

# Trace Semantics for Coalgebras

Bart Jacobs

*Department of Computer Science, University of Nijmegen*  
*P.O. Box 9010, 6500 GL Nijmegen, The Netherlands*  
*Email: bart@cs.kun.nl URL: <http://www.cs.kun.nl/~bart>*

---

## Abstract

Traditionally, traces are the sequences of labels associated with paths in transition systems  $X \rightarrow \mathcal{P}(A \times X)$ . Here we describe traces more generally, for coalgebras of the form  $X \rightarrow \mathcal{P}(F(X))$ , where  $F$  is a polynomial functor. The main result states that  $F$ 's final coalgebra  $Z \xrightarrow{\cong} F(Z)$  gives rise to a weakly final coalgebra with state space  $\mathcal{P}(Z)$ , in a suitable category of coalgebras. Weak finality means that there is a coalgebra map  $X \rightarrow \mathcal{P}(Z)$ , but there is no uniqueness. We show that there is a canonical choice among these maps  $X \rightarrow \mathcal{P}(Z)$ , namely the largest one, describing the traces in a suitably abstract formulation. A crucial technical ingredient in our construction is a general distributive law  $F\mathcal{P} \Rightarrow \mathcal{P}F$ , obtained via relation lifting.

---

## 1 Introduction

A final coalgebra  $Z \xrightarrow{\cong} F(Z)$  of a functor  $F$  is an abstract model consisting of behaviours of the kind described by the functor (or interface)  $F$ . The elements of  $Z$  are “structured” in the sense that the transition structure is incorporated. The associated notion of equivalence is “strong” bisimulation. It arises as the natural notion of equivalence within the theory of coalgebras.

Trace semantics involves only linear behaviours consisting of sequences of consecutive actions, see for instance [2] for a discussion. It is much more problematic within the theory of coalgebras. Partial results may be found in [5], for coalgebras of the form  $X \rightarrow \mathcal{P}(1 + (A \times X))$ . Here we generalise the approach from [5] to coalgebras of the form  $X \rightarrow \mathcal{P}(F(X))$ . This is achieved by generalising the distributive law that is used in [5] to one of the form  $F\mathcal{P} \Rightarrow \mathcal{P}F$ .

Once the appropriate abstraction level is achieved, we are not done, since the technical elaboration appears non-trivial. Much of the work focusses on general traces of coalgebras  $X \rightarrow \mathcal{P}(F(X))$  and on their properties.

---

<sup>1</sup> Proceedings of *Coalgebraic Methods in Computer Science* (CMCS) 2004.

*This is a preliminary version. The final version will be published in*  
*Electronic Notes in Theoretical Computer Science*  
*URL: [www.elsevier.nl/locate/entcs](http://www.elsevier.nl/locate/entcs)*

This paper starts with two preliminary sections, one on traditional trace semantics for labeled transition systems, and one on relation lifting associated with an endofunctor. Then, Section 4 describes the general distributive law  $F\mathcal{P} \Rightarrow \mathcal{P}F$  that forms the basis of our work, and Section 5 introduces an appropriate category of coalgebras over a Kleisli category for our trace semantics. Our main result about a weakly final coalgebra of traces is in Section 6.

## 2 Traditional trace semantics

This section sketches the basics of traditional trace semantics for labeled transition systems (LTSs). Such a system is described as a coalgebra  $\alpha: X \rightarrow \mathcal{P}(A \times X)$ , where

- $X$  is the set of states
- $A$  is the set of labels (or actions)
- $\alpha$  describes the transitions  $x \xrightarrow{a} x'$ , namely as  $(a, x') \in \alpha(x)$ .

For such an LTS one can form the set  $U$  of infinite paths as:

$$U = \{(\langle x_0, a_0 \rangle, \langle x_1, a_1 \rangle, \dots) \in (X \times A)^\mathbb{N} \mid \forall n \in \mathbb{N}. x_n \xrightarrow{a_n} x_{n+1}\}$$

With each state  $x \in X$  one can then associate an infinite **trace**  $\ell(x) \in \mathcal{P}(A^\mathbb{N})$ , namely

$$(1) \quad \ell(x) \stackrel{\text{defn}}{=} \{\pi_2^\mathbb{N}(u) \mid u \in U \text{ with } \pi_1(u_0) = x\}.$$

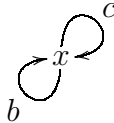
Here we write  $\pi_2^\mathbb{N}$  for the element-wise extension of the second projection  $\pi_2: X \times A \rightarrow A$  to a function  $(X \times A)^\mathbb{N} \rightarrow A^\mathbb{N}$ .

It is not hard to see that this function  $\ell: X \rightarrow \mathcal{P}(A^\mathbb{N})$  satisfies:

$$(2) \quad \ell(x) = \{a \cdot \sigma \mid \exists x' \in X. x \xrightarrow{a} x' \text{ and } \sigma \in \ell(x')\}.$$

where the dot in  $a \cdot \sigma$  describes prefixing.

The following simple example shows that the equation (2) does not uniquely determine  $\ell(x)$ . Consider the LTS given by the following transitions:



Hence this LTS is captured by  $X = \{x\}$ ,  $A = \{b, c\}$  and  $\alpha(x) = \{(b, x), (c, x)\}$ . The set  $\ell(x)$  is then  $\{b, c\}^\mathbb{N}$ , *i.e.* contains all infinite sequences with elements  $b$  and  $c$ . However, the proper subset

$$(3) \quad k(x) = \{\sigma \in \ell(x) \mid b \text{ occurs infinitely often in } \sigma\}$$

also satisfies (2). For the inclusion ( $\subseteq$ ) assume  $\sigma \in k(x)$ , so that  $\sigma$  contains infinitely many  $b$ 's. If we write  $\sigma = a \cdot \sigma'$ , where  $a \in \{b, c\}$ , then clearly the tail

$\sigma'$  must also contain infinitely many  $b$ 's. The reverse inclusion ( $\supseteq$ ) is obtained similarly.

We have two remarks.

- This last example shows that uniqueness is not part of trace semantics (in this manner), and thus that a trace function  $\ell$  will not arise as map to a final coalgebra. However, as our main result (Theorem 6.5) shows, it does arise as canonical map to a weakly final coalgebra. This “weakness” means existence, but not unique existence.

Despite this lack of uniqueness, trace semantics is useful for the comparison of states, namely via trace inclusion or trace equality between states. This paper however, focuses exclusively on the abstract theory, and not on applications.

