathbb{C}) \;=\; H \otimes \mathbb{C}^n. \end{aligned}$$

For an isometry (dagger mono) $f \colon H \rightarrowtail K$ in **Hilb** we have $\mathcal{L}(f) = f^\dagger \circ (-) \circ f \colon \mathcal{L}(K) \to \mathcal{L}(H)$. We use $\mathcal{M}_n(B(\mathcal{H})) \cong B(H \otimes \mathbb{C}^n)$, with the map corresponding to $\mathcal{M}_n(\mathcal{L}(f))$ being:

$$\mathcal{L}(K \otimes \mathbb{C}^n) \xrightarrow{(f^\dagger \otimes \mathrm{id}) \circ (-) \circ (f \otimes \mathrm{id})} \mathcal{L}(H \otimes \mathbb{C}^n).$$

We show that if $g \in \mathcal{L}(K \otimes \mathbb{C}^n)$ is positive, then so is $(f^\dagger \otimes \mathrm{id})g(f \otimes \mathrm{id}) \in \mathcal{L}(H \otimes \mathbb{C}^n)$. For a vector $v \in H \otimes \mathbb{C}^n$, write $w = (f \otimes \mathrm{id})v$; then, using that g is positive:

$$\langle\, v \,|(f^\dagger \otimes \mathrm{id})g(f \otimes \mathrm{id})|\, v \,\rangle = \langle\, (f \otimes \mathrm{id})v \,|g|\, (f \otimes \mathrm{id})v \,\rangle = \langle\, w \,|g|\, w \,\rangle \geq 0.$$

The effects associated with the C^*-algebra $\mathcal{L}(H)$ are the effects $\mathcal{E}f(H) = [0,1]_{\mathcal{L}(H)}$ described Example 3.5. Thus the triangle on the left below commutes.

$$\begin{tikzcd} & \mathbf{EMod}^{\mathrm{op}} & \\ \mathbf{Hilb}_{\mathrm{isom}} \arrow[ur] \arrow[rr, "\mathcal{L}"'] & & (\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}} \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{L}(H) & \\ \mathcal{L}(\mathcal{B}_n(H)) \arrow[ur, "\mathcal{L}(\mathit{char}_A)"] \arrow[rr, equal, "\sim"] & & \mathcal{M}_n(H) \arrow[ul, "\mathit{char}_A"'] \end{tikzcd}$$

For an n-test $A = (A_1, \ldots, A_n)$ in $\mathcal{E}f(H)$, the triangle on the right also commutes in $(\mathbf{Cstar}_{\mathrm{PU}})$. The *char* map on the left is as in Example 3.5, and the one on the right as in Proposition 5.3. As described above, a map

$f\colon \mathcal{B}_n(H) \to \mathcal{B}_n(H)$ corresponds to a matrix M_f. Then:

$$\begin{aligned}
char^{\mathcal{M}}_A(M_f) &= (\sqrt{A_1} \dots \sqrt{A_n}) \begin{pmatrix} \pi_1 \circ f \circ \kappa_1 & \cdots & \pi_1 \circ f \circ \kappa_n \\ \vdots & & \vdots \\ \pi_n \circ f \circ \kappa_1 & \cdots & \pi_n \circ f \circ \kappa_n \end{pmatrix} \begin{pmatrix} \sqrt{A_1} \\ \vdots \\ \sqrt{A_n} \end{pmatrix} \\
&\overset{(*)}{=} [\sqrt{A_1} \dots \sqrt{A_n}] \circ f \circ \langle \sqrt{A_1} \dots \sqrt{A_n} \rangle \\
&= \langle \sqrt{A_1} \dots \sqrt{A_n} \rangle^\dagger \circ f \circ \langle \sqrt{A_1} \dots \sqrt{A_n} \rangle \\
&= \mathcal{L}(\langle \sqrt{A_1} \dots \sqrt{A_n} \rangle)(f) \\
&= \mathcal{L}(char^{\mathcal{B}}_A)(f).
\end{aligned}$$

The marked equation $\overset{(*)}{=}$ involves some elementary calculations with biproducts $\oplus$ in **Hilb**. □

Example 5.5 Consider the "identity" and "negation" matrices $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as elements $I_2, X \in \mathcal{L}(\mathbb{C}^2)$. The "map" operation from Proposition 5.1 (ii) yields:

$$\mathcal{M}_2(\mathcal{L}(\mathbb{C}^2)) \xrightarrow{map(I_2,X)} \mathcal{M}_2(\mathcal{L}(\mathbb{C}^2)) \quad \text{given by} \quad M \longmapsto \left(\begin{smallmatrix} I_2 & 0 \\ 0 & X \end{smallmatrix}\right) M \left(\begin{smallmatrix} I_2 & 0 \\ 0 & X \end{smallmatrix}\right)$$

