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Comprehension for Coalgebras

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Abstract

The notion of an endofunctor having "greatest subcoalgebras" is introduced as a form of comprehension. This notion is shown to be instrumental in giving a systematic and abstract proof of the existence of limits for coalgebras—proved earlier by Worrell and by Gumm & Schröder. These insights, in dual form, are used to reinvestigate colimits for algebras in terms of "least quotient algebras"—leading to a uniform approach to limits of coalgebras and colimits of algebras. Finally, at an abstract level of fibrations, an equivalence is established between having greatest subcoalgebras (in a base category of types) and greatest invariants (in a total category of predicates).

1 Introduction

The setting of this paper is the modern theory of (co)algebras [12,19,4]. The paper concentrates on two technical topics for these coalgebras: (1) limits, and (2) comprehension and invariants. It starts with coalgebras $X \to F(X)$ of an endofunctor $F: \mathbb{C} \to \mathbb{C}$ on an arbitrary category $\mathbb{C}$. In such a situation it is well-known, and easy to see, that the category $\mathrm{CoAlg}(F)$ of coalgebras of the functor F has the same type of colimits that $\mathbb{C}$ has. Technically, the forgetful functor $\mathrm{CoAlg}(F) \to \mathbb{C}$ creates colimits. But limits for coalgebras are a different story. This situation is dual to the more familiar situation for algebras, where limits are easy, but colimits require work, see for instance [1, §3.4, Theorem 1 and §9.3, Proposition 4] or [2, Volume 2, §4.3]. Limits for coalgebras have been studied earlier by Worrell [20] (see also [13]) and by Gumm & Schröder [5,4]. Here we use an axiomatic approach, and show how a notion of comprehension for coalgebras is an essential ingredient for the existence of limits.

This comprehension can be understood in terms of "greatest subcoalgebras", induced by predicates. The second topic of this paper investigates this aspect of comprehension in the framework of fibrations (also called fibred categories) [10]. It is shown that these greatest coalgebras typically come about via greatest invariants. The latter form a notion which can be expressed appropriately in the logic of a fibration via so-called predicate lifting [6].

The contributions of this paper are five-fold.

- It introduces a notion of comprehension for coalgebras, expressed in terms of greatest subcoalgebras, see Definition 2.1. This notion immediately transfers equalisers to a category of coalgebras, from the underlying category, see Theorem 4.1.
- It generalises a standard adjunction result for coalgebras [19, Theorem 17.1], making crucial use of greatest subcoalgebras, see Theorem 3.1.
- It uses this generalised theorem in a new, snappy proof of the existence of products of coalgebras, see Theorem 4.2. It thus identifies greatest subcoalgebras as a key ingredient for the transfer of limits.
- It dualises this approach to obtain existence results for colimits of algebras, see Section 5.
- It gives the connection between the logical notion of "greatest invariants" and the notion of "greatest subcoalgebras" from universal coalgebra. Both are described as suitable right adjoints in a context of fibred categories, where there is a clean separation between a logical and a type theoretical level. This connection forms the most technical part of the paper.

Further, the paper briefly discusses towards the end of Section 4 the earlier proofs of the existence of limits of coalgebras.

2 Comprehension and greatest subcoalgebras

This section starts by recalling the notion of comprehension (or subset types) from categorical logic, see [10, §4.6]. It is shown how this notion for coalgebras gives rise to greatest subcoalgebras. Then, in an example, it becomes clear how greatest invariants give rise to greatest subcoalgebras.

Comprehension involves the mapping of a predicate to a set, namely the set of elements satisfying the predicate. We shall first give a categorical description for the standard logic of sets. Let **Sets** therefore be the usual category of sets and functions. We write **Pred** for the category of predicates $(P \subseteq X)$ on sets. Its morphisms $(P \subseteq X) \xrightarrow{f} (Q \subseteq Y)$ are functions $f: X \to Y$ mapping P to Q: $f(x) \in Q$ for all $x \in P$. There is then an obvious forgetful functor C: **Pred** $\to$ **Sets** mapping a predicate $(P \subseteq X)$ to its carrier X. It is a "fibration", see [10], but that is not very relevant at this stage.

There is a "truth" functor $\top$: **Sets** $\to$ **Pred** sending a set X to the truth predicate $\top(X) = (X \subseteq X)$. It is not hard to see that $\top$ is right adjoint to C.

Additionally, there is a "comprehension" or "subset type" functor $\{-\}$: **Pred** $\to$ **Sets** sending predicates to sets, via $(P \subseteq X) \mapsto P$. Interestingly, $\{-\}$ is right adjoint to $\top$, so that there is a situation, like on the left below.

(1)
$$
\begin{tikzcd}
\mathbf{Pred} \arrow[d, bend right=60, "C"'] \arrow[d, bend left=60, "\{-\}"] \\
\mathbf{Sets} \arrow[u, "\dashv\top\dashv" description]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{Sub}(\mathbb{C}) \arrow[d, bend right=60, "\mathrm{cod}"'] \arrow[d, bend left=60, "\mathrm{dom}"] \\
\mathbb{C} \arrow[u, "\dashv\top\dashv" description]
\end{tikzcd}
$$

We generalise this a bit. Let $\mathbb{C}$ be an arbitrary category. A subobject in $\mathbb{C}$ is an equivalence class of monos with a common codomain, usually written simply as $(P \rightarrowtail X)$, for objects $X, P \in \mathbb{C}$. Such a subobject is considered as a predicate on $X \in \mathbb{C}$, generalising the above predicates on $X \in \mathbf{Sets}$.

We shall write $\mathrm{Sub}(\mathbb{C})$ for the category of such subobjects in $\mathbb{C}$. A morphism $f\colon (P \rightarrowtail X) \to (Q \rightarrowtail Y)$ in $\mathrm{Sub}(\mathbb{C})$ is given by a morphism $f\colon X \to Y$ in $\mathbb{C}$ for which there is a necessarily unique dashed map $P \to Q$ yielding a commuting diagram. This yields a category. For $\mathbb{C} = \mathbf{Sets}$, we get $\mathrm{Sub}(\mathbb{C}) = \mathbf{Pred}$.

There is an obvious forgetful functor $\mathrm{cod}\colon \mathrm{Sub}(\mathbb{C}) \to \mathbb{C}$, which is again a fibration. Also, there is a right adjoint $\top\colon \mathbb{C} \to \mathrm{Sub}(\mathbb{C})$ to this forgetful functor, sending an object $X \in \mathbb{C}$ to the truth predicate $(X \rightarrowtail X)$ given by the identity map. Also, truth has a right adjoint $\{-\} = \mathrm{dom}\colon \mathrm{Sub}(\mathbb{C}) \to \mathbb{C}$ describing comprehension: it is given by $(P \rightarrowtail X) \mapsto P$. Thus the adjoint situation on the right, above, generalises the one one the left. This description of comprehension as right adjoint to truth is due to [3], and goes ultimately back to [16]. See also [18].

Next assume we have an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$. Its category of coalgebras, comes with a forgetful functor $U\colon \mathrm{CoAlg}(F) \to \mathbb{C}$. Thus we can form the pullback of functors:

(2)
$$
\begin{tikzcd}
\mathrm{CoAlgSub}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sub}(\mathbb{C}) \arrow[d, "\mathrm{cod}"] \\
\mathrm{CoAlg}(F) \arrow[r, "U"'] & \mathbb{C}
\end{tikzcd}
$$

This means that the objects of the category $\mathrm{CoAlgSub}(F)$ are given by subobjects on carriers of coalgebras, written as $(P \rightarrowtail X \to F(X))$. A morphism $(P \rightarrowtail X \to F(X)) \longrightarrow (Q \rightarrowtail Y \to F(Y))$ is a morphism $X \to Y$ in $\mathbb{C}$ which is at the same time a morphism of subobjects $(P \rightarrowtail X) \longrightarrow (Q \rightarrowtail Y)$ and of coalgebras $(X \to F(X)) \longrightarrow (Y \to F(Y))$. Notice that the subobjects $P \rightarrowtail X$ in objects $(P \rightarrowtail X \to F(X))$ in $\mathrm{CoAlgSub}(F)$ are subobjects in $\mathbb{C}$, and not in $\mathrm{CoAlg}(F)$. The latter are called **subcoalgebras**.

There is also an (induced) truth functor $\top_F\colon \mathrm{CoAlg}(F) \to \mathrm{CoAlgSub}(F)$, sending a coalgebra $(X \to F(X))$ to $(X \rightarrowtail X \to F(X))$ with subobject given by the identity map on X. Thus, the square (2) can be extended to a commuting square with truth functors $\top$.

Can we also add comprehension? It does not come for free.

