The Expectation Monad in Quantum Foundations

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Abstract

The expectation monad is introduced and related to known monads: it sits between on the one hand the distribution and ultrafilter monad, and on the other hand the continuation monad. The Eilenberg-Moore algebras of the expectation monad are characterized as convex compact Hausdorff spaces, using a theorem of Świrszcz. These convex compact Hausdorff spaces are dually equivalent to Banach (complete) order unit spaces, via a result of Kadison, which in turn are equivalent to Banach effect modules. In this way we obtain a close 'triangle' relationship between predicates and states for the expectation monad. Moreover, the approach leads to a new re-formulation of Gleason's theorem, expressing that effects on a Hilbert space are free effect modules on projections, obtained via tensoring with the unit interval.

1 Introduction

Techniques that have been developed over the last decades for the semantics of programming languages and programming logics are gaining wider significance. In this way a new interdisciplinary area has emerged where researchers from mathematics, (theoretical) physics and (theoretical) computer science collaborate, notably on quantum computation and quantum foundations. The article [5] uses the phrase "Rosetta Stone" for the language and concepts of category theory that form an integral part of this common area.

The present article is also part of this new field. It uses results from programming semantics, topology and (convex) analysis, category theory (esp. monads), logic and probability, and quantum foundations. The origin of this article is an illustration of the connections involved. Previously, the authors have worked on effect algebras and effect modules [25,20,21,15] from quantum logic, which are fairly general structures incorporating both logic (Boolean and

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orthomodular lattices) and probability (the unit interval $[0,1]$ and fuzzy predicates). By reading completely different work, on formal methods in computer security (in particular the thesis [41]), the expectation monad was noticed. The monad is used in [41,8] to give semantics to a probabilistic programming language that helps to formalize (complexity) reduction arguments from security proofs in a theorem prover. In [41] (see also [4,38]) the expectation monad is defined in a somewhat *ad hoc* manner (see Section 9 for details). Soon it was realized that a more systematic definition of this expectation monad could be given via the (dual) adjunction between convex sets and effect modules. Subsequently the two main parts of the present paper emerged.

(1) The expectation monad turns out to be related to several known monads as described in the following diagram.

$$\begin{tikzcd}[row sep=small] \big(\text{distribution } \mathcal{D}\big) \arrow[dr, tail] & & \\ & \big(\text{expectation } \mathcal{E}\big) \arrow[r, tail] & \big(\text{continuation } \mathcal{C}\big) \\ \big(\text{ultrafilter } \mathcal{U}\big) \arrow[ur, tail] & & \end{tikzcd} \qquad (1)$$

The continuation monad $\mathcal{C}$ also comes from programming semantics. But here we are more interested in the connection with the distribution and ultrafilter monads $\mathcal{D}$ and $\mathcal{U}$. Since the algebras of the distribution monad are convex sets and the algebras of the ultrafilter monad are compact Hausdorff spaces (a result known as Manes theorem) it follows that the algebras of the expectation monad must be some subcategory of convex compact Hausdorff spaces. This is made precise by a theorem of Świrszcz, describing convex compact Hausdorff spaces as monadic/algebraic over sets, via the monad that sends a set X to the states of the order unit space $\ell^\infty(X)$ of bounded real-valued functions on X. It turns out that the expectation monad is isomorphic to this monad used by Świrszcz. We give a more concrete desciption of the algebras of the monad using basic notions from Choquet theory, notably barycenters of measures.

(2) Kadison duality describes the dual equivalence between convex compact Hausdorff spaces and Banach complete order unit spaces. Here it is shown how these order unit spaces correspond to effect modules. This allows us to give a proper categorical description of the duality between states and effects (predicates) that is fundamental in quantum theory.