- One may wish to include finite traces—and not only infinite ones. A typical way of doing this is via auxiliary transitions that are repeated over and over again once a final state is reached. Then one may also restrict oneself to coalgebras of the form  $X \rightarrow \mathcal{P}_*(A \times X)$  for the powerset  $\mathcal{P}_*$  of non-empty sets. We shall not do so.

### 2.1 Trace semantics for LTSs, more abstractly

Before we proceed to a coalgebraic approach to trace semantics, we like to identify the crucial steps of this more abstract approach for an LTS  $\alpha: X \rightarrow \mathcal{P}(A \times X)$  as considered above. We first notice that its functor  $\mathcal{P}(A \times -)$  can be written as  $\mathcal{P}F$ , there  $F = A \times -$ . The final coalgebra of  $F$  is of course the set  $A^{\mathbb{N}}$  of infinite sequences, given by the head and tail operation  $\langle h, t \rangle: A^{\mathbb{N}} \xrightarrow{\cong} A \times A^{\mathbb{N}}$ . Interestingly, the set  $U$  of infinite traces carries an  $F$ -coalgebra structure  $U \rightarrow A \times U$ , given by  $(\langle x_0, a_0 \rangle, \langle x_1, a_1 \rangle, \dots) \mapsto \langle a_0, (\langle x_1, a_1 \rangle, \dots) \rangle$ . The induced map to the final coalgebra is  $\pi_2^{\mathbb{N}}: U \rightarrow A^{\mathbb{N}}$ , as used in the definition (1).

In the more abstract approach in the remainder of this paper we shall allow more general functors  $F$  for coalgebra  $X \rightarrow \mathcal{P}(FX)$ . A crucial step is to give an appropriate definition of the set  $U$  of infinite paths, with an  $F$ -coalgebra structure. This yields a map  $U \rightarrow Z$  to the final  $F$ -coalgebra  $Z$ , and hence a trace map  $X \rightarrow \mathcal{P}(Z)$ .

## 3 Coalgebraic preliminaries

We shall restrict ourselves to coalgebras  $X \rightarrow F(X)$  of endofunctors  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$ . Later, in Section 6 we shall further restrict these  $F$ 's to so-called polynomial functors, built up from the identity, constant functors, product, co-products and exponents  $(-)^A$  with a constant set, see for instance [3].

Associated with a functor  $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$  we can define what is usually called **relation lifting**. It is an extension  $\text{Rel}(F)$  of  $F$  from sets to relations, in the sense that  $R \subseteq X \times Y$  is mapped to  $\text{Rel}(F)(R) \subseteq F(X) \times F(Y)$ . It may be defined in general terms via epi-mono factorisation: if we have

$\langle r_1, r_2 \rangle: R \hookrightarrow X \times Y$ , then:

$$\begin{array}{ccc} F(R) & \xrightarrow{\quad} & \text{Rel}(F)(R) \\ & \searrow \langle F(r_1), F(r_2) \rangle & \downarrow \\ & & F(X) \times F(Y) \end{array}$$

For polynomial functors one can describe this relation lifting conveniently via an inductive definition, see [3].

The following list sums up the basic results that we need about relation lifting.

- (i) Equality is preserved:  $\text{Rel}(F)(=_X) = =_{F(X)}$ .
- (ii) Composition is preserved: for  $R \subseteq X \times Y$  and  $S \subseteq Y \times Z$ , the relational composition  $S \circ R = \{(x, z) \mid \exists y. R(x, y) \wedge S(y, z)\}$  satisfies:

$$\text{Rel}(F)(S \circ R) = \text{Rel}(F)(S) \circ \text{Rel}(F)(R).$$

- (iii) Inclusions are preserved: if  $R \subseteq S$  then  $\text{Rel}(F)(R) \subseteq \text{Rel}(F)(S)$ .
- (iv) Inverse images (or substitution, or reindexing) and direct images preserved: for functions  $f_1: X_1 \rightarrow Y_1$ ,  $f_2: X_2 \rightarrow Y_2$  and relations  $S \subseteq Y_1 \times Y_2$  and  $R \subseteq X_1 \times X_2$  we have:

$$\begin{aligned} \text{Rel}(F)\left((f_1 \times f_2)^{-1}(S)\right) &= (F(f_1) \times F(f_2))^{-1}\left(\text{Rel}(F)(S)\right) \\ \text{Rel}(F)\left(\coprod_{f_1 \times f_2}(R)\right) &= \coprod_{F(f_1) \times F(f_2)}\left(\text{Rel}(F)(R)\right), \end{aligned}$$

where, in general  $\coprod_h(P) = \{x \mid \exists y. x = h(y) \wedge y \in P\} = \{h(y) \mid y \in P\}$ .

All these properties hold for functors  $F$  that preserve weak pullbacks, see [6], and hence in particular for polynomial functors.

We shall need the following non-standard result about relation lifting.

**Lemma 3.1** *Assume a relation  $\langle r_1, r_2 \rangle: R \hookrightarrow X \times Y$ , with associated relation lifting  $\langle p_1, p_2 \rangle: \text{Rel}(F)(R) \hookrightarrow F(X) \times F(Y)$ . Assume we have a diagram*

$$\begin{array}{ccccc} T & \xleftarrow{\quad} & W & \xrightarrow{\quad} & S \\ \downarrow & & \downarrow \lrcorner & & \downarrow \\ Y \times Z & \xleftarrow{r_1 \times id} & R \times Z & \xrightarrow{r_2 \times id} & X \times Z \end{array}$$

Then we have an inclusion, as indicated by the dashed arrow, in:

$$\begin{array}{ccccc} \text{Rel}(F)(T) & \dashrightarrow & \bullet & \xleftarrow{\quad} & \text{Rel}(F)(S) \\ & \searrow & \downarrow & \downarrow \lrcorner & \downarrow \\ FY \times FZ & & \text{Rel}(F)(R) \times FZ & \xrightarrow{p_2 \times id} & FX \times FZ \\ & & \xleftarrow{p_1 \times id} & & \end{array}$$

**Proof** Since relation lifting preserves both direct and inverse images, we can apply it to the first diagram in the lemma and get:

$$\begin{array}{ccccc}
 \text{Rel}(F)(T) & \longleftarrow & \text{Rel}(F)(W) & \longrightarrow & \text{Rel}(F)(S) \\
 \downarrow & & \downarrow \lrcorner & & \downarrow \\
 FY \times FZ & \xleftarrow{F(r_1) \times \text{id}} & FR \times FZ & \xrightarrow{F(r_2) \times \text{id}} & FX \times FZ
 \end{array}$$