Definition 2.1 *We say that a functor $F\colon \mathbb{C} \to \mathbb{C}$ has* ***greatest subcoalgebras*** *if the* *truth functor* $\top_F\colon$ *CoAlg*$(F) \to$ *CoAlgSub*(F) *has a right adjoint* $[-]$*, as in:*

$$\begin{tikzcd}
\mathit{CoAlgSub}(F) \arrow[rrr] \arrow[d, bend right=60] \arrow[d, bend left=60, "{[-]}"] & & & \mathit{Sub}(\mathbb{C}) \arrow[d, bend right=60, "\mathit{cod}"'] \arrow[d, bend left=60, "\mathit{dom}"] \\
\mathit{CoAlg}(F) \arrow[rrr] \arrow[u, "\dashv\ \top_F\ \dashv"] & & & \mathbb{C} \arrow[u, "\dashv\ \top\ \dashv"]
\end{tikzcd}$$

Note that we do not require that the functors $[-]$ *and dom make this diagram commute.*

Explicitly, we shall use the following notation for such an adjoint $[-]$. It sends an object $(P \rightarrowtail X \to F(X))$ to a coalgebra written as $[c]\colon [P] \to F([P])$. By unravelling the adjunction it becomes clear what is happening. The adjunction involves a bijective correspondence:

$$\frac{\left(Y \overset{\mathrm{id}}{\rightarrowtail} Y \overset{d}{\to} F(Y)\right) \xrightarrow{\quad f \quad} \left(P \overset{m}{\rightarrowtail} X \overset{c}{\to} F(X)\right)}{\left(Y \overset{d}{\to} F(Y)\right) \xrightarrow[\quad g \quad]{} \left([P] \overset{[c]}{\to} F([P])\right)}$$

That is:

$$\begin{tikzcd}
F(Y) \arrow[r, "F(f)"] & F(X) \\
Y \arrow[u, "d"] \arrow[r, "f"'] & X \arrow[u, "c"'] \\
Y \arrow[u, tail, "\mathrm{id}"] \arrow[r, dashed] & P \arrow[u, tail, "m"']
\end{tikzcd}$$

$$\overline{\overline{\hspace{8em}}}$$

$$\begin{tikzcd}
F(Y) \arrow[r, "F(g)"] & F([P]) \\
Y \arrow[u, "d"] \arrow[r, "g"'] & {[P]} \arrow[u, "{[c]}"']
\end{tikzcd}$$

This means that a coalgebra homomorphism factors through the subobject P if and only if it yields a coalgebra homomorphism to $[P]$. This can be made more explicit by involving the counit $\varepsilon\colon \top_F \circ [-] \Rightarrow \mathrm{id}$ of the adjunction:

$$\begin{tikzcd}
F([P]) \arrow[r, "F(\varepsilon)"] & F(X) \\
{[P]} \arrow[u, "{[c]}"] \arrow[r, "\varepsilon"'] \arrow[dr, dashed] & X \arrow[u, "c"'] \\
& P \arrow[u, tail, "m"']
\end{tikzcd}$$

This ε is a monomorphism in CoAlg(F), making $[P]$ a subcoalgebra. The argument is like in [10, Lemma 4.6.2 (i)]: assume a coalgebra $Z \to F(Z)$ with two homomorphisms $h, k: Z \to [P]$ satisfying $\varepsilon \circ h = \varepsilon \circ k$. The corresponding transposes $h', k': (Z \rightarrowtail Z \to F(Z)) \to (P \rightarrowtail X \to F(X))$ are then equal. But then h, k must be equal.

Summarising, an adjunction $\top_F \dashv [-]$ as in Definition 2.1 yields for each subobject $P \rightarrowtail X$ on the carrier of a coalgebra $c: X \to F(X)$ a subcoalgebra $[c]: [P] \to F([P])$ of c with the following universal property. A map $f: Y \to X$ forming a coalgebra homomorphism $(Y \xrightarrow{d} F(Y)) \longrightarrow (X \xrightarrow{c} F(X))$ factors through the subobject $P \rightarrowtail X$ if and only if it factors through the subcoalgebra $\varepsilon: [c] \rightarrowtail c$ via a (necessarily unique) coalgebra homomorphism $d \to [c]$.

Later, in Section 6, we shall consider greatest subcoalgebras with respect to a more general fibration than the subobject fibration $\mathrm{Sub}(\mathbb{C}) \to \mathbb{C}$. But in the remainder of this section we consider examples of functors with greatest subcoalgebras as in Definition 2.1.

Example 2.2 (i) *Let F: **Sets** $\to$ **Sets** be a polynomial functor as in [11]. For a coalgebra $c: X \to F(X)$ there is an associated notion of **invariant** for a predicate $P \subseteq X$, stating that c maintains P. The notation $\Box P \subseteq X$ is used for the greatest invariant contained in P. It is the carrier of the greatest subcoalgebra associated with $(P \hookrightarrow X \to F(X))$. A bit formally, the adjoint $[P]$ from Definition 2.1 is obtained as $\{\Box P \subseteq X\}$, involving a greatest invariant and comprehension, as in (1).*

(ii) *Let $F: \mathbb{C} \to \mathbb{C}$ be a functor preserving weak pullbacks (see [19]) on a category $\mathbb{C}$ with pullbacks along monos. It is easy to see that the functor F then sends monos to monos, and that F preserves pullbacks along such monos: $F(f^*(m)) \cong F(f)^*(F(m))$, for a mono m and arbitrary map f.*

We assume that arbitrary joins $\bigvee$ and meets $\bigwedge$ exist of subobjects in $\mathbb{C}$. This allows us to construct greatest subcoalgebras, via images and greatest invariants:

(a) *Given an arbitrary map $f: X \to Y$ in $\mathbb{C}$ and a subobject $Q \rightarrowtail X$ we write*

$$\textstyle\coprod_f(Q) = \bigwedge\{S \leq Y \mid Q \leq f^*(S)\}.$$

It satisfies:

$$\textstyle\coprod_f(Q) \leq P \Longleftrightarrow Q \leq f^*(P)$$

using that pullback functors f^* preserve arbitrary meets.

(b) *For a subobject $P \rightarrowtail X$ on the carrier of a coalgebra $c: X \to F(X)$, we define a new subobject $\Box P \rightarrowtail X$ as:*

$$\Box P = \bigvee\{S \leq X \mid S \leq P \text{ and } S \leq c^*(F(S))\}.$$

Clearly, $\Box P \leq P$. Also, $\Box P$ inherits a subcoalgebra structure $[c]: \Box P \to F(\Box P)$, since for each subobject $S \leq P$ with $S \leq c^*(F(S))$ we have $S \leq \Box P$, and thus by $S \leq c^*(F(\Box P))$. But then $\Box P \leq c^*(F(\Box P))$, as required.

Now we are in a position to prove that we have greatest subcoalgebras. Assume $P \rightarrowtail X$, and a coalgebra homomorphism $(Y \stackrel{d}{\to} F(Y)) \stackrel{f}{\longrightarrow} (X \stackrel{c}{\to} F(X))$ which factors through P. The latter can be expressed as $id_Y \leq f^*(P)$, or equivelently as $\coprod_f(id_Y) \leq P$—where id_Y is the identity (top) subobject on Y. We have to prove $\coprod_f(id_Y) \leq \Box P$, which amounts to proving $\coprod_f(id_Y) \leq c^*(F(\coprod_f(id_Y)))$, or equivalently, $id_Y \leq f^*c^*(F(\coprod_f(id_Y)))$. This holds because:

$$\begin{aligned}
f^*c^*(F(\textstyle\coprod_f(id_Y))) &= d^*F(f)^*(F(\textstyle\coprod_f(id_Y))) \\
&\qquad \text{since } f \text{ is a homomorphism} \\
&= d^*F\Big(f^*(\textstyle\coprod_f(id_Y))\Big) \\
&\qquad \text{because } F \text{ preserves pullbacks} \\
&= d^*F(id_Y) \\
&\qquad \text{since } id_Y \leq f^*(\textstyle\coprod_f(id_Y)) \\
&= d^*(id_{F(Y)}) \\
&= id_Y.
\end{aligned}$$

What we see is that the greatest subcoalgebras $[P]$ arise via a combination $\{\Box P\}$ of comprehension $\{-\}$ and greatest invariants $\Box$. Similarly one expects least subcoalgebras via a combination of comprehension and least invariants. A properly abstract investigation of these phenomena using fibred category theory is given later in Section 6.

3 Adjoint lifting for coalgebras

This section introduces an auxiliary result about lifting adjoints to categories of coalgebras. It generalises [19, Theorem 17.1]. The result makes use of "cofree coalgebras", which will be explained first.

Let $\mathbb{C}$ be an arbitrary category, with an endofunctor $F: \mathbb{C} \to \mathbb{C}$. It gives rise to a category of coalgebras $\mathrm{CoAlg}(F)$ with a forgetful functor $U: \mathrm{CoAlg}(F) \to \mathbb{C}$. We say that F admits **cofree** coalgebras if this functor U has a right adjoint $G: \mathbb{C} \to \mathrm{CoAlg}(F)$, sending an object $X \in \mathbb{C}$ to a coalgebra $UGX \to F(UGX)$.