These two parts of the paper may be summarized as follows. There are classical results describing the category $\mathcal{EM}(\mathcal{U})$ of Eilenberg-Moore algebra of the ultrafilter monad $\mathcal{U}$ as:

$$\mathcal{EM}(\mathcal{U}) \overset{\text{[Manes]}}{\simeq} \begin{pmatrix} \text{compact} \\ \text{Hausdorff spaces} \end{pmatrix} \overset{\text{[Gelfand]}}{\simeq} \begin{pmatrix} \text{commutative} \\ C^*\text{-algebras} \end{pmatrix}^{\mathrm{op}}$$

Here we give the following "probabilistic" analogues for the expection monad

$\mathcal{E}$:

$$\mathcal{EM}(\mathcal{E}) \overset{[\text{Świrszcz}]}{\simeq} \begin{pmatrix}\text{convex compact}\\ \text{Hausdorff spaces}\end{pmatrix} \overset{[\text{Kadison}]}{\simeq} \begin{pmatrix}\text{Banach order}\\ \text{unit spaces}\end{pmatrix}^{\mathrm{op}}$$
$$\simeq \begin{pmatrix}\text{Banach}\\ \text{effect modules}\end{pmatrix}^{\mathrm{op}}$$

The role played by the two-element set $\{0,1\}$ in these classical results—*e.g.* as "schizophrenic" object—is played in our probabilistic analogues by the unit interval $[0,1]$.

Quantum mechanics is notoriously non-intuitive. Hence a proper mathematical understanding of the relevant phenomena is important, certainly within the emerging field of quantum computation. It seems fair to say that such an all-encompassing understanding of quantum mechanics does not exist yet. For instance, the categorical analysis in [1,2] describes some of the basic underlying structure in terms of monoidal categories, daggers, and compact closure. However, an integrated view of logic and probability is still missing. Here we certainly do not provide this integrated view, but possibly we do contribute a bit. The states of a Hilbert space $\mathcal{H}$, described as density matrices $\mathrm{DM}(\mathcal{H})$, fit within the category of convex compact Hausdorff spaces investigated here. Also, the effects $\mathrm{Ef}(\mathcal{H})$ of the space fit in the associated dual category of Banach Hausdorff spaces. The duality we obtain between convex compact Hausdorff spaces and Banach effect algebras precisely captures the translations back and forth between states and effects, as expressed by the isomorphisms:

$$\mathrm{Hom}\Big(\mathrm{Ef}(\mathcal{H}), [0,1]\Big) \cong \mathrm{DM}(H) \qquad \mathrm{Hom}\Big(\mathrm{DM}(\mathcal{H}), [0,1]\Big) \cong \mathrm{Ef}(H).$$

These isomorphisms (implicitly) form the basis for the quantum weakest precondition calculus described in [12].

In this context we shed a bit more light on the relation between quantum logic—as expressed by the projections $\mathrm{Pr}(\mathcal{H})$ on a Hilbert space—and quantum probability—via its effects $\mathrm{Ef}(\mathcal{H})$. In Section 8 it will be shown that Gleason's famous theorem, expressing that states are probability measures, can equivalently be expressed as an isomorphism relating projections and effects:

$$[0,1] \otimes \mathrm{Pr}(\mathcal{H}) \cong \mathrm{Ef}(\mathcal{H}).$$

This means that the effects form the free effect module on projections, via the free functor $[0,1] \otimes (-)$. More loosely formulated: quantum probabilities are freely obtained from quantum predicates.

We briefly describe the organization of the paper. It starts with a quick recap on monads in Section 2, including descriptions of the monads relevant in the rest of the paper. Section 3 gives a brief introduction to effect algebras and effect modules. It also establishes equivalences between (Banach) order unit spaces and (Banach) Archimedean effect modules, and the relevant theorems of Świrszcz and Kadison. In Section 4 we give several descriptions of the expectation monad in terms effect modules, states and measures. We also describe the map between the expectation monad and the continuation monad from (1). Sections 5 and 6 deal with the construction of the other two monad maps from Diagram (1): those from the ultrafilter and distribution monads to the expectation monad. Here we also explore some of the implications of these maps. Using the abstract monadicity theorem of Świrszcz we know that algebras of the expectation monad are convex compact Hausdorff spaces. Section 7 describes these algebras more concretely in terms of barycenters. In Section 8 we turn to quantum logic and prove that the isomorphism $[0,1] \otimes \mathrm{Pr}(\mathcal{H}) \cong \mathrm{Ef}(\mathcal{H})$ is an algebraic reformulation of Gleason's theorem. Finally in Section 9 we summarize our findings in a triangle diagram (17) that closely connects the Kleisli category and the Eilenberg-Moore category of the expection monad to effect modules, relating computations, states and predicates. We further examine how the expectation monad has appeared in earlier work on programming semantics.