The surjection  $e: F(R) \twoheadrightarrow \text{Rel}(F)(R)$  with  $p_i \circ e = F(r_i)$  yields a mediating map  $\text{Rel}(F)(W) \rightarrow \star$ . The required map is then obtained via orthogonality (or so-called “diagonal-fill-in”) in:

$$\begin{array}{ccc}
 \text{Rel}(F)(W) & \longrightarrow & \text{Rel}(F)(T) \\
 \downarrow & \searrow \text{---} & \downarrow \\
 \star & & FY \times FZ \\
 \downarrow & \nearrow \text{---} & \bullet \\
 \bullet & \longrightarrow & 
 \end{array}$$

□

## 4 A distributive law

We now assume we have a functor  $F$  with associated relation lifting  $\text{Rel}(F)$ . Applying this relation lifting to the inhabitation relation  $\in_X \hookrightarrow X \times \mathcal{P}(X)$  on a set  $X$ , yields a new relation  $\text{Rel}(F)(\in_X) \hookrightarrow F(X) \times F(\mathcal{P}(X))$ . By transposition it gives a function:

$$(4) \quad \begin{array}{ccc}
 F(\mathcal{P}(X)) & \xrightarrow{\lambda_X} & \mathcal{P}(F(X)) \\
 u \mapsto & \longrightarrow & \{a \in F(X) \mid (a, u) \in \text{Rel}(F)(\in_X)\}
 \end{array}$$

We check that we get a natural transformation  $\lambda: F\mathcal{P} \Rightarrow \mathcal{P}F$ , using some of the above special properties of relation lifting. For  $f: X \rightarrow Y$  and  $u \in F(\mathcal{P}(X))$ ,

$$\begin{aligned}
 (\lambda_Y \circ F\mathcal{P}(f))(u) &= \{b \in F(Y) \mid (b, F\mathcal{P}(f)(u)) \in \text{Rel}(F)(\in_Y)\} \\
 &= \{b \mid (b, u) \in (F(\text{id}) \times F(\mathcal{P}(f)))^{-1}(\text{Rel}(F)(\in_Y))\} \\
 &= \{b \mid (b, u) \in \text{Rel}(F)((\text{id} \times \mathcal{P}(f))^{-1}(\in_Y))\} \\
 &\stackrel{*}{=} \{b \mid (b, u) \in \text{Rel}(F)(\coprod_{f \times \text{id}}(\in_X))\} \\
 &= \{b \mid (b, u) \in \coprod_{F(f) \times F(\text{id})}(\text{Rel}(F)(\in_X))\} \\
 &= \{F(f)(a) \mid (a, u) \in \text{Rel}(F)(\in_X)\} \\
 &= (\mathcal{P}F(f) \circ \lambda_X)(u).
 \end{aligned}$$

The marked equation  $\stackrel{*}{=}$  holds because:

$$\begin{aligned}
(\text{id} \times \mathcal{P}(f))^{-1}(\in_Y) &= \{(y, v) \mid y \in \mathcal{P}(f)(v)\} \\
&= \{(y, v) \mid \exists x \in v. y = f(x)\} \\
&= \{(f(x), v) \mid x \in v\} \\
&= \coprod_{f \times \text{id}}(\in_X).
\end{aligned}$$

It is well-known that the powerset functor  $\mathcal{P}$  forms a monad with singleton map as unit  $\eta = \{-\}$  and union as multiplication  $\mu = \bigcup$ . This monad structure is also preserved by the distributive law  $\lambda$ , as will be shown next.

$$\begin{aligned}
&(\lambda_X \circ F\{-\})(u) \\
&= \{a \mid (a, F(\{-\})(u)) \in \text{Rel}(F)(\in)\} \\
&= \{a \mid (a, u) \in \text{Rel}(F)((\text{id} \times \{-\})^{-1}(\in))\} \\
&= \{a \mid (a, u) \in \text{Rel}(F)(=_{\mathcal{P}(X)})\} \\
&= \{a \mid (a, u) \in =_{F(X)}\} \\
&= \{u\}. \\
&(\mu_{F(X)} \circ \mathcal{P}(\lambda_X) \circ \lambda_{\mathcal{P}(X)})(u) \\
&= \bigcup \{\lambda(a) \mid (a, u) \in \text{Rel}(F)(\in)\} \\
&= \{b \mid \exists a. (b, a) \in \text{Rel}(F)(\in) \wedge (a, u) \in \text{Rel}(F)(\in)\} \\
&= \{b \mid (b, u) \in \text{Rel}(F)(\in) \circ \text{Rel}(F)(\in)\} \\
&= \{b \mid (b, u) \in \text{Rel}(F)(\in \circ \in)\} \\
&\stackrel{*}{=} \{b \mid (b, u) \in \text{Rel}(F)((\text{id} \times \mu)^{-1}(\in))\} \\
&= \{b \mid (b, u) \in (F(\text{id}) \times F(\mu))^{-1}(\text{Rel}(F)(\in))\} \\
&= (\lambda_X \circ F(\mu))(u).
\end{aligned}$$

The equation  $\stackrel{*}{=}$  holds because:

$$\begin{aligned}
(x, z) \in (\in \circ \in) &\iff \exists y. x \in y \wedge y \in z \\
&\iff x \in \bigcup z \\
&\iff (x, z) \in (\text{id} \times \mu)^{-1}(\in).
\end{aligned}$$

The definition (4) of our distributive law also makes sense for the non-empty powerset  $\mathcal{P}_*$ , in the sense that  $\lambda: F\mathcal{P} \Rightarrow \mathcal{P}F$  restricts to  $\lambda_*: F\mathcal{P}_* \Rightarrow$

$\mathcal{P}_*F$ . It is well-defined, since:

$$\begin{array}{ccc}
 & \in & F(\in) \longrightarrow \text{Rel}(F)(\in) \\
 & \uparrow \langle e_1, e_2 \rangle & \uparrow \langle F(e_1), F(e_2) \rangle \\
 \mathcal{P}_*(X) \xleftarrow{\pi_2} X \times \mathcal{P}_*(X) & & F(\mathcal{P}_*(X)) \xleftarrow{\pi_2} F(X) \times F(\mathcal{P}_*(X))
 \end{array}$$

This says that for each  $u \in F\mathcal{P}_*(X)$  there is an  $a \in F(X)$  with  $(a, u) \in \text{Rel}(F)(\in)$ . Hence  $\lambda(u) \subseteq F(X)$  is non-empty.