As an aside: it is a folklore result that in this case the composition $UG: \mathbb{C} \to \mathbb{C}$ is a comonad, whose category of Eilenberg-Moore coalgebras of the comonad is isomorphic to the original category $\mathrm{CoAlg}(F)$ of functor-coalgebras, see for instance [9, Theorem 1].

We recall [19, Theorem 17.1]. It assumes two functors $F, H: \mathbf{Sets} \to \mathbf{Sets}$ with a natural transformation $\sigma: F \Rightarrow H$ between them. The latter gives rise to a functor $\sigma \circ (-): \mathrm{CoAlg}(F) \to \mathrm{CoAlg}(H)$. The theorem then says that if F has cofree coalgebras, then $\sigma \circ (-)$ has a right adjoint.

Below, in Theorem 3.1, we shall generalise this result in several dimensions, by:

- dropping the restriction to **Sets**;
- letting F, H be endofunctors on two different categories;
- replacing the special form of functor $\sigma \circ (-)$ by an arbitrary functor (commuting with the forgetful functors);
- dropping the use of bisimulations from the proof in [19], and making the implicit use of greatest subcoalgebras explicit.

The resulting theorem will be used to prove our main result in the next section.

Theorem 3.1 *Consider two categories $\mathbb{C}$ and $\mathbb{D}$, each with an endofunctor, say $F : \mathbb{C} \to \mathbb{C}$ and $H : \mathbb{D} \to \mathbb{D}$. Suppose there are functors K, L, M as in the commuting square below.*

$$\begin{tikzcd}
\mathit{CoAlg}(F) \arrow[rr, "M"] \arrow[d, "U_F"'] \arrow[d, phantom, shift left=2, "\dashv"] & & \mathit{CoAlg}(H) \arrow[d, "U_H"] \\
\mathbb{C} \arrow[u, bend right, "G"'] \arrow[rr, "L"'] \arrow[rr, phantom, shift left=2, "\top"] \arrow[loop left, "F"] & & \mathbb{D} \arrow[ll, bend right, "K"'] \arrow[loop right, "H"]
\end{tikzcd}$$

Then: if $\mathbb{C}$ has equalisers and F has both greatest subcoalgebras and cofree coalgebras (as already indicated in the diagram), then M has a right adjoint.

Proof. The proof is laborious, but not really difficult. We describe the essentials. First, we write the bijective correspondences associated with the adjunction $L \dashv K$ as:

$$\begin{tikzcd}
\mathbb{D}(LX, Y) \arrow[rr, bend left, "(-)^{\vee}"] \arrow[rr, phantom, "\cong"] & & \mathbb{C}(X, KY) \arrow[ll, bend left, "(-)^{\wedge}"]
\end{tikzcd}$$

with unit η and counit ε—so that $f^{\vee} = K(f) \circ \eta_X$ and $g^{\wedge} = \varepsilon_Y \circ L(g)$. We start the construction of a right adjoint N to M by assuming an arbitrary H-coalgebra $d : Y \to H(Y)$. Let us write the cofree F-coalgebra on $KY \in \mathbb{C}$ as:

(3) $$U_F GKY \xrightarrow{\;e\;} F(U_F GKY),$$

with associated counit component $\pi : U_F GKY \to KY$. Note that it can be transposed to $\pi^{\wedge} = \varepsilon_Y \circ L(\pi) : LU_F GKY \to Y$.

If we apply the functor M to the cofree F-coalgebra e in (3) we get an H-coalgebra:

$$LU_F GKY \xrightarrow{\;M(e)\;} H(LU_F GKY)$$

This map can be transposed, which yields a morphism:

$$U_F GKY \xrightarrow{\;M(e)^{\vee}\;} KHLU_F GKY$$

This allows us to form the following equaliser $E \rightarrowtail U_F GKY$, with greatest F-

subcoalgebra on $[E]$.

$$\begin{tikzcd}
 & & & KHLU_FGKY \arrow[dr, "KH(\pi^\wedge)"] & \\
{[E]} \arrow[r, "n"] & E \arrow[r, tail, "m"] & U_FGKY \arrow[ur, "M(e)^\vee"] \arrow[dr, "\pi"'] & & KHY \\
 & & & KY \arrow[ur, "K(d)"'] &
\end{tikzcd}$$

By construction we then have a subcoalgebra on $[E]$, which we call $N(d)$:

$$\begin{tikzcd}
F([E]) \arrow[r, "F(m \circ n)"] & F(U_FGKY) \\
{[E]} \arrow[u, "N(d)"] \arrow[r, "m \circ n"'] & U_FGKY \arrow[u, "e"']
\end{tikzcd}$$

This gives the required right adjoint. □

4 Limits of coalgebras

This section combines the results from the previous two sections to prove the existence of limits of coalgebras, under suitable assumptions. The existence of equalisers follows directly from greatest subcoalgebras, but the existence of products follows indirectly via Theorem 3.1. At the end of this section the proofs we give are compared with earlier proofs.

Theorem 4.1 *Let $F: \mathbb{C} \to \mathbb{C}$ be a functor with greatest subcoalgebras. Then: if $\mathbb{C}$ has equalisers, then so has CoAlg(F).*

Proof. Consider two coalgebras $c: X \to F(X)$, $d: Y \to F(Y)$ with two homomorphisms $f, g: X \to Y$ between them. Then we can form the equaliser $E \rightarrowtail X$ of f, g in $\mathbb{C}$, and consider the induced greatest subcoalgebra $[E] \to F([E])$. This is the equaliser in CoAlg(F). □

We proceed with products of coalgebras.

Theorem 4.2 *Let $F: \mathbb{C} \to \mathbb{C}$ be a functor with greatest subcoalgebras and cofree coalgebras, on a category $\mathbb{C}$ with equalisers. Then: if $\mathbb{C}$ has I-indexed products, then so has CoAlg(F), for each set I.*

Proof. Assume an index set I, and consider the I-fold product category $\mathbb{C}^I$ of I-indexed collections $(X_i)_{i \in I}$ of objects $X_i \in \mathbb{C}$. A morphism $(X_i)_{i \in I} \xrightarrow{f} (Y_i)_{i \in I}$ in $\mathbb{C}^I$ consists of an I-indexed collection $(X_i \xrightarrow{f_i} Y_i)_{i \in I}$ of morphisms f_i in $\mathbb{C}$. The I-indexed products in $\mathbb{C}$ correspond to a right adjoint $\prod_I$ to the obvious diagonal functor $\Delta: \mathbb{C} \to \mathbb{C}^I$, see *e.g.* [15, IV, 2].

One can easily extend $F\colon \mathbb{C} \to \mathbb{C}$ to $F^I\colon \mathbb{C}^I \to \mathbb{C}^I$ by $(X_i)_{i\in I} \longmapsto (F(X_i))_{i\in I}$. The key observation is:

$$\mathrm{CoAlg}(F^I) \;=\; \mathrm{CoAlg}(F)^I.$$

Indeed, there is no difference between an F^I-coalgebra $(X_i)_{i\in I} \to (F(X_i))_{i\in I}$ and an I-indexed collection of F-coalgebras $(X_i \to F(X_i))_{i\in I}$. Thus we have a diagram like in Theorem 3.1:

$$\begin{tikzcd}
\mathrm{CoAlg}(F) \arrow[rr, "\Delta"] \arrow[d, ""{name=U}] & & \mathrm{CoAlg}(F^I) = \mathrm{CoAlg}(F)^I \arrow[d] \\
\mathbb{C} \arrow[u, bend right, ""{name=V}] \arrow[rr, "\Delta"'] \arrow[loop left, "F"] & \arrow[phantom, "\top"] & \mathbb{C}^I \arrow[ll, bend right, "\prod_I"'] \arrow[loop right, "F^I"]
\arrow[from=U, to=V, phantom, "\dashv"]
\end{tikzcd}$$

Thus, Theorem 3.1 gives us a right adjoint $\prod_I\colon \mathrm{CoAlg}(F)^I \to \mathrm{CoAlg}(F)$ to the (top) diagonal Δ, showing that $\mathrm{CoAlg}(F)$ has I-indexed products. □

Corollary 4.3 *Let $F\colon \mathbb{C} \to \mathbb{C}$ be a functor with greatest subcoalgebras and cofree coalgebras. If $\mathbb{C}$ is complete, then so is CoAlg(F).*

It may be helpful to see explicitly what the product of two coalgebras $c_i\colon X_i \to F(X_i)$ according to Theorems 4.2 and 3.1 amounts to. One first takes the equaliser $E \rightarrowtail U_F G(X_1 \times X_2)$ of the cofree coalgebra on the product of the underlying objects, as in the diagram below, and then forms the greatest subcoalgebra $[E]$. This is the product coalgebra.

$$\begin{tikzcd}[column sep=small]
 & & & F(U_F G(X_1 \times X_2)) \arrow[dr, "{\langle F(\pi_1 \circ \pi), F(\pi_2 \circ \pi)\rangle}"] & \\
{[E]} \arrow[r] & E \arrow[r, tail] & U_F G(X_1 \times X_2) \arrow[ur, "e"] \arrow[dr, "\pi"'] & & F(X_1) \times F(X_2) \\
 & & & X_1 \times X_2 \arrow[ur, "c_1 \times c_2"'] &
\end{tikzcd}$$

In the remainder of this section we discuss earlier versions of Theorems 4.1 and 4.2 about the existence of equalisers and products of coalgebras.