2 A recap on monads

This section recalls the basics of the theory of monads, as needed here. For more information, see *e.g.* [34,7,33,9]. Some specific examples will be elaborated later on.

A monad is a functor $T\colon \mathbf{C} \to \mathbf{C}$ together with two natural transformations: a unit $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow T$ and multiplication $\mu\colon T^2 \Rightarrow T$. These are required to make the following diagrams commute, for $X \in \mathbf{C}$.

$$\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, equal] & T^2(X) \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\eta_X)"'] \arrow[dl, equal] \\
& T(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] & T^2(X) \arrow[d, "\mu_X"] \\
T^2 \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}$$

Each adjunction $F \dashv G$ gives rise to a monad GF.

Given a monad T one can form a category $\mathcal{EM}(T)$ of (Eilenberg-Moore) algebras. Objects of this category are maps of the form $a\colon T(X) \to X$, making the first two diagram below commute.

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "T(a)"] \arrow[d, "\mu"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd} \qquad \begin{tikzcd} TX \arrow[r, "T(f)"] \arrow[d, "a"'] & TY \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

A homomorphism of algebras $(X, a) \to (Y, b)$ is a map $f : X \to Y$ in $\mathbf{C}$ between the underlying objects making the diagram above on the right commute. The diagram in the middle thus says that the map a is a homomorphism $\mu \to a$. The forgetful functor $U : \mathcal{EM}(T) \to \mathbf{C}$ has a left adjoint, mapping an object $X \in \mathbf{X}$ to the (free) algebra $\mu_X : T^2(X) \to T(X)$ with carrier $T(X)$.

Each category $\mathcal{EM}(T)$ inherits limits from the category $\mathbf{C}$. In the special case where $\mathbf{C} = \mathbf{Sets}$, the category of sets and functions (our standard universe), the category $\mathcal{EM}(T)$ is not only complete but also cocomplete (see [7, § 9.3, Prop. 4]).

A map of monads $\sigma : T \Rightarrow S$ is a natural transformation that commutes with the units and multiplications, as in:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\eta_X"'] & X \arrow[d, "\eta_X"] \\ T(X) \arrow[r, "\sigma_X"'] & S(X) \end{tikzcd} \qquad \begin{tikzcd} T^2(X) \arrow[r, "\sigma_{TX}"] \arrow[d, "\mu_X"'] & S(T(X)) \arrow[r, "S(\sigma_X)"] & S^2(X) \arrow[d, "\mu_X"] \\ T(X) \arrow[rr, "\sigma_X"'] & & S(X) \end{tikzcd} \tag{2}$$

Lemma 1 *Let $\sigma : T \Rightarrow S$ be a map of monads.*

(1) There is a functor $(-) \circ \sigma : \mathcal{EM}(S) \to \mathcal{EM}(T)$ that commutes with the forgetful functors.

(2) If the category $\mathcal{EM}(S)$ has sufficiently many coequalizers—like when the underlying category is **Sets**—*this functor has a left adjoint $\mathcal{EM}(T) \to \mathcal{EM}(S)$; it maps an algebra $a : T(X) \to X$ to the following coequalizer a_σ in $\mathcal{EM}(S)$.*

$$\begin{pmatrix} S^2(TX) \\ \downarrow \mu \\ S(TX) \end{pmatrix} \underset{S(a)}{\overset{\mu \circ S(\sigma)}{\rightrightarrows}} \begin{pmatrix} S^2(X) \\ \downarrow \mu \\ S(X) \end{pmatrix} \xrightarrow{\;c\;} \begin{pmatrix} S(X_\sigma) \\ \downarrow a_\sigma \\ X_\sigma \end{pmatrix} \qquad \square$$

Proof We need to establish a bijective correspondence between algebra maps:

$$\frac{\begin{pmatrix} S(X_\sigma) \\ \downarrow a_\sigma \\ X_\sigma \end{pmatrix} \xrightarrow{\;f\;} \begin{pmatrix} S(Y) \\ \downarrow b \\ Y \end{pmatrix}}{\begin{pmatrix} T(X) \\ \downarrow a \\ X \end{pmatrix} \xrightarrow[\;g\;]{} \begin{pmatrix} T(Y) \\ \downarrow b \circ \sigma \\ Y \end{pmatrix}}$$

This works as follows. Given f, one takes $\overline{f} = f \circ c \circ \eta \colon X \to Y$. And given g one obtains $\overline{g} \colon X_\sigma \to Y$ because $b \circ T(g) \colon S(X) \to Y$ coequalizes the above parallel pair $\mu \circ S(\sigma)$ and $S(a)$. Remaining details are left to the interested reader. $\square$

It is well-known that adjoints, if they exist, are unique up to natural isomorphism. Here we need a stronger result, namely that there is also an monad isomorphism between the induced monads.