**Example 4.1** Let  $F$  be the deterministic automaton functor  $(-)^A \times B$ . The associated distributive law  $\lambda: \mathcal{P}(X)^A \times B \Rightarrow \mathcal{P}(X^A \times B)$  is given

$$\lambda_X(f, b) = \{(g, c) \in X^A \times B \mid c = b \wedge \forall a \in A. g(a) \in f(a)\}.$$

As noted in [1], for the special case where  $B$  is the two element lattice  $\{0, 1\}$  there is also a distributive law  $\kappa$  in the reverse direction, given by:

$$\kappa_X(U) = \langle \lambda x \in A. \{h(x) \mid \exists b. (h, b) \in U\}, \bigvee \{b \mid \exists h. (h, b) \in U\} \rangle$$

It is easy to see that  $\kappa$  restricts to  $\kappa_*: \mathcal{P}_*((-)^A \times B) \Rightarrow \mathcal{P}_*(-)^A \times B$ . It has  $\lambda_*$  as section:

$$\begin{aligned}
 (\kappa_* \circ \lambda_*)(f, b) &= \kappa_* (\{(g, b) \mid \forall a. g(a) \in f(a)\}) \\
 &= \langle \lambda x \in A. \{h(x) \mid \forall a. h(a) \in f(a)\}, \bigvee \{b\} \rangle \\
 &= \langle f, b \rangle, \quad \text{since each } f(a) \text{ is non-empty.}
 \end{aligned}$$

For future use we note the following result.

**Lemma 4.2** For  $n \in \mathbb{N}$  define the natural transformation  $\lambda^n: F^n\mathcal{P} \Rightarrow \mathcal{P}F^n$  by

$$\lambda_X^n = \begin{cases} id_{\mathcal{P}(X)} & \text{if } n = 0 \\ \lambda_{F^{n-1}(X)} \circ F(\lambda_X^{n-1}) & \text{otherwise.} \end{cases}$$

Then, for each  $n \in \mathbb{N}$  we have a pullback square:

$$\begin{array}{ccc}
 \text{Rel}(F)^n(\in_X) & \longrightarrow & \in_{F^n(X)} \\
 \downarrow \lrcorner & & \downarrow \\
 F^n(X) \times F^n(\mathcal{P}(X)) & \xrightarrow{id \times \lambda_X^n} & F^n(X) \times \mathcal{P}(F^n(X))
 \end{array}$$

**Proof** By induction on  $n$ . The base case  $n = 0$  is obvious. If  $n > 0$ , then by applying relation lifting  $\text{Rel}(F)$  to the pullback square for  $n - 1$  we get the

square below.

$$\begin{array}{ccc}
 \text{Rel}(F)^n(\in_X) & \xrightarrow{\quad} & \text{Rel}(F)(\in_{F^{n-1}(X)}) \\
 \downarrow \lrcorner & & \downarrow \\
 F^n(X) \times F^n(\mathcal{P}(X)) & \xrightarrow{\text{id} \times F(\lambda_X^{n-1})} & F^n(X) \times F(\mathcal{P}(F^{n-1}(X)))
 \end{array}$$

The next square is a pullback by definition of  $\lambda$ .

$$\begin{array}{ccc}
 \text{Rel}(F)(\in_{F^{n-1}(X)}) & \xrightarrow{\quad} & \in_{F^n(X)} \\
 \downarrow \lrcorner & & \downarrow \\
 F^n(X) \times F(\mathcal{P}(F^{n-1}(X))) & \xrightarrow{\text{id} \times \lambda_{F^{n-1}(X)}} & F^n(X) \times \mathcal{P}(F^n(X))
 \end{array}$$

Hence by putting these two squares side-by-side we get the required pullback.  $\square$

## 5 Coalgebras and Kleisli categories

In this section we assume we have a functor  $F: \mathbb{C} \rightarrow \mathbb{C}$ , a monad  $T = (T, \eta, \mu)$  on  $\mathbb{C}$ , and a distributive law  $\lambda: FT \Rightarrow TF$ . The latter satisfies  $\lambda \circ F\eta = \eta$  and  $\lambda \circ F\mu = \mu \circ T\lambda \circ \lambda$ . Later we shall concentrate on the special case where  $T$  is powerset  $\mathcal{P}$ , but here it is clearer to use a general monad for clarity and abstraction. As an aside, the monad is not to be seen as a “term formation monad” like in [7], but as a “computational monad”, as in [4].

We shall write  $\mathbb{C}_T$  for the Kleisli category associated with  $\mathbb{C}$ . Its objects are the objects  $X \in \mathbb{C}$ , and its morphisms  $X \rightarrow Y$  are maps  $X \rightarrow TY$  in  $\mathbb{C}$ . The identity map  $X \rightarrow X$  in  $\mathbb{C}_T$  is the unit  $\eta_X$ , and the composition in  $\mathbb{C}_T$  of  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  is the map  $\mu_Z \circ Tg \circ f$ . There is an obvious inclusion functor  $J: \mathbb{C} \rightarrow \mathbb{C}_T$  given by  $X \mapsto X$  and  $f \mapsto \eta \circ f$ . It has a right adjoint  $K: \mathbb{C}_T \rightarrow \mathbb{C}$  described by  $Y \mapsto TY$  and  $g \mapsto \mu \circ Tg$ .

The functor  $F: \mathbb{C} \rightarrow \mathbb{C}$  can be lifted to a functor  $F_T: \mathbb{C}_T \rightarrow \mathbb{C}_T$  on the Kleisli category, by:

$$X \mapsto FX \quad \text{and} \quad f \mapsto \lambda \circ Ff.$$

We now have the following situation,

$$(5) \quad \begin{array}{ccc}
 \mathbf{CoAlg}(F) & \xrightarrow{J_T} & \mathbf{CoAlg}(F_T) \\
 \downarrow & \curvearrowright K & \downarrow \\
 \mathbb{C} & \xrightarrow{J} & \mathbb{C}_T \\
 \uparrow F & & \uparrow F_T
 \end{array}$$

where  $J_T$  is the obvious lifting of  $J$ .



## 6 The weakly final coalgebra of traces

In this section we return to the concrete situation where the monad  $T$  from the previous section is the powerset  $\mathcal{P}$ . The resulting Kleisli category  $\mathbf{Sets}_{\mathcal{P}}$  is the category of sets with relations between them as morphisms. The main result of this section is a description of (the carrier of) a weakly final coalgebra in the category  $\mathbf{CoAlg}(F_{\mathcal{P}})$  as powerset of (the carrier of) the final coalgebra of  $F$ .