Equalisers

As far as we know, equalisers for coalgebras were first discussed in [20], for "bounded" endofunctors on **Sets**. This was generalised in [5, Theorem 5.1] (see also [4]), still over **Sets**, by using an explicitly constructed greatest coalgebra, like in Theorem 4.1 above. The approach was further generalised in [8] to endofunctors on an arbitrary category with several additional assumptions enabling (implicitly) the construction of greatest subcoalgebras. Our contribution lies in the axiomatisation of the greatest subcoalgebra construction on which this (easy) result essentially relies.

Products

The existence of products of coalgebras was first established in [20], relying on a (dualisation) of a general result on cocompleteness of categories of Eilenberg-Moore algebras, see [2, Volume 2, Proposition 4.3.4]. This evolved into an explicit construction in [13, Proposition 3.4], in a category of Eilenberg-Moore coalgebras—which is isomorphic to a category of functor coalgebras if cofree coalgebras exist, as mentioned in the beginning of Section 3. Still, this formulation in terms of comonad-coalgebras is slightly unsatisfying if one works with functor coalgebras, as is usually done in the theory of coalgebras in theoretical computer science.

A construction for functor coalgebras (over **Sets**) appears in [5] (see also [4]), using the explicit construction for the greatest invariant as a suitable union, like in Example 2.2 (ii). The proof we give of Theorem 4.2 implicitly follows this construction, but generalises and axiomatises it. Finally, there is also an unpublished construction of Kurz [14], using factorisations of sinks (collections of maps with a common codomain) to construct limits of coalgebras. Implicitly, this factorisation of sinks amounts to a greatest subcoalgebra construction.

5 Colimits for algebras

Colimits of algebras are well-studied topic in the categorical literature, see for instance [17,1,2]. That work concentrates on algebras for monads. This section obtains existence results for colimits of algebras of functors, by dualising the approach of the previous sections. Especially, it introduces a notion of "least quotient algebra", as dual to "greatest subcoalgebra". It also gives alternative descriptions of these least quotient algebras and greatest subcoalgebras as left and right adjoints to suitable equality functors.

Definition 5.1 *We say that an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$ has* ***least quotient algebras*** *if the associated functor $F^{op}\colon \mathbb{C}^{op} \to \mathbb{C}^{op}$ has greatest subcoalgebras.*

What this amounts to is described in the following diagram.

$$\begin{tikzcd}
\mathrm{AlgQuot}(F)^{\mathrm{op}} = \mathrm{CoAlgSub}(F^{\mathrm{op}}) \arrow[rr] \arrow[d, bend right, ""'{name=L}] \arrow[d, bend left, "/"{name=R}] & & \mathrm{Sub}(\mathbb{C}^{\mathrm{op}}) = \mathrm{Quot}(\mathbb{C})^{\mathrm{op}} \arrow[d] \\
\mathrm{Alg}(F)^{\mathrm{op}} = \mathrm{CoAlg}(F^{\mathrm{op}}) \arrow[u, "\dashv\ \dashv"] \arrow[rr] & & \mathbb{C}^{\mathrm{op}}
\end{tikzcd}$$

The category $\mathrm{Quot}(\mathbb{C})$ thus has quotients (equivalence classes of epis) as objects, with obvious maps between them. Similarly, the objects of the category $\mathrm{AlgQuot}(F)$ consists of algebras-with-quotients $F(X) \to X \twoheadrightarrow X'$. By taking the $(-)^{\mathrm{op}}$ of the whole diagram we get that the functor $\mathrm{Alg}(F) \to \mathrm{AlgQuot}(F)$ given by $(F(X) \to X) \longmapsto (F(X) \to X \overset{\mathrm{id}}{\twoheadrightarrow} X)$ has a *left* adjoint $/$. When applied to an object $F(X) \xrightarrow{a} X \overset{e}{\twoheadrightarrow} X'$ it yields a quotient algebra, written as:

$$\begin{tikzcd} F(X) \arrow[r, "F(\eta)"] \arrow[d, "a"'] & F(X/e) \arrow[d, "a/e"] \\ X \arrow[r, "\eta"'] & X/e \end{tikzcd}$$

with the following universal property. For each algebra map $(F(X) \xrightarrow{a} X) \xrightarrow{f} (F(Y) \xrightarrow{b} Y)$, if f "factors through"[1] e, then f factors through η via a unique algebra homomorphism $a/e \to b$.

With this definition in place we can dualise everything from the previous two sections, from coalgebras to algebras. In particular, assume F has least quotient algebras. Coequalisers in $\mathbb{C}$ then yield coequalisers in Alg(F). And in the presence of coequalisers in $\mathbb{C}$ and free F-algebras, I-indexed coproducts in $\mathbb{C}$ yield I-indexed coproducts in Alg(F). The latter can be proved via the dual version of Theorem 3.1 for algebras.

One usually defines quotients with respect to relations. We conclude this section by showing that such formulations are equivalent to the ones we have been using. In order to do so, assume a functor $F: \mathbb{C} \to \mathbb{C}$ on a category $\mathbb{C}$ with binary products $\times$. We form the category Rel$(\mathbb{C})$ of binary relations $R \rightarrowtail X \times X$ in $\mathbb{C}$. A morphism $(R \rightarrowtail X \times X) \longrightarrow (S \rightarrowtail Y \times Y)$ is a map $f: X \to Y$ in $\mathbb{C}$ such that $f \times f: X \times X \to Y \times Y$ restricts to $R \to S$. There is an obvious forgetful functor Rel$(\mathbb{C})$ mapping a relation to its carrier. Now we can form the following two pullbacks.

$$\begin{tikzcd} \mathrm{CoAlgRel}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Rel}(\mathbb{C}) \arrow[d] & \mathrm{AlgRel}(F) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ \mathrm{CoAlg}(F) \arrow[r] & \mathbb{C} & \mathrm{Alg}(F) \arrow[l] \end{tikzcd}$$

There is an equality functor Eq: $\mathbb{C} \to$ Rel$(\mathbb{C})$, mapping $X \in \mathbb{C}$ to the diagonal relation $\delta_X = \langle \mathrm{id}, \mathrm{id} \rangle: X \rightarrowtail X \times X$. It yields two equality functors Eq: CoAlg$(F) \to$ CoAlgRel(F) and Eq: Alg$(F) \to$ AlgRel(F), by providing a (co)algebra with the equality relation on its carrier.

In this setting there are similar descriptions of greatest subcoalgebras and least quotient algebras.

Proposition 5.2 *In the above situation, assume that $\mathbb{C}$ has equalisers. Then*

(i) *The functor F has greatest subcoalgebras if and only if the induced equality functor Eq: CoAlg$(F) \to$ CoAlgRel(F) has a right adjoint.*

(ii) *Assume additionally that $\mathbb{C}$ has coequalisers, in such a way that each epi in $\mathbb{C}$ is the coequaliser of its own kernel pair. Then F has least quotient algebras if*

[1] We mean: there is a (necessarily unique) map $g: X' \to Y$ with $f = g \circ e$. It is unclear what the standard terminology (if any) is for this property.

and only if the equality functor Eq: $Alg(F) \to AlgRel(F)$ *has a left adjoint.*

6 Greatest subcoalgebras and greatest invariants for fibrations

In Section 2 we have seen the definition, and several examples, of greatest subcoalgebras. Especially, in Example 2.2 (ii) we saw how a definition of a greatest invariant predicate $\Box P$ gave rise to a greatest subcoalgebra. In that setting predicates are identified with subobjects.

More generally, in categorical logic [10], predicates are described as objects of one category which is "fibred" over another category of types. In such a setting one can explicitly describe comprehension as an (adjoint) functor from predicates to types. And invariants can be described as predicates carrying a certain coalgebra structure (following [6]). Such a setting provides the proper level of abstraction to describe the relation between greatest invariants $\Box$ and greatest subcoalgebras $[-]$. We shall have to assume that the reader is familiar with these fibred notions, and refer to [10,6] for further background information.

The theorem of this section relates greatest invariants and greatest subcoalgebras for fibrations, generalising the construction in Example 2.2 (ii). Earlier in this paper we have relied on greatest subcoalgebras, for subobject fibrations. Greatest invariants for such fibrations have been used in [8,7]. But the context of subobjects is not optimal because it does not distinguish clearly between subcoalgebras and predicates. Also, the poset order between predicates does not involve a notion of proof(term) as arrow between predicates, which is present in a non-preorder fibration. Therefore we use fibred category theory, in which predicates exist as objects of a separate (not necessarily poset) category, and comprehension is an explicit operation.