Lemma 2 *Consider a functor $G \colon \mathbf{C} \to \mathbf{D}$ with two left adjoints: $F \dashv G$ and $F' \dashv G$. The induced isomorphism $F \cong F'$ also yields an isomorphism $GF \cong GF'$ of monads on $\mathbf{D}$.*

Proof Let's write η, ε for the unit and counit of the adjunction $F \dashv G$, and similarly η', ε' for $F' \dashv G$. The multiplication maps for the induced monads GF and GF' are then given by $\mu_X = G(\varepsilon_{FX}) \colon GFGF(X) \to GF(X)$ and $\mu'_X = G(\varepsilon'_{F'X})$. There is then a natural isomorphism $\sigma \colon F \Rightarrow F'$ with components:

$$\sigma_X = \left(F(X) \xrightarrow{F(\eta'_X)} FGF'(X) \xrightarrow{\varepsilon_{F'X}} F'(X) \right)$$

Then $G\sigma \colon GF \Rightarrow GF'$ is a isomorphism of monads. By using the triangle identities we get:

$$\begin{aligned}
G\sigma \circ \eta &= G(\varepsilon F') \circ FG(\eta') \circ \eta \\
&= G(\varepsilon F') \circ \eta GF' \circ \eta' \\
&= \eta' \\
\mu' \circ G\sigma GF' \circ GFG\sigma &= G\varepsilon' F \circ G\varepsilon F' GF' \circ GF\eta' GF' \circ GFG\varepsilon F' \circ GFGF\eta' \\
&= G\varepsilon F' \circ GFG\varepsilon' F' \circ GF\eta' GF' \circ GFG\varepsilon F' \circ GFGF\eta' \\
&= G\varepsilon F' \circ GF(G\varepsilon' \circ \eta' G)F' \circ GFG\varepsilon F' \circ GFGF\eta' \\
&= G\varepsilon F' \circ GFG\varepsilon F' \circ GFGF\eta' \\
&= G\varepsilon F' \circ GF\eta' \circ G\varepsilon F \\
&= G\sigma \circ \mu.
\end{aligned}$$

$\square$

2.1 The Distribution monad

We shall write $\mathcal{D}$ for the discrete probability distribution monad on **Sets**. It maps a set X to the set of formal convex combinations $r_1 x_1 + \cdots + r_n x_n$, where $x_i \in X$ and $r_i \in [0, 1]$ with $\sum_i r_1 = 1$. Alternatively,

$$\mathcal{D}(X) \;=\; \{\varphi \colon X \to [0,1] \mid \mathit{supp}(\varphi) \text{ is finite, and } \textstyle\sum_x \varphi(x) = 1\},$$

where $supp(\varphi) \subseteq X$ is the support of φ, containing all x with $\varphi(x) \neq 0$. The functor $\mathcal{D}\colon \mathbf{Sets} \to \mathbf{Sets}$ forms a monad with the Dirac function as unit in:

$$\begin{array}{ccc} X & \xrightarrow{\quad\eta\quad} & \mathcal{D}X \\ x & \longmapsto & 1x = \lambda y.\begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \end{array} \qquad \begin{array}{ccc} \mathcal{D}\mathcal{D}X & \xrightarrow{\quad\mu\quad} & \mathcal{D}X \\ \Psi & \longmapsto & \lambda y.\, \sum_{\varphi \in \mathcal{D}X} \Psi(\varphi) \cdot \varphi(y). \end{array}$$

[Here we use the "lambda" notation from the lambda calculus [6]: the expression $\lambda x.\, \cdots$ is used for the function $x \mapsto \cdots$. We also use the associated application rule $(\lambda x.\, f(x))(y) = f(y)$.]