We assume that our functor  $F$  is polynomial, in the sense that it is built up from the identity functor  $X \mapsto X$  and constant functors  $X \mapsto A$ , using products  $\times$ , coproducts  $+$  and exponents  $(-)^A$ . Such functors are guaranteed to have a final coalgebra, which we shall write as  $\zeta: Z \xrightarrow{\cong} FZ$ .

An important starting point is the next technical lemma. It involves elements  $u \in \prod_{n \in \mathbb{N}} A_n$  in a countable product of sets  $A_n$ . For such an  $u$  we shall write  $u_i \in A_i$  for the  $i$ -th projection  $p_i(u)$ , where  $p_i: (\prod_{n \in \mathbb{N}} A_n) \rightarrow A_i$ .

**Lemma 6.1** *Assume we have countable collections  $(X_n)_{n \in \mathbb{N}}$  of sets with relations  $(R_n \subseteq X_{n+1} \times X_n)_{n \in \mathbb{N}}$  between them. We form the sets of sequences,*

$$\begin{aligned} V &= \{v \in \prod_{n \in \mathbb{N}} X_n \mid \forall n \in \mathbb{N}. (v_{n+1}, v_n) \in R_n\} \\ W_F &= \{w \in \prod_{n \in \mathbb{N}} F(X_n) \mid \forall n \in \mathbb{N}. (w_{n+1}, v_n) \in \text{Rel}(F)(R_n)\}. \end{aligned}$$

*The tuple of projection functions  $\langle F(p_n) \rangle_{n \in \mathbb{N}}: F(\prod_{n \in \mathbb{N}} X_n) \rightarrow \prod_{n \in \mathbb{N}} F(X_n)$  then restricts to an isomorphism*

$$F(V) \xrightarrow{\cong} W_F.$$

**Proof** By induction on the structure of  $F$ .

- If  $F$  is the identity functor  $Id$  the result is obvious because  $V = W_{Id}$ .
- If  $F$  is the constant functor  $A$ , then  $W_A$  consists of the sequences  $w \in \prod_{n \in \mathbb{N}} A$  with  $w_0 = w_1 = w_2 = \dots$ , because  $(w_{n+1}, w_n) \in \text{Rel}(F)(R_n) = =_A$ . Hence  $W_A$  is isomorphic to  $A = F(V)$ .
- If  $F = F_1 \times F_2$ , then each  $w \in W_{F_1 \times F_2}$  has projections  $w_n = (w_n^1, w_n^2)$  with

$$(w_{n+1}^1, w_n^1) \in \text{Rel}(F_1)(R_n) \quad \text{and} \quad (w_{n+1}^2, w_n^2) \in \text{Rel}(F_2)(R_n).$$

Hence  $w \in W_{F_1 \times F_2}$  corresponds to two sequences  $w^1 \in W_{F_1}$  and  $w^2 \in W_{F_2}$ . But by the induction hypothesis, these  $w^1$  and  $w^2$  correspond to elements in  $F_1(V)$  and  $F_2(V)$ , and thus to a single element in  $F_1(V) \times F_2(V) = F(V)$ .

- If  $F = F_1 + F_2$ , then each  $w \in W_{F_1 + F_2}$  has elements  $w_n \in F_1(X_n) + F_2(X_n)$ . But:

$$\begin{aligned} (w_{n+1}, w_n) \in \text{Rel}(F)(R_n) &= \{(\kappa_1 a, \kappa_1 b) \mid (a, b) \in \text{Rel}(F_1)(R_n)\} \\ &\quad \cup \{(\kappa_1 a, \kappa_1 b) \mid (a, b) \in \text{Rel}(F_1)(R_n)\}. \end{aligned}$$

As a result, we have either of the following two situations.

- Each  $w_n$  is of the form  $\kappa_1 z_n$ , with the sequence  $z = (z_n)_{n \in \mathbb{N}} \in W_{F_1}$ . Such a  $z$  corresponds by the induction hypothesis to an element in  $F_1(V)$ .
- Each  $w_n$  is of the form  $\kappa_2 z_n$  with  $z \in W_{F_2}$  corresponding to an element in  $F_2(V)$ .

Combining these two points yields the required isomorphism  $W_{F_1+F_2} \cong F_1(V) + F_2(V)$ .

- If  $F = G^A$ , then each  $w \in W_{G^A}$  consists of  $w_n \in G(X)^A$  consisting of  $w_n(a) \in G(X)$ , for each  $a \in A$ . Each sequence  $(w_n(a))_{n \in \mathbb{N}}$  is then in  $W_G$ , and thus corresponds to an element in  $G(V)$ . This yields a correspondence of  $w$  to an element in  $G(V)^A = F(V)$ .

□

We use this result in our next step towards a weakly final coalgebra in the category  $\mathbf{CoAlg}(F_{\mathcal{P}})$ . It uses a coalgebra  $X \rightarrow \mathcal{P}(F(X))$ , *i.e.* an object in the category  $\mathbf{CoAlg}(F_{\mathcal{P}})$

**Lemma 6.2** *For a coalgebra  $\alpha: X \rightarrow \mathcal{P}(F(X))$  consider the induced relation  $(id \times \alpha)^{-1}(\in_{FX})$  in:*

$$\begin{array}{ccc}
 (id \times \alpha)^{-1}(\in_{FX}) & \xrightarrow{\quad} & \in_{FX} \\
 \downarrow \lrcorner & & \downarrow \\
 FX \times X & \xrightarrow{id \times \alpha} & FX \times \mathcal{P}FX
 \end{array}$$

together with the repeated applications of relation lifting:

$$R_n \stackrel{\text{defn}}{=} \text{Rel}(F)^n((id \times \alpha)^{-1}(\in_{FX})) \hookrightarrow F^{n+1}(X) \times F^n(X)$$

The set

$$U = \{u \in \prod_{n \in \mathbb{N}} F^n(X) \mid \forall n \in \mathbb{N}. (u_{n+1}, u_n) \in R_n\}$$

then carries a coalgebra structure given by a surjective function  $\alpha_U: U \rightarrow F(U)$  with

$$\begin{array}{ccc}
 U & \xrightarrow{\alpha_U} & F(U) \\
 & \searrow p_{n+1} & \downarrow F(p_n) \\
 & & F^{n+1}(X)
 \end{array}$$