In this section we shall work with the following situation.

(i) An endofunctor $F: \mathbb{B} \to \mathbb{B}$ on a category $\mathbb{B}$ whose objects will be considered as types.

(ii) A fibration $p: \mathbb{E} \to \mathbb{B}$ on $\mathbb{B}$, which is a special functor from a category $\mathbb{E}$ of predicates, over the category $\mathbb{B}$ of types. It provides the logic to reason about $\mathbb{B}$. Crucially, associated with each morphism $u: X \to Y$ in $\mathbb{B}$ is a substitution functor $u^*: \mathbb{E}_Y \to \mathbb{E}_X$ between the "fibre" subcategories of objects and arrows in $\mathbb{E}$ that are sent by p to Y and X. In Section 2 we have used **Pred** $\to$ **Sets** and $\mathrm{Sub}(\mathbb{C}) \to \mathbb{C}$ as two concrete examples of such a fibration. The reader who is not so comfortable with fibred category theory may wish to first consider the constructions below for these two concrete fibrations.

(iii) A "lifting" $\mathrm{Pred}(F): \mathbb{E} \to \mathbb{E}$ of F from types to predicates, forming a fibred functor in a situation:

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\mathrm{Pred}(F)"] \arrow[d, "p"'] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{B} \arrow[r, "F"'] & \mathbb{B} \end{tikzcd}$$

We do not assume that this lifting is defined inductively on the structure of a "polynomial" functor F (like in Example 2.2 (i) or in [6,11]). Instead we shall treat this situation axiomatically. We could have written G instead of Pred(F), but the notation Pred(F) is used because in many concrete situation this lifting is derived from F. The fact that Pred(F) is fibred means that it preserves substitution functors. For functors $F\colon \mathbb{C} \to \mathbb{C}$ preserving monos and pullbacks of monos, like in Example 2.2 (ii), one can define predicate lifting Pred(F): Sub$(\mathbb{C}) \to \mathbb{C}$ by $(Q \stackrel{m}{\rightarrowtail} X) \longmapsto (FQ \stackrel{Fm}{\rightarrowtail} FX)$. This is used implicitly in Example 2.2 (ii).

Given what we already have, we can set up the following structure.

$$
\begin{tikzcd}
\mathrm{CoAlg}(\mathrm{Pred}(F)) \arrow[drrr, bend left, "U_{\mathrm{Pred}(F)}"] \arrow[dr, dotted, "I"] \arrow[ddr, bend right, "\mathrm{CoAlg}(p)"'] & & & \\
& \mathrm{CoAlgPred}(F) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathbb{E} \arrow[d, "p"] \arrow[loop right, "\mathrm{Pred}(F)"] \\
& \mathrm{CoAlg}(F) \arrow[rr, "U_F"'] & & \mathbb{B} \arrow[loop right, "F"]
\end{tikzcd}
\tag{4}
$$

The objects of the category CoAlgPred(F) are thus pairs $(X \to F(X), P \in \mathbb{E}_X)$ consisting of an F-coalgebra with an (arbitrary) predicate on its state space. The two functors CoAlg$(F) \longleftarrow$ CoAlgPred$(F) \longrightarrow \mathbb{E}$ are then the obvious projections. By construction, the first projection CoAlgPred$(F) \to$ CoAlg(F) is also a fibration. It may be understood as describing a logic for reasoning about coalgebras.

The objects of the category CoAlg(Pred(F)) are coalgebras on predicates, of the form $Q \to \mathrm{Pred}(F)(Q)$. They are mapped by the functor CoAlg(p) to the underlying F-coalgebra $p(Q) \to p(\mathrm{Pred}(F)(Q)) = F(p(Q))$. The predicate Q is understood as an **invariant** for this F-coalgebra. One can show that also CoAlg(p) is a fibration, and that the pair of forgetful functors $(U_F, U_{\mathrm{Pred}(F)})$ is a morphism of fibrations CoAlg$(p) \to p$. The functor I maps $Q \to \mathrm{Pred}(F)(Q)$ to the pair $(pQ \to F(pQ), Q)$. It is a faithfull fibred functor.

(iv) A "truth" functor $1\colon \mathbb{B} \to \mathbb{E}$ with $p \dashv 1$ and $p \circ 1 = \mathrm{Id}$. Then, for each object $X \in \mathbb{B}$, the object $1X$ is terminal in the fibre category $\mathbb{E}_X$ of objects over X. Moreover, the predicate lifting functor Pred(F) should preserve this terminal object functor in the sense that the canonical (vertical) natural transformation

$$\mathrm{Pred}(F)\,1 \xRightarrow{\alpha} 1\,F$$

is an isomorphism. It is obtained by transposing the identity $p\mathrm{Pred}(F)1 \Rightarrow F$.

This terminal object functor 1 for the fibration p gives rise to two new terminal object functors 1_F and $1_{\mathrm{Pred}(F)}$ for the two other fibrations in (4), namely:

(5)
$$\begin{tikzcd}
\mathrm{CoAlg}(\mathrm{Pred}(F)) \arrow[rr, "I"] \arrow[dr] & & \mathrm{CoAlgPred}(F) \arrow[r] \arrow[dl] & \mathbb{E} \arrow[d] \arrow[d, phantom, shift left=2, "\dashv"] \\
& \mathrm{CoAlg}(F) \arrow[ul, bend left, "1_{\mathrm{Pred}(F)}"] \arrow[ul, phantom, "\dashv"] \arrow[ur, bend right, "1_F"'] \arrow[ur, phantom, "\dashv"] \arrow[rr] & & \mathbb{B} \arrow[u, bend right, "1"']
\end{tikzcd}$$

These functors are defined by:

$$\begin{aligned}
1_F(X \xrightarrow{c} F(X)) &= (X \xrightarrow{c} F(X), 1X) \\
1_{\mathrm{Pred}(F)}(X \xrightarrow{c} F(X)) &= (1X \xrightarrow{1c} 1F(X) \overset{\alpha}{\cong} \mathrm{Pred}(F)(1X)).
\end{aligned}$$

And they commute with the functor $I\colon \mathrm{CoAlg}(\mathrm{Pred}(F)) \to \mathrm{CoAlgPred}(F)$ and with the projection $\mathrm{CoAlgPred}(F) \to \mathbb{E}$.

(v) The fibration $\mathbb{E} \to \mathbb{B}$ has comprehension, in the form of a right adjoint $1 \dashv \{-\}$ to its terminal object functor 1. Moreover, predicate lifting preserves comprehension, in the sense that the canonical map

$$F\,\{-\} \overset{\beta}{\Longrightarrow} \{-\}\,\mathrm{Pred}(F)$$

is an isomorphism. It is obtained by transposing the following composition,

(6)
$$1\,F\,\{-\} \underset{\cong}{\overset{\alpha^{-1}}{\Longrightarrow}} \mathrm{Pred}(F)\,1\,\{-\} \overset{\mathrm{Pred}(F)\varepsilon}{\Longrightarrow} \mathrm{Pred}(F)$$

where ε is the counit of the adjunction $1 \dashv \{-\}$, with unit η. For objects $X \in \mathbb{B}$ and $Q \in \mathbb{E}_X$ we shall write $\pi_Q \overset{\mathrm{def}}{=} p(\varepsilon_Q)\colon \{Q\} \to X$ for the canonical projection associated with the comprehension of Q. Informally, one thinks of this as the inclusion $\{x\colon X \mid Q(x)\} \hookrightarrow X$. Indeed, the projection π_Q is a mono if the fibration p is preordered, see [10, Lemma 4.6.2 (i)].

For arbitrary objects $X, Y \in \mathbb{B}$ and $Q \in \mathbb{E}_X$ the following diagrams commute.

(7)
$$\begin{tikzcd}
F\{Q\} \arrow[rr, "\beta_Q", "\cong"'] \arrow[dr, "F(\pi_Q)"'] & & \{\mathrm{Pred}(F)(Q)\} \arrow[dl, "\pi_{\mathrm{Pred}(F)(Q)}"] \\
& X &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\{1Y\} \arrow[d, "\pi_{1Y}"'] \arrow[d, phantom, shift left=2, "\cong"] \\
Y \arrow[u, bend right, "\eta_Y"']
\end{tikzcd}$$

Indeed,

$$\begin{aligned}
\pi_{\mathrm{Pred}(F)(Q)} \circ \beta_Q &= p(\varepsilon_{\mathrm{Pred}(F)(Q)} \circ 1\beta_Q) \\
&= p(\mathrm{Pred}(F)(\varepsilon_Q) \circ \alpha^{-1}) && \text{by (6)} \\
&= F(p(\varepsilon_Q)) && \text{since } \alpha \text{ is vertical, } \textit{i.e. } p(\alpha) = \mathrm{id} \\
&= F(\pi_Q).
\end{aligned}$$

And the unit η is an isomorphism since the terminal object functor $1\colon \mathbb{B} \to \mathbb{E}$ is full and faithful (see [15, dual of IV 3, Theorem 1]). It is inverse is $\pi\,1$, since $\pi_{1Y} \circ \eta_Y = p(\varepsilon_Y \circ 1\eta_Y) = \mathrm{id}$.