Via the isomorphism $\mathcal{M}_2(\mathcal{L}(\mathbb{C}^2)) \cong \mathcal{L}(\mathbb{C}^4)$ this is the operation $\mathcal{L}(\mathsf{CNOT})\colon \mathcal{L}(\mathbb{C}^4) \to \mathcal{L}(\mathbb{C}^4)$, where CNOT is the "conditional negation" matrix:

$$\mathsf{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left(\begin{smallmatrix} I_2 & 0 \\ 0 & X \end{smallmatrix}\right).$$

Remark 5.6 The category $\mathbf{Cstar}_{\mathrm{cPU}}$ also has monoidal structure. In fact, there is a "minimal" and a "maximal" tensor $A \otimes B$, but as long as either A or B is finite-dimensional, they coincide (with $\mathbb{C}$ as tensor unit). Via these tensors we can see a closer analogy between the copower and matrix block structures $\mathcal{B}$ and $\mathcal{M}$ on C^*-algebras, namely:

$$\mathcal{B}_n(A) = A^n \cong \mathbb{C}^n \otimes A = \mathcal{B}_n(\mathbb{C}) \otimes A \quad \text{and} \quad \mathcal{M}_n(A) \cong \mathcal{M}_n(\mathbb{C}) \otimes A.$$

In particular, for a Hilbert space H tensors are preserved:

$$\mathcal{L}(\mathbb{C}^n \otimes H) \cong \mathcal{M}_n(H) \cong \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{L}(H) = \mathcal{L}(\mathbb{C}^n) \otimes \mathcal{L}(H).$$

6 Examples and discussion

So far we have seen examples of block structures in a non-deterministic and probabilistic setting — in the Kleisli categories $\mathcal{K}\ell(\mathcal{P})$ and $\mathcal{K}\ell(\mathcal{D})$ — and also in a quantum setting, in the categories of Hilbert spaces and of C^*-algebras. In the latter setting we have seen two block structures, namely copower $\mathcal{B}_n(A) = n \cdot A$ and matrix $\mathcal{M}_n(A)$. It seems that $\mathcal{B}_n$ is most appropriate in a commutative/probabilistic setting, and $\mathcal{M}_n$ in a quantum setting, because:

- $\mathcal{B}_n$ is a comonad, involving a copying operation; $\mathcal{M}_n$ is not a comonad, since copying is impossible in a non-commutative setting, see *e.g.* [17].
- The functor $\mathcal{K}\ell(\mathcal{D}) \to (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ putting probabilistic transitions in a C^*-algebraic context commutes with $\mathcal{B}_n$.
- The functor $\mathbf{Hilb}_{\mathrm{isom}} \to (\mathbf{Cstar}_{\mathrm{cPU}})^{\mathrm{op}}$ from (16) commutes with $\mathcal{M}_n$.

The issue of which block structure to use, in which situation, remains unclear and will be further explored in follow-up research. In the remainder of this section we briefly investigate how block structures can be used to describe familiar quantum protocols like superdense coding and teleportation as maps in the category of C^*-algebras. Such descriptions can be used to represent the protocols in computer algebra tools, for simulation and verification. The whole point that we are trying to suggest is that logical blocks may form a clean language construct in a future (quantum) programming language.

We start by recalling some basic material. The Bell basis of $\mathbb{C}^4$ is given by the vectors:

$$\begin{array}{ll} |\,b_1\,\rangle = \frac{1}{\sqrt{2}}(|\,00\,\rangle + |\,11\,\rangle) \qquad & |\,b_2\,\rangle = \frac{1}{\sqrt{2}}(|\,01\,\rangle + |\,10\,\rangle) \\ |\,b_3\,\rangle = \frac{1}{\sqrt{2}}(|\,00\,\rangle - |\,11\,\rangle) \qquad & |\,b_4\,\rangle = \frac{1}{\sqrt{2}}(|\,01\,\rangle - |\,10\,\rangle) \end{array}$$

The associated projections $e_i = |\,b_i\,\rangle\langle\,b_i\,| \in \mathcal{E}\!f(\mathbb{C}^4)$ can be described by the matrices:

$$e_1 = \frac{1}{2}\begin{pmatrix} 1&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&1 \end{pmatrix} \quad e_2 = \frac{1}{2}\begin{pmatrix} 0&0&0&0 \\ 0&1&1&0 \\ 0&1&1&0 \\ 0&0&0&0 \end{pmatrix} \quad e_3 = \frac{1}{2}\begin{pmatrix} 1&0&0&-1 \\ 0&0&0&0 \\ 0&0&0&0 \\ -1&0&0&1 \end{pmatrix} \quad e_4 = \frac{1}{2}\begin{pmatrix} 0&0&0&0 \\ 0&1&-1&0 \\ 0&-1&1&0 \\ 0&0&0&0 \end{pmatrix}$$