Moreover, it gives rise to the unique map of  $F$ -coalgebra  $\bar{\alpha}: U \rightarrow Z$  in:

$$\begin{array}{ccc}
 F(U) & \xrightarrow{F(\bar{\alpha})} & F(Z) \\
 \alpha_U \uparrow & & \cong \uparrow \zeta \\
 U & \xrightarrow{\bar{\alpha}} & Z
 \end{array}$$

**Proof** We apply the previous lemma with  $X_n = F^n(X)$ ,  $V = U$  and

$$W_F = U_- = \{\langle u_{n+1} \rangle_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} F(X_n) \mid u \in U\}$$

Hence  $U_-$  contains the sequences  $u \in U$  with the first element  $u_0 \in X$  removed. We then get our coalgebra  $\alpha_U: U \rightarrow F(U)$  using the isomorphism  $\varphi$  from the previous lemma:

$$(6) \quad \alpha_U = \left( U \xrightarrow{\langle p_{i+1} \rangle_{i \in \mathbb{N}}} U_- \xrightarrow[\cong]{\varphi} F(U) \right)$$

Then clearly,  $F(p_n) \circ \alpha_U = F(p_n) \circ \varphi \circ \langle p_{i+1} \rangle_{i \in \mathbb{N}} = p_n \circ \langle p_{i+1} \rangle_{i \in \mathbb{N}} = p_{n+1}$ .  $\square$

**Example 6.3** We shall illustrate the constructions so far for the functor  $F(X) = A \times X$ , with a coalgebra  $\alpha: X \rightarrow \mathcal{P}(A \times X)$  describing a labelled transition system. The set  $U$  from the previous lemma then contains elements  $u \in \prod_{n \in \mathbb{N}} A^n \times X$  with the following property. If  $u_n = (\vec{a}, x)$ , then  $u_{n+1} = (\vec{a} \cdot b, y)$ , where  $(b, y) \in \alpha(x)$ . The latter is usually written as a transition  $x \xrightarrow{b} y$ . Hence the set  $U$  contains the infinite transition sequences, together with their labels, like in Section 2. The above map  $\bar{\alpha}$  from  $U$  to the final  $F$ -coalgebra  $A^{\mathbb{N}}$  maps such a sequence of transitions to the sequence of labels involved.

**Remark 6.4** An alternative way to understand the coalgebra map  $\bar{\alpha}: U \rightarrow Z$  in Lemma 6.2 is as follows. Since  $F$  is a polynomial functor we know that it preserves limits of  $\omega$ -chains, so that its final coalgebra  $Z$  can be described simply as  $\omega$ -limit, namely of the sequence

$$1 \longleftarrow F(1) \xleftarrow{F(!)} F^2(1) \xleftarrow{F^2(!)} \dots$$

The associated projections  $\pi_n: Z \rightarrow F^n(1)$  satisfy  $F^n(!) \circ \pi_{n+1} = \pi_n$ . The final coalgebra  $\zeta: Z \xrightarrow{\cong} F(Z)$  satisfies  $F(\pi_n) \circ \zeta = \pi_{n+1}$ .

A sequence  $u \in U$  consists of elements  $u_n \in F^n(X)$ , which yield  $u'_n = F^n(!)(u_n) \in F^n(1)$  forming a chain. The chain property requires that:

$$u'_n = F^n(!_{F(1)})(u'_{n+1}),$$

i.e. that

$$\begin{aligned} F^n(!_X)(u_n) &= F^n(!_{F(1)})(F^{n+1}(!_X)(u'_{n+1})) \\ &= F^n(!_X)(u_{n+1}). \end{aligned}$$

The latter equation holds for the following reason. Obviously, the following diagram commutes.

$$\begin{array}{ccc} (id \times \alpha)^{-1}(\in_{FX}) & \longrightarrow & 1 \\ \downarrow & & \downarrow \langle id, id \rangle \\ FX \times X & \xrightarrow{!_{FX} \times !_X} & 1 \times 1 \end{array}$$

Hence by repeatedly applying relation lifting and using that relation lifting preserves the equality relation  $1 \mapsto 1 \times 1$  on the right, we get:

$$\begin{array}{ccc} \text{Rel}(F)^n((\text{id} \times \alpha)^{-1}(\in_{FX})) & \longrightarrow & \text{Rel}(F)^n(1) = F^n(1) \\ \downarrow & & \downarrow \langle \text{id}, \text{id} \rangle \\ F^{n+1}(X) \times F^n(X) & \xrightarrow{F^n(!_{FX}) \times F^n(!_X)} & F^n(1) \times F^n(1) \end{array}$$

This shows that  $F^n(!_X)(u_n) = F^n(!_X)(u_{n+1})$ .

As a result we get an element  $\bar{u} \in Z$  in the final coalgebra satisfying  $\pi_n(\bar{u}) = F^n(!_X)(u_n)$ . This is  $\bar{\alpha}(u)$  as in Lemma 6.2, since:

$$(7) \quad \pi_n \circ \bar{\alpha} = F^n(!_X) \circ p_n.$$

This is easily checked by induction.

We are now in a position to state and prove the main result of this paper.

**Theorem 6.5** *Let  $F$  be a polynomial functor.*

- (i) *The functor  $J_{\mathcal{P}}: \mathbf{CoAlg}(F) \rightarrow \mathbf{CoAlg}(F_{\mathcal{P}})$  from (5) turns a final  $F$ -coalgebra  $\zeta: Z \xrightarrow{\cong} FZ$  into a weakly final  $F_{\mathcal{P}}$ -coalgebra.*

*Concretely, this means that for each coalgebra  $\alpha: X \rightarrow \mathcal{P}(F(X))$  there is a “trace” map  $\ell: X \rightarrow \mathcal{P}(Z)$  making the following diagram commute.*

$$\begin{array}{ccccccc} \mathcal{P}FX & \xrightarrow{\mathcal{P}F\ell} & \mathcal{P}F\mathcal{P}Z & \xrightarrow{\mathcal{P}\lambda} & \mathcal{P}^2FZ & \xrightarrow{\mu} & \mathcal{P}FZ \\ \alpha \uparrow & & & & & & \cong \uparrow \mathcal{P}\zeta \\ X & \xrightarrow{\ell} & & & & & \mathcal{P}Z \end{array}$$

That is, for each  $x \in X$ ,

$$(8) \quad \{\zeta(z) \mid z \in \ell(x)\} = \bigcup \{\lambda(F(\ell)(y)) \mid y \in \alpha(x)\}.$$

- (ii) *There is a canonical choice for this coalgebra map  $\ell$ , namely the largest one:*

$$\ell(x) = \{\bar{\alpha}(u) \mid u \in U \text{ and } u_0 = x\}$$

where  $U$  and  $\bar{\alpha}$  are as defined in Lemma 6.2.