Finally, one says that comprehension is **full** if the induced functor $\mathbb{E} \to \mathbb{B}^{\to}$ given by $Q \mapsto (\{Q\} \stackrel{\pi_Q}{\to} pQ)$ is full and faithful. In that case the predicate lifting functor $\mathrm{Pred}(F)$ is entirely determined, because one can show that for each predicate $P \in \mathbb{E}$ the map $\mathrm{Pred}(F)(\varepsilon_P) \circ \alpha_P^{-1} \colon 1F\{P\} \to \mathrm{Pred}(F)(P)$ from (6) is opcartesian.

We first prove an auxiliary result. It is an "adjoint lifting" result, in the style considered in [6].

Lemma 6.1 *The fibration $\mathit{CoAlg}(\mathit{Pred}(F)) \to \mathit{CoAlg}(F)$ inherits comprehension from $\mathbb{E} \to \mathbb{B}$: the terminal object functor $1_{\mathit{Pred}(F)} \colon \mathit{CoAlg}(F) \to \mathit{CoAlg}(\mathit{Pred}(F))$ from (5) has a right adjoint $\langle - \rangle$, given by*

$$\Big(Q \xrightarrow{a} \mathit{Pred}(F)(Q)\Big) \longmapsto \Big(\{Q\} \xrightarrow{\{a\}} \{\mathit{Pred}(F)(Q)\} \xrightarrow[\cong]{\beta_Q^{-1}} F(\{Q\})\Big)$$

Proof. Assume an F-coalgebra $c \colon X \to F(X)$ and a $\mathrm{Pred}(F)$-coalgebra $a \colon Q \to \mathrm{Pred}(F)(Q)$. We have to establish a bijective correspondence between maps of $\mathrm{Pred}(F)$- and F-coalgebras:

$$\begin{array}{c}
1_{\mathrm{Pred}(F)}(X \xrightarrow{c} F(X)) \longrightarrow (Q \xrightarrow{a} \mathrm{Pred}(F)(Q)) \\
\hline\hline
\begin{tikzcd} \mathrm{Pred}(F)(1X) \arrow[r, "\mathrm{Pred}(F)(f)"] & \mathrm{Pred}(F)(Q) \\ 1X \arrow[u, "\alpha_X^{-1} \circ 1c"] \arrow[r, "f"'] & Q \arrow[u, "a"'] \end{tikzcd} \\
\hline\hline
\begin{tikzcd} FX \arrow[r, "F(g)"] & F(\{Q\}) \\ X \arrow[u, "c"] \arrow[r, "g"'] & \{Q\} \arrow[u, "\beta_Q^{-1} \circ \{a\}"'] \end{tikzcd} \\
\hline\hline
(X \xrightarrow{c} F(X)) \longrightarrow \Big\langle Q \xrightarrow{a} \mathrm{Pred}(F)(Q) \Big\rangle
\end{array} \quad (*)$$

Thus we still have to prove the correspondence (*). It results of course from the adjunction $1 \dashv \{-\}$, but the homomorphism requirements have to be checked. This is done as follows.

- Given a homomorphism $f \colon 1X \to Q$ of $\mathrm{Pred}(F)$-coalgebras, the transpose $\{f\} \circ$

$\eta_X : X \to \{Q\}$ is a homomorphism of F-coalgebras:

$$\begin{aligned}
&\beta_Q^{-1} \circ \{a\} \circ \{f\} \circ \eta_X \\
&= \beta_Q^{-1} \circ \{\mathrm{Pred}(F)(f) \circ \alpha_X^{-1} \circ 1c\} \circ \eta_X \\
&= F(\{f\}) \circ \beta_{1X}^{-1} \circ \{\alpha_X^{-1}\} \circ \eta_{FX} \circ c \\
&= F(\{f\}) \circ \beta_{1X}^{-1} \circ \{\alpha_X^{-1}\} \circ \pi_{1FX}^{-1} \circ c \qquad \text{by (7)} \\
&= F(\{f\}) \circ \beta_{1X}^{-1} \circ \pi_{\mathrm{Pred}(F)(1X)}^{-1} \circ p(\alpha_X^{-1}) \circ c \\
&= F(\{f\}) \circ F(\pi_{1X}^{-1}) \circ c \qquad \text{by (7)} \\
&= F(\{f\} \circ \eta_X) \circ c.
\end{aligned}$$

- Conversely, given a homomorphism of F-coalgebras $g : X \to \{Q\}$, its transpose $\varepsilon_Q \circ 1g : 1X \to Q$ is a homomorphism of $\mathrm{Pred}(F)$-coalgebras:

$$\begin{aligned}
\mathrm{Pred}(F)(\varepsilon_Q \circ 1g) \circ \alpha_X^{-1} \circ 1c &= \mathrm{Pred}(F)(\varepsilon_Q) \circ \alpha_{FX}^{-1} \circ 1F(g) \circ 1c \\
&= \varepsilon_{\mathrm{Pred}(F)(Q)} \circ 1\beta_Q \circ 1(\beta_Q^{-1} \circ \{a\} \circ g) \\
&= a \circ \varepsilon_Q \circ 1g.
\end{aligned}$$

□

We are now in a position to state the main theorem of this paper, relating greatest invariants—a right adjoint to the functor $I : \mathrm{CoAlg}(\mathrm{Pred}(F)) \to \mathrm{CoAlgPred}(F)$—and greatest coalgebras, or comprehension—a right adjoint to the terminal object functor $1_F : \mathrm{CoAlg}(F) \to \mathrm{CoAlgPred}(F)$.

Theorem 6.2 *Consider the situation as described in the above points 1–5. Then*

(i) *Greatest invariants imply greatest subcoalgebras: a fibred right adjoint $I \dashv \Box$ yields a right adjoint $1_F \dashv [-]$.*

(ii) *Greatest subcoalgebras imply greatest invariants: under the additional assumptions that comprehension is full and that p is a bifibration, a right adjoint $1_F \dashv [-]$ yields a fibred right adjoint $I \dashv \Box$.*

The statement that $\Box$ is a *fibred* adjoint says that $\Box$ commutes with substitution, see for instance [11, Lemma 3.6 (iii)] or [7, Theorem 5.2]. And an equivalent way to say that p is a bifibration is that it has left adjoints $\coprod_u \dashv u^*$ to its substitution functors u^*, see [10, Lemma 9.1.2].

Proof. 1. Assume a right adjoint $I \dashv \Box$. By using Lemma 6.1 we have an adjoint $1_{\mathrm{Pred}(F)} \dashv \langle - \rangle$. Hence by composition of adjoints we get a right adjoint $1_F = I \circ 1_{\mathrm{Pred}(F)} \dashv \langle - \rangle \circ \Box$.

2. Next, assume that our fibration p is a bifibration, that comprehension is full, and that there is a right adjoint $1_F \dashv [-]$, say with unit κ and counit θ. (We still use η and ε for the unit and counit of $1 \dashv \{-\}$.) We start with some preliminary observations. First, let us write this right adjoint $[-] : \mathrm{CoAlgPred}(F) \to \mathrm{CoAlg}(F)$

as:

$$\Big[(X \stackrel{c}{\longrightarrow} F(X), P \in \mathbb{E}_X)\Big] = ([P] \stackrel{[c]}{\longrightarrow} F([P])),$$

For convenience we shall often write the counit $\theta: 1_F\,[-] \Rightarrow \mathrm{Id}$ at an object $(X \to F(X), P) \in \mathrm{CoAlgPred}(F)$ simply as θ_P, omitting the coalgebra. We then get a a homomorphism of F-coalgebras:

$$\begin{tikzcd} F([P]) \arrow[r, "F(p(\theta_P))"] & F(X) \\ {[P]} \arrow[u, "{[c]}"] \arrow[r, "p(\theta_P)"'] & X \arrow[u, "c"'] \end{tikzcd}$$

Transposing the counit $\theta_P: 1[P] \to P$ accross the adjunction $1 \dashv \{-\}$ yields a map:

$$\begin{tikzcd} {[P]} \arrow[rr, "\overline{\theta_P} = \{\theta_P\} \circ \eta_{[P]}"] \arrow[dr, "p(\theta_P)"'] & & \{P\} \arrow[dl, "\pi_P"] \\ & X & \end{tikzcd}$$

Indeed,

$$\begin{aligned} \pi_P \circ \overline{\theta_P} &= p(\varepsilon_P \circ 1\{\theta_P\} \circ 1\eta_{[P]}) \\ &= p(\theta_P \circ \varepsilon_{1[P]} \circ 1\eta_{[P]}) \\ &= p(\theta_P). \end{aligned}$$