They satisfy $e_1 \ovee e_2 \ovee e_3 \ovee e_4 = \mathrm{id}$ and thus form a 4-test in $\mathcal{E}\!f(\mathbb{C}^4) = [0,1]_{\mathcal{L}(\mathbb{C}^4)}$. Since $e_i^2 = e_i$ we have $\sqrt{e_i} = e_i$. Further, because the Bell basis is orthogonal, we have $e_i e_j = 0$ for $i \neq j$. We shall write $\mathit{char}_{\mathit{Bell}}$ in $(\mathbf{Cstar}_{\mathrm{cPU}})^{\mathrm{op}}$ for the associated measurement operation $\mathcal{L}(\mathbb{C}^4) \to \mathcal{M}_4(\mathcal{L}(\mathbb{C}^4))$.

Next we need the four Pauli matrices in $\mathcal{L}(\mathbb{C}^2)$:

$$\sigma_1 = \left(\begin{smallmatrix} 1&0 \\ 0&1 \end{smallmatrix}\right) \qquad \sigma_2 = X = \left(\begin{smallmatrix} 0&1 \\ 1&0 \end{smallmatrix}\right) \qquad \sigma_3 = Z = \left(\begin{smallmatrix} 1&0 \\ 0&-1 \end{smallmatrix}\right) \qquad \sigma_4 = XZ = \left(\begin{smallmatrix} 0&-1 \\ 1&0 \end{smallmatrix}\right).$$

They all satisfy $\sigma_i^\dagger \sigma_i = I_2$, and may thus be used in "map" constructions, like in Propositions 4.3 and 5.1

6.1 Superdense coding

What the superdense coding algorithm of [2] achieves is sending two classical bits via one (entangled) qubit. Two parties, Alice and Bob each possess one qubit of a shared entangled (Bell) state. Alice applies one of 4 operations σ_i to her qubit — thus encoding one the four options $i \in 4$ given by 2 classical bits — and sends the result to Bob. Through the local operations, represented as $\sigma_i \otimes \mathrm{id} \in \mathcal{L}(\mathbb{C}^4)$, Alice affects the shared state. By performing a Bell measurement Bob can find out which of the four operations σ_i was applied by Alice, and thus which $i \in 4$ is transmitted.

Our block-based representation of the superdense coding protocol consists of the following four maps in the category $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathcal{L}(\mathbb{C}^4) \arrow[d, "in_4^{\mathcal{B}}"'] & & \mathcal{B}_4(\mathcal{L}(\mathbb{C}^4)) \\
\mathcal{B}_4(\mathcal{L}(\mathbb{C}^4)) \arrow[r, "map(\overrightarrow{\sigma_i \otimes \mathrm{id}})"] & \mathcal{B}_4(\mathcal{L}(\mathbb{C}^4)) \arrow[r, "\mathcal{B}_4(char_{Bell})"] & \mathcal{B}_4(\mathcal{M}_4(\mathcal{L}(\mathbb{C}^4))) \arrow[u, "\mathcal{B}_4(out_4^{\mathcal{M}})"']
\end{tikzcd} \tag{17}$$

First a copower 4-block is opened to deal with the four classical options (corresponding to the two classical bits at hand). In each of these four options Alice performs one of the operations σ_i, only to her part of the shared state, via $\sigma_i \otimes \mathrm{id}$. These operations are combined in a single one via "map". At this stage Alice transfers her qubit to Bob, and Bob owns the whole state. In each of the four block options he performs a Bell measurement. Then he closes the outer block. The outcome of these Bell measurements distinguishes the various block options and enables Bob to recognise these options.

The computation (17) in $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$ consists of a computation $\mathcal{B}_4(\mathcal{L}(\mathbb{C}^4)) \to \mathcal{L}(\mathbb{C}^4)$ that computes the weakest precondition. We shall compute it with the above Bell projections (e_1, e_2, e_3, e_4) as input to this computation going backwards:

$$\begin{aligned}
&\big(\mathcal{B}_4(out_4^{\mathcal{M}}) \circ^{\mathrm{op}} \mathcal{B}_4(char_{Bell}) \circ^{\mathrm{op}} map(\overrightarrow{\sigma_i \otimes \mathrm{id}}) \circ^{\mathrm{op}} in_4^{\mathcal{B}}\big)(e_1, e_2, e_3, e_4) \\
&\;= \big(in_4^{\mathcal{B}} \circ map(\overrightarrow{\sigma_i \otimes \mathrm{id}}) \circ (char_{Bell})^4\big)\big(e_1 I_4, e_2 I_4, e_3 I_4, e_4 I_4\big) \\
&\;= \big(in_4^{\mathcal{B}} \circ map(\overrightarrow{\sigma_i \otimes \mathrm{id}})\big)\big(\textstyle\sum_i \sqrt{e_i} e_1 \sqrt{e_i}, \sum_i \sqrt{e_i} e_2 \sqrt{e_i}, \sum_i \sqrt{e_i} e_3 \sqrt{e_i}, \sum_i \sqrt{e_i} e_4 \sqrt{e_i}\big) \\
&\;= \big(in_4^{\mathcal{B}} \circ map(\overrightarrow{\sigma_i \otimes \mathrm{id}})\big)(e_1, e_2, e_3, e_4) \\
&\;= in_4^{\mathcal{B}}\big((\sigma_1 \otimes \mathrm{id})^\dagger e_1 (\sigma_1 \otimes \mathrm{id}), (\sigma_2 \otimes \mathrm{id})^\dagger e_2 (\sigma_2 \otimes \mathrm{id}), \\
&\qquad\qquad (\sigma_3 \otimes \mathrm{id})^\dagger e_3 (\sigma_3 \otimes \mathrm{id})(\sigma_4 \otimes \mathrm{id})^\dagger e_4 (\sigma_4 \otimes \mathrm{id})\big) \\
&\;\overset{(*)}{=} in_4^{\mathcal{B}}(e_1, e_1, e_1, e_1) \\
&\;= e_1.
\end{aligned}$$

The equalities $(\sigma_i^\dagger \otimes \mathrm{id}) e_i (\sigma_i \otimes \mathrm{id}) = e_1$ used in marked equation $\overset{(*)}{=}$ are left to the reader.

This calculation for (17) can be interpreted as follows. In order to get as postcondition (e_1, e_2, e_3, e_4), one has to start the computation with precondition e_1. This precondition $e_1 = |\, b_1 \,\rangle\langle\, b_1 \,|$ for $|\, b_1 \,\rangle = \frac{1}{\sqrt{2}}(|\, 00 \,\rangle + |\, 11 \,\rangle)$ is the shared Bell state that usually serves as starting point for super dense coding.

6.2 Teleportation

For the teleportation protocol (see *e.g.* [18]) we open a "matrix" block via initialisation. The bell basis vector $|\, b_1 \,\rangle = \frac{1}{\sqrt{2}}(|\, 00 \,\rangle + |\, 11 \,\rangle) \in \mathbb{C}^4$ gives rise to a (dagger monic) map $\mathrm{id} \otimes |\, b_1 \,\rangle \colon \mathbb{C}^2 \rightarrowtail \mathbb{C}^2 \otimes \mathbb{C}^4$. By applying the functor $\mathcal{L}$ from Proposition 5.4 we obtain:

$$\mathcal{L}(\mathbb{C}^2) \longrightarrow \mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^4) \cong \mathcal{M}_4(\mathcal{L}(\mathbb{C}^2))$$

This is the first map in the protocol below.

$$\begin{tikzcd}
\mathcal{L}(\mathbb{C}^2) \arrow[d] & & & \mathcal{L}(\mathbb{C}^2) \\
\mathcal{M}_4(\mathcal{L}(\mathbb{C}^2)) \arrow[r, "char_{Bell}"] & \mathcal{B}_4(\mathcal{M}_4(\mathcal{L}(\mathbb{C}^2))) \arrow[r, "\mathcal{B}_4(out_4^{\mathcal{M}})"] & \mathcal{B}_4(\mathcal{L}(\mathbb{C}^2)) \arrow[r, "map(\overrightarrow{\sigma_i})"] & \mathcal{B}_4(\mathcal{L}(\mathbb{C}^2)) \arrow[u, "out_4^{\mathcal{B}}"']
\end{tikzcd}$$

In this case, after initialisation Alice does a measurement $char_{Bell}$ giving a copower block $\mathcal{B}_4$ in order to transfer two bits of information to Bob. Here we consider the above matrices e_i as matrices over $\mathcal{L}(\mathbb{C}^2)$. In each of the resulting 4 block options Bob does an adjustment, with the Pauli matrices σ_i. It can be shown that the resulting map $\mathcal{L}(\mathbb{C}^2) \to \mathcal{L}(\mathbb{C}^2)$ is the identity.

Conclusions

This paper presents the first steps towards understanding the structure and role of blocks and predicates in non-deterministic / probabilistic / quantum programming. The opening of blocks via characteristic maps (measurements) induced by n-tests in effect algebras is common in these approaches. For the particular case of "von Neumann" n-tests of projections this can be described via Eilenberg-Moore coalgebras. In the general case there is much variation that requires further investigation.

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