**Proof** The canonical description of  $\ell$  means:

$$\begin{aligned} (z, x) \in (\text{id} \times \ell)^{-1}(\in_Z) &\iff z \in \ell(x) \\ &\iff \exists u \in U. z = \bar{\alpha}(u) \text{ and } p_0(u) = x \\ &\iff (z, x) \in \coprod_{\bar{\alpha} \times \text{id}} \langle \text{id}_U, p_0 \rangle. \end{aligned}$$

This can also be expressed in a diagram:

$$\begin{array}{ccccc}
 U & \longrightarrow & (\text{id} \times \ell)^{-1}(\in_Z) & \longrightarrow & \in_Z \\
 \langle \text{id}, p_0 \rangle \downarrow & & \downarrow \lrcorner & & \downarrow \\
 U \times X & \xrightarrow{\bar{\alpha} \times \text{id}} & Z \times X & \xrightarrow{\text{id} \times \ell} & Z \times \mathcal{P}(Z)
 \end{array}$$

Since relation lifting preserves direct and inverse images it yields:

$$\begin{array}{ccccc}
 \text{Rel}(F)(U) = F(U) & \longrightarrow & \text{Rel}(F)((\text{id} \times \ell)^{-1}(\in_Z)) & \longrightarrow & \text{Rel}(F)(\in_Z) \\
 \langle \text{id}, F(p_0) \rangle \downarrow & & \downarrow \lrcorner & & \downarrow \\
 F(U) \times F(X) & \xrightarrow{F(\bar{\alpha}) \times \text{id}} & F(Z) \times F(X) & \xrightarrow{\text{id} \times F(\ell)} & F(Z) \times F(\mathcal{P}(Z))
 \end{array}$$

We use the last diagram in the step marked by (\*) below in a derivation of (8).

$$\begin{aligned}
 a &\in \bigcup \{ \lambda(F(\ell)(y)) \mid y \in \alpha(x) \} \\
 &\iff \exists y \in \alpha(x). a \in \lambda(F(\ell)(y)) \\
 &\iff \exists y \in \alpha(x). (a, F(\ell)(y)) \in \text{Rel}(F)(\in_Z) \\
 &\iff \exists y \in \alpha(x). (a, y) \in (\text{id} \times F(\ell))^{-1} \text{Rel}(F)(\in_Z) \\
 &\stackrel{(*)}{\iff} \exists y \in \alpha(x). (a, y) \in \coprod_{F(\bar{\alpha}) \times \text{id}} \text{Rel}(F)(\langle \text{id}, F(p_0) \rangle) \\
 &\iff \exists y \in \alpha(x). \exists b \in F(U). F(\bar{\alpha})(b) = a \text{ and } F(p_0)(b) = y \\
 &\stackrel{(6)}{\iff} \exists y \in \alpha(x). \exists u \in U. F(\bar{\alpha})(\alpha_U(u)) = a \text{ and } F(p_0)(\alpha_U(u)) = y \\
 &\iff \exists y \in \alpha(x). \exists u \in U. \zeta(\bar{\alpha}(u)) = a \text{ and } p_1(u) = y \\
 &\stackrel{(**)}{\iff} \exists u \in U. \zeta(\bar{\alpha}(u)) = a \text{ and } p_0(u) = x \\
 &\iff \exists z \in \ell(x). \zeta(z) = a \\
 &\iff a \in \{ \zeta(z) \mid z \in \ell(x) \}.
 \end{aligned}$$

The step marked with (\*\*) is justified because  $\alpha(u)$ , and thus also  $\bar{\alpha}(u)$ , does not depend on  $u_0$ , see (6).

We still have to prove the second point of the theorem, namely that  $\ell$  is the largest coalgebra map. Suppose  $k: X \rightarrow \mathcal{P}(Z)$  also satisfies (8), *i.e.* for all  $x \in X$ ,

$$\{ \zeta(z) \mid z \in k(x) \} = \bigcup \{ \lambda(F(k)(y)) \mid y \in \alpha(x) \}.$$

We first massage this equation into an equation between relations, so that we can apply relation lifting to both sides. The left-hand-side (LHS) is in fact the relation on  $F(Z) \times X$  given by

$$\begin{aligned}
 \text{LHS} &= \{ (w, x) \in F(Z) \times X \mid \exists z \in Z. w = \zeta(z) \wedge z \in k(x) \} \\
 &= \coprod_{\zeta \times \text{id}} (\text{id} \times k)^{-1}(\in_Z),
 \end{aligned}$$

as in the diagram:

$$\begin{array}{ccccc}
 \text{LHS} & \longleftarrow & \bullet & \longrightarrow & \in_Z \\
 \downarrow & & \downarrow \lrcorner & & \downarrow \\
 FZ \times X & \xleftarrow{\zeta \times \text{id}} & Z \times X & \xrightarrow{\text{id} \times k} & Z \times \mathcal{P}(Z)
 \end{array}$$

Hence

$$\begin{aligned}
 \text{Rel}(F)^n(\text{LHS}) &= \coprod_{F^n(\zeta) \times \text{id}} (\text{id} \times F^n(k))^{-1} \text{Rel}(F)^n(\in_Z) \\
 &= \coprod_{F^n(\zeta) \times \text{id}} (\text{id} \times F^n(k))^{-1} (\text{id} \times \lambda^n)^{-1} (\in_{F^n Z}) \\
 &\quad \text{by Lemma 4.2} \\
 &= \coprod_{F^n(\zeta) \times \text{id}} (\text{id} \times k^n)^{-1} (\in_{F^n Z}) \\
 &= \{(F^n(\zeta)(v), w) \mid v \in k^n(w)\},
 \end{aligned}$$

where we have abbreviated  $k^n = \lambda^n \circ F^n(k): F^n X \rightarrow \mathcal{P}(F^n Z)$ , with  $\lambda^n$  as defined in Lemma 4.2.