Our first aim is to show that we thus get an isomorphism $\overline{\theta}: [-] \circ I \stackrel{\cong}{\Longrightarrow} \langle - \rangle$ between comprehension functors (see Lemma 6.1) in:

$$\begin{tikzcd} \mathrm{CoAlg}(\mathrm{Pred}(F)) \arrow[rr, "I"] \arrow[dr, "\langle - \rangle"'] & \cong & \mathrm{CoAlgPred}(F) \arrow[dl, "{[-]}"] \\ & \mathrm{CoAlg}(F) & \end{tikzcd}$$

For the inverse of $\overline{\theta}$, assume we have a $\mathrm{Pred}(F)$-coalgebra $Q \stackrel{a}{\to} \mathrm{Pred}(F)(Q)$ in $\mathbb{E}$, say above $Y \stackrel{d}{\to} F(Y)$ in $\mathbb{B}$. Lemma 6.1 describes $\langle - \rangle$ as an F-coalgebra structure on the comprehension $\{Q\}$. In the proof of this lemma we see that the counit $\varepsilon_Q: 1\{Q\} \to Q$ forms a homomorphism in $\mathrm{CoAlgPred}(F)$. Applying the functor I yields:

$$\begin{array}{ccc} I1_{\mathrm{Pred}(F)}\Big\langle Q \xrightarrow{\;a\;} \mathrm{Pred}(F)Q\Big\rangle & \xrightarrow{\;\varepsilon_Q\;} & I\Big(Q \xrightarrow{\;a\;} \mathrm{Pred}(F)(Q)\Big) \\ \| & & \| \\ 1_F\Big(\{Q\} \xrightarrow{\;\beta_Q^{-1} \circ \{a\}\;} F(\{Q\})\Big) & & \Big(Y \xrightarrow{\;d\;} F(Y), Q\Big) \end{array} \tag{8}$$

Hence by transposition accross the adjunction $1_F \dashv [-]$ we get a coalgebra map

$\overline{\varepsilon_Q} = [\varepsilon_Q] \circ \kappa_{\{Q\}}: \{Q\} \to [Q]$ in:

$$\begin{tikzcd}
F(\{Q\}) \arrow[r, "F(\overline{\varepsilon_Q})"] & F([Q]) \arrow[r, "F(p(\theta_Q))"] & F(Y) \\
\{Q\} \arrow[u, "\beta_Q^{-1} \circ \{a\}"] \arrow[r, "\overline{\varepsilon_Q}"] \arrow[rr, bend right, "\pi_Q"'] & {[Q]} \arrow[u, "{[d]}"] \arrow[r, "p(\theta_Q)"] & Y \arrow[u, "d"']
\end{tikzcd}$$

In this way we indeed get an isomorphism:

$$\begin{aligned}
\overline{\theta_Q} \circ \overline{\varepsilon_Q} &= \{\theta_Q\} \circ \eta_{[Q]} \circ \overline{\varepsilon_Q} \\
&= \{\theta_Q \circ 1\overline{\varepsilon_Q}\} \circ \eta_{\{Q\}} \\
&= \{\varepsilon_Q\} \circ \eta_{\{Q\}} \\
&= \mathrm{id}_{\{Q\}}.
\end{aligned}$$

And $\overline{\varepsilon_Q} \circ \overline{\theta_Q} = \mathrm{id}_{[Q]}$ because the transpose of this composition is the counit:

$$\begin{aligned}
\theta_Q \circ 1(\overline{\varepsilon_Q} \circ \overline{\theta_Q}) &= \varepsilon_Q \circ 1\{\theta_Q\} \circ 1\eta_{[Q]} \\
&= \theta_Q \circ \varepsilon_{1[Q]} \circ 1\eta_{[Q]} \\
&= \theta_Q.
\end{aligned}$$

Only now we start the construction of a right adjoint to the functor I, using that p is a bifibration. For an object $(X \overset{c}{\to} F(X), P \in \mathbb{E}_X) \in \mathrm{CoAlgPred}(F)$ we can factorise the counit $\theta_P: 1[P] \to P$ as a composition:

(9)
$$\begin{tikzcd}
 & P \\
1[P] \arrow[ur, "\theta_P"] \arrow[r, "o_P"'] & \Box P \arrow[u, "i_P"']
\end{tikzcd}$$

where o_P is opcartesian over $p(\theta_P)$ and i_P is vertical. We first show that this $\Box P \in \mathbb{E}_X$ carries a $\mathrm{Pred}(F)$-coalgebra structure, which we call $\Box c$, over $c: X \to F(X)$, using that o_P is opcartesian in:

(10)
$$\begin{tikzcd}
 & \mathrm{Pred}(F)(1[P]) \arrow[rr, "\mathrm{Pred}(F)(o_P)"] & & \mathrm{Pred}(F)(\Box P) \\
1[P] \arrow[ur, "\alpha_{[P]}^{-1} \circ 1[c]"] \arrow[rr, "o_P"] & & \Box P \arrow[ur, dashed, "\Box c"'] & \\
 & F([P]) \arrow[rr, "F(p(\theta_P))"] & & F(X) \\
{[P]} \arrow[ur, "{[c]}"] \arrow[rr, "p(\theta_P)"'] & & X \arrow[ur, "c"'] &
\end{tikzcd}$$

In a similar way one defines $\Box$ on morphisms.

Next we note that if we have an invariant $Q \xrightarrow{a} \mathrm{Pred}(F)(Q)$, say over $Y \xrightarrow{d} F(Y)$, then applying the $\Box$-construction to $I(Q \to \mathrm{Pred}(F)(Q))$ makes the map $i_Q \colon \Box Q \to Q$ in (9) an isomorphism. The proof uses fullness of the comprehension operation $\{-\}$. It requires some work.

First we consider the following commuting diagram in $\mathbb{B}$.

$$\begin{tikzcd}
\{Q\} \arrow[r, "\overline{\varepsilon_Q}", "\cong"'] \arrow[d, "\pi_Q"'] & {[Q]} \arrow[r, "\eta_{[Q]}", "\cong"'] & \{1[Q]\} \arrow[r, "\{o_Q\}"] & \{\Box Q\} \arrow[d, "\pi_{\Box Q}"] \\
Y \arrow[rrr, equal] & & & Y
\end{tikzcd}$$

By fullness of comprehension, it yields a unique vertical morphism $j_Q \colon Q \to \Box Q$ with $\{j_Q\} = \{o_Q\} \circ \eta_{[Q]} \circ \overline{\varepsilon_Q}$. We then get $i_Q \circ j_Q = \mathrm{id}_Q$, again by fullness:

$$\begin{aligned}
\{i_Q \circ j_Q\} &= \{i_Q \circ o_Q\} \circ \eta_{[Q]} \circ [\varepsilon_Q] \circ \kappa_{\{Q\}} \\
&= \{\theta_Q \circ 1[\varepsilon_Q] \circ 1\kappa_{\{Q\}}\} \circ \eta_{\{Q\}} \\
&= \{\varepsilon_Q \circ \theta_{1\{Q\}} \circ 1\kappa_{\{Q\}}\} \circ \eta_{\{Q\}} \\
&= \{\varepsilon_Q\} \circ \eta_{\{Q\}} \\
&= \mathrm{id}_{\{Q\}}.
\end{aligned}$$

For the reverse equation $j_Q \circ i_Q = \mathrm{id}_{\Box Q}$ we extend the diagram (9) defining $\Box Q$ as follows.

$$\begin{tikzcd}
& \Box Q \\
& Q \arrow[u, "j_Q"'] \\
1[Q] \arrow[uur, "o_Q"] \arrow[ur, "\theta_Q"] \arrow[r, "o_Q"'] & \Box Q \arrow[u, "i_Q"']
\end{tikzcd}$$

Since o_Q is opcartesian by construction, it suffices to show that the upper triangle commutes. Again we use fullness:

$$\begin{aligned}
\{j_Q \circ \theta_Q\} &= \{o_Q\} \circ \eta_{[Q]} \circ \overline{\varepsilon_Q} \circ \{\theta_Q\} \\
&= \{o_Q\} \circ \left(\overline{\theta_Q} \circ \pi_{1[Q]}\right)^{-1} \circ \{\theta_Q\} \\
&= \{o_Q\} \circ \left(\{\theta_Q\} \circ \eta_{[Q]} \circ \pi_{1[Q]}\right)^{-1} \circ \{\theta_Q\} \\
&= \{o_Q\} \circ \left(\{\theta_Q\}\right)^{-1} \circ \{\theta_Q\} \\
&= \{o_Q\}.
\end{aligned}$$

Hence we have established $i_Q \colon \Box Q \xrightarrow{\cong} Q$. This means that the counit $\theta_Q \colon 1[Q] \to Q$ is opcartesian.

We can apply these considerations to an invariant $\Box P$ as defined in (9). It yields an isomorphism $i_{\Box P} \colon \Box\Box P \xrightarrow{\cong} \Box P$, with inverse $j_{\Box P}$. We shall investigate this situation a bit more, and show that:

(11) $$i_{\Box P} \;=\; \Box(i_P).$$

The $\Box(-)$ map on the right hand side is the unique vertical map on the left in:

$$\begin{tikzcd}
1[P] \arrow[r, "o_P"] & \Box P \\
1[\Box P] \arrow[u, "{1[i_P]}"] \arrow[r, "o_{\Box P}"'] & \Box\Box P \arrow[u, dotted, "\Box(i_P)"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1[\Box P] \arrow[r, "o_{\Box P}"] & \Box\Box P \\
1[P] \arrow[u, "1\overline{o_P}"] \arrow[r, "o_P"'] & \Box P \arrow[u, dotted, "j_{\Box P}"']
\end{tikzcd}$$

In order to prove (11) we first show that $j_{\Box P}$ is the unique vertical map in the above diagram on the right, where $\overline{o_P} = [o_P] \circ \kappa_{[P]} \colon [P] \to [\Box P]$ is obtained by transposition. This is enough because by pasting the two diagrams together—the left one on top of the right one—and using that $1[i_P] \circ 1(\overline{o_P}) = 1([i_P \circ o_P] \circ \kappa_{[P]}) = 1([\theta_P] \circ \kappa_{[P]}) = \mathrm{id}_{1[P]}$—one gets $\Box(i_P) \circ j_{\Box P} = \mathrm{id}_{\Box P}$, and thus $\Box(i_P) = i_{\Box P}$, as in (11).

We use fullness to prove that $j_{\Box P}$ makes the diagram on the right commute.

$$\begin{aligned}
\{j_{\Box P} \circ o_P\} &= \{o_{\Box P}\} \circ \eta_{[\Box P]} \circ [\varepsilon_{\Box P}] \circ \kappa_{\{\Box P\}} \circ \{o_P\} \\
&= \{o_{\Box P}\} \circ \eta_{[\Box P]} \circ [\varepsilon_{\Box P} \circ 1\{o_P\}] \circ \kappa_{\{1[P]\}} \\
&= \{o_{\Box P}\} \circ \eta_{[\Box P]} \circ [o_P \circ \varepsilon_{1[P]}] \circ \kappa_{\{1[P]\}} \\
&= \{o_{\Box P} \circ 1[o_P]\} \circ \eta_{[1[P]]} \circ [1(\eta_{[P]}^{-1})] \circ \kappa_{\{1[P]\}} \\
&= \{o_{\Box P} \circ 1[o_P]\} \circ \eta_{[1[P]]} \circ \kappa_{[P]} \circ \eta_{[P]}^{-1} \\
&= \{o_{\Box P} \circ 1[o_P] \circ 1\kappa_{[P]}\} \circ \eta_{[P]} \circ \eta_{[P]}^{-1} \\
&= \{o_{\Box P} \circ 1(\overline{o_P})\}.
\end{aligned}$$

Finally we come to the required bijective correspondences for $I \dashv \Box$.

$$\begin{array}{c}
I(Q \xrightarrow{a} \mathrm{Pred}(F)(Q)) \longrightarrow (X \xrightarrow{c} F(X), P) \\
\| \\
(Y \xrightarrow{d} F(Y), Q) \\
\hline\hline
Q \xrightarrow{f} P \quad \text{with} \quad
\begin{tikzcd}
FY \arrow[r, "F(pf)"] & F(X) \\
Y \arrow[u, "d"] \arrow[r, "pf"'] & X \arrow[u, "c"']
\end{tikzcd} \\
\hline\hline \text{(*)} \\
\begin{tikzcd}
\mathrm{Pred}(F)(Q) \arrow[r, "\mathrm{Pred}(F)(g)"] & \mathrm{Pred}(F)(\Box P) \\
Q \arrow[u, "a"] \arrow[r, "g"'] & \Box P \arrow[u, "\Box c"']
\end{tikzcd} \\
\hline\hline
(Q \xrightarrow{a} \mathrm{Pred}(F)(Q)) \longrightarrow \Box(X \xrightarrow{c} F(X), P)
\end{array}$$

We shall elaborate the marked correspondence (*).

- The direction upwards is easy, so we do it first. Assume a homorphism of $\mathrm{Pred}(F)$-coalgebras $g\colon Q \to \Box P$ as indicated. Then $\overline{g} = i_P \circ g\colon Q \to P$ satisfies:

$$\begin{aligned}
c \circ p\overline{g} &= p(\Box c \circ g) && \text{since } i_P \text{ is vertical, and } \Box c \text{ is over } c \\
&= p(\mathrm{Pred}(F)(g) \circ a) \\
&= F(pg) \circ d \\
&= F(p\overline{g}) \circ d.
\end{aligned}$$

- For the downward direction assume $f\colon Q \to P$ with $c \circ pf = F(pf) \circ d$. Then $\overline{f} = \Box(f) \circ j_Q\colon Q \to \Box P$ is the required map. We use that the counit θ_Q is opcartesian in order to show that $\overline{f}$ is an appropriate homomorphism of $\mathrm{Pred}(F)$-coalgebras, in a non-trivial calculation:

$$\begin{aligned}
&\mathrm{Pred}(F)(\overline{f}) \circ a \circ \theta_Q \\
&= \mathrm{Pred}(F)(\Box(f) \circ j_Q) \circ a \circ \varepsilon_Q \circ 1(\overline{\theta_Q}) \\
&= \mathrm{Pred}(F)(\Box(f) \circ j_Q) \circ \varepsilon_{\mathrm{Pred}(F)(Q)} \circ 1(\{a\} \circ (\overline{\varepsilon_Q})^{-1}) \\
&= \mathrm{Pred}(F)(\Box(f) \circ j_Q) \circ \varepsilon_{\mathrm{Pred}(F)(Q)} \circ 1(\beta_Q \circ F((\overline{\varepsilon_Q})^{-1}) \circ [d]) && \text{by (8)} \\
&= \mathrm{Pred}(F)(\Box(f) \circ j_Q \circ \varepsilon_Q) \circ \alpha^{-1}_{\{Q\}} \circ 1F(\overline{\theta_Q}) \circ 1[d] && \text{by (6)} \\
&= \mathrm{Pred}(F)(\Box(f) \circ j_Q \circ \varepsilon_Q \circ 1(\overline{\theta_Q})) \circ \alpha^{-1}_{[Q]} \circ 1[d] \\
&= \mathrm{Pred}(F)(\Box(f) \circ j_Q \circ \theta_Q) \circ \alpha^{-1}_{[Q]} \circ 1[d] \\
&= \mathrm{Pred}(F)(\Box(f) \circ o_Q) \circ \alpha^{-1}_{[Q]} \circ 1[d] \\
&= \mathrm{Pred}(F)(o_P \circ 1[f]) \circ \alpha^{-1}_{[Q]} \circ 1[d] \\
&= \mathrm{Pred}(F)(o_P) \circ \alpha^{-1}_{[P]} \circ 1(F([f]) \circ [d]) \\
&= \mathrm{Pred}(F)(o_P) \circ \alpha^{-1}_{[P]} \circ 1([c] \circ [f]) \\
&= \Box c \circ o_P \circ 1[f] && \text{by (10)} \\
&= \Box c \circ \Box(f) \circ o_Q \\
&= \Box c \circ \overline{f} \circ \theta_Q.
\end{aligned}$$

Next, these mappings $f \mapsto \overline{f}$ and $g \mapsto \overline{g}$ are each other's inverses:

$$\begin{aligned}
\overline{\overline{f}} \circ \theta_Q &= i_P \circ \Box(f) \circ j_Q \circ \theta_Q \\
&= i_P \circ \Box(f) \circ o_Q \\
&= i_P \circ o_P \circ 1[f] \\
&= \theta_P \circ 1[f] \\
&= f \circ \theta_Q. \\
\overline{\overline{g}} \circ \theta_Q &= \Box(i_P \circ g) \circ j_Q \circ \theta_Q \\
&= \Box(i_P \circ g) \circ o_Q \\
&= \Box(i_P) \circ o_{\Box P} \circ 1[g] \\
&= i_{\Box P} \circ o_{\Box P} \circ 1[g] \qquad \text{by (11)} \\
&= \theta_{\Box P} \circ 1[g] \\
&= g \circ \theta_Q.
\end{aligned}$$

This completes the verifications of the adjunction $I \dashv \Box$. What remains is to remark that $\Box$ is automatically a fibred functor, see [10, Exercise 1.8.5], because it commutes with the fibrations and because the unit and counit of the adjunction $I \dashv \Box$ are vertical. $\Box$

7 Conclusions and future work

Comprehension for coalgebras and quotients for algebras have been identified as key ingredients in proving the existence of limits of coalgebras and colimits and algebras. This comprehension is related to greatest invariants. There is probably a similar result relating quotients for algebras to least congruence relations (*i.e.* to algebras of a "relation lifting" funtor).

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