The right-hand-side (RHS) is more complicated. It can be described as:

$$\begin{array}{ccccccc}
 \text{RHS} & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longrightarrow & \in_{FX} \\
 \downarrow & & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\
 FZ \times X & \xleftarrow{e_1 \times \text{id}} & \in_{FZ} \times X & \xrightarrow{e_2 \times \text{id}} & \mathcal{P}FZ \times X & \xleftarrow{k^1 \times \text{id}} & FX \times X & \xrightarrow{\text{id} \times \alpha} & FX \times \mathcal{P}FX
 \end{array}$$

where  $\in_{FZ}$  is written as relation  $\langle e_1, e_2 \rangle: \in_{FZ} \rightarrow FZ \times \mathcal{P}FZ$ . We now use Lemma 3.1 and write  $\langle p_1, p_2 \rangle: \text{Rel}(F)^n(\in_{FZ}) \rightarrow F^{n+1}Z \times F^n \mathcal{P}FZ$ , so that we get an inclusion:

$$\begin{aligned}
 \text{Rel}(F)^n(\text{RHS}) &\subseteq \coprod_{p_1 \times \text{id}} (p_2 \times \text{id})^{-1} \coprod_{F^n(k^1) \times \text{id}} (R_n) \\
 &\quad \text{where } R_n = \text{Rel}(F)^n((\text{id} \times \alpha)^{-1}(\in_{FX})) \\
 &= \{(v, w) \mid \exists a \in \text{Rel}(F)^n(\in_{FZ}). p_1(a) = v \wedge \\
 &\quad (p_2(a), w) \in \coprod_{F^n(k^1) \times \text{id}} (R_n)\} \\
 &= \{(v, w) \mid \exists b \in F^n \mathcal{P}FZ. v \in \lambda^n(b) \wedge \\
 &\quad (b, w) \in \coprod_{F^n(k^1) \times \text{id}} (R_n)\} \quad \text{by Lemma 4.2} \\
 &= \{(v, w) \mid \exists b \in F^n \mathcal{P}FZ. v \in \lambda^n(b) \wedge \\
 &\quad \exists y \in F^{n+1}X. b = F^n(k^1)(y) \wedge (y, w) \in R_n\} \\
 &= \{(v, w) \mid \exists y \in F^{n+1}X. v \in k^{n+1}(y) \wedge (y, w) \in R_n\}.
 \end{aligned}$$

Once we know these facts about  $\text{Rel}(F)^n(\text{LHS})$  and  $\text{Rel}(F)^n(\text{RHS})$  we can start to show that  $k$  is included in  $\ell$ . Assume therefore that  $z \in k(x_0)$ . Our aim is to prove that  $z \in \ell(x_0)$  by showing that there is a sequence  $u = (x_0, x_1, x_2, \dots) \in U$  with  $\bar{\alpha}(u) = z$ .

We proceed step by step. We shall use the abbreviation  $\zeta^n: Z \xrightarrow{\cong} F^n Z$  for:

$$\zeta^n = \begin{cases} \text{id} & \text{if } n = 0 \\ F^{n-1}(\zeta) \circ \zeta^{n-1} & \text{otherwise.} \end{cases}$$

- (i)  $z \in k(x_0)$  yields  $(\zeta(z), x_0) \in \text{LHS} = \text{RHS}$ . Hence there is an  $x_1 \in F(X)$  with  $\zeta(z) \in k^1(x_1)$  and  $(x_1, x_0) \in R_0$ .
- (ii)  $\zeta(z) = \zeta^1(z) \in k^1(x_1)$  yields  $(\zeta^2(z), x_1) \in \text{Rel}(F)(\text{LHS}) = \text{Rel}(F)(\text{RHS})$ . Hence there is an  $x_2 \in F^2(X)$  with  $\zeta^2(z) \in k^2(x_2)$  and  $(x_2, x_1) \in R_1$ .
- (iii)  $\dots$

In this way we obtain a sequence  $u = (x_0, x_1, x_2, \dots) \in U$  with  $\zeta^n(z) \in k^n(x_n)$ . The latter amounts to  $(\zeta^n(z), F^n(k)(x_n)) \in \text{Rel}(F)^n(\in_Z)$ , using Lemma 4.2. But then, like in Remark 6.4,

$$F^n(!_Z)(\zeta^n(z)) = F^n(!_P Z)(F^n(k)(x_n)).$$

The right-hand-side of this equation is clearly equal to  $F^n(!_X)(x_n)$ . And the left-hand-side is equal to  $\pi_n z$ , since it is not hard to check that  $F^n(!_Z) \circ \zeta^n = \pi_n$ . This shows, as noted in Remark 6.4, that  $z = \bar{\alpha}(u)$ .  $\square$

## 7 Conclusions

We have shown how non-trivial use of relation lifting leads to a general formulation of traces for coalgebras  $X \rightarrow \mathcal{P}(FX)$ , and to a weak finality result. Suitable relational reasoning for establishing trace inclusion or equality is left to future work.

### *Acknowledgments*

The counterexample to uniqueness in Section 2 came up in discussion with Jesse Hughes.

## References

- [1] F. Bartels. *On generalised coinduction and probabilistic specification formats. Distributive laws in coalgebraic modelling*. PhD thesis, Free Univ. Amsterdam, 2004.  
<http://homepages.cwi.nl/~bartels/>.
- [2] B. Bloom, S. Istrail, and A.R. Meyer. Bisimulation can't be traced. *Journ. ACM*, 42(1):232–268, 1988.
- [3] B. Jacobs. Exercises in coalgebraic specification. In R. Crole R. Backhouse and J. Gibbons, editors, *Algebraic and Coalgebraic Methods in the Mathematics of Program Construction*, number 2297 in Lect. Notes Comp. Sci., pages 237–280. Springer, Berlin, 2002.

- [4] E. Moggi. Notions of computation and monads. *Inf. & Comp.*, 93(1):55–92, 1991.
- [5] J. Power and D. Turi. A coalgebraic foundation for linear time semantics. In M. Hofmann D. Pavlović and G. Rosolini, editors, *Category Theory and Computer Science 1999*, number 29 in Elect. Notes in Theor. Comp. Sci. Elsevier, Amsterdam, 1999.  
[www.elsevier.nl/locate/entcs/volume29.html](http://www.elsevier.nl/locate/entcs/volume29.html).
- [6] J. Rutten. Universal coalgebra: a theory of systems. *Theor. Comp. Sci.*, 249:3–80, 2000.
- [7] D. Turi and G. Plotkin. Towards a mathematical operational semantics. In *Logic in Computer Science*, pages 280–291. IEEE, Computer Science Press, 1997.