Objects of the category $\mathcal{EM}(\mathcal{D})$ of (Eilenberg-Moore) algebras of this monad $\mathcal{D}$ can be identified as *convex sets*, in which sums $\sum_i r_i x_i$ of convex combinations exists. Morphisms are called affine functions, preserving such convex sums, see [20]. Hence we also write $\mathcal{EM}(\mathcal{D}) = \mathbf{Conv}$, where $\mathbf{Conv}$ is the category of convex sets and affine functions.

The prime example of a convex set is the unit interval $[0,1] \subseteq \mathbb{R}$ of probabilities. Also, for an arbitrary set X, the set of functions $[0,1]^X$, or fuzzy predicates on X, is a convex set, via pointwise convex sums.

2.2 The ultrafilter monad

A particular monad that plays an important role in this paper is the ultrafilter monad $\mathcal{U}\colon \mathbf{Sets} \to \mathbf{Sets}$, given by:

$$\begin{array}{rcl} \mathcal{U}(X) & = & \{\mathcal{F} \subseteq \mathcal{P}(X) \mid \mathcal{F} \text{ is an ultrafilter}\} \\ & \cong & \{f\colon \mathcal{P}(X) \to \{0,1\} \mid f \text{ is a homomorphism of Boolean algebras}\} \end{array} \tag{3}$$

Such an ultrafilter $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfies, by definition, the following three properties.

- It is an upset: $V \supseteq U \in \mathcal{F} \Rightarrow V \in \mathcal{F}$;
- It is closed under finite intersections: $X \in \mathcal{F}$ and $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$;
- For each set U either $U \in \mathcal{F}$ or $\neg U = \{x \in X \mid x \notin U\} \in \mathcal{F}$, but not both. As a consequence, $\emptyset \notin \mathcal{F}$.

For a function $f\colon X \to Y$ one obtains $\mathcal{U}(f)\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ by:

$$\mathcal{U}(f)(\mathcal{F}) \;=\; \{V \subseteq Y \mid f^{-1}(V) \in \mathcal{F}\}.$$

Taking ultrafilters is a monad, with unit $\eta\colon X \to \mathcal{U}(X)$ given by principle ultrafilters:

$$\eta(x) \;=\; \{U \subseteq X \mid x \in U\}.$$

The multiplication $\mu\colon \mathcal{U}^2(X) \to \mathcal{U}(X)$ is:

$$\mu(\mathcal{A}) \;=\; \{U \subseteq X \mid D(U) \in \mathcal{A}\} \qquad \text{where} \quad D(U) \;=\; \{\mathcal{F} \in \mathcal{U}(X) \mid U \in \mathcal{F}\}.$$

The set $\mathcal{U}(X)$ of ultrafilters on a set X is a topological space with basic (compact) clopens given by subsets $D(U) = \{\mathcal{F} \in \mathcal{U}(X) \mid U \in \mathcal{F}\}$, for $U \subseteq X$. This makes $\mathcal{U}(X)$ into a compact Hausdorff space. The unit $\eta\colon X \to \mathcal{U}(X)$ is a dense embedding.

The following result shows the importance of the ultrafilter monad, see *e.g.* [32], [26, III.2], or [9, Vol. 2, Prop. 4.6.6].

Theorem 3 (Manes) $\mathcal{EM}(\mathcal{U}) \simeq \mathbf{CH}$, *i.e. the category of algebras of the ultrafilter monad is equivalent to the category* $\mathbf{CH}$ *of compact Hausdorff spaces and continuous maps.*

The proof is complicated and will not be reproduced here. We only extract the basic constructions. For a compact Hausdorff space Y one uses denseness of the unit η to define a unique continuous extensions $f^{\#}$ as in:

$$\begin{tikzcd} X \arrow[r, tail, "\eta"] \arrow[dr, "f"'] & \mathcal{U}(X) \arrow[d, dashed, "f^{\#}"] \\ & Y \end{tikzcd} \tag{4}$$

One defines $f^{\#}(\mathcal{F})$ to be the unique element in $\bigcap\{\overline{V} \mid V \subseteq Y \text{ with } f^{-1}(V) \in \mathcal{F}\}$. This intersection is a singleton precisely because Y is a compact Hausdorff space. In such a way one obtains an algebra $\mathcal{U}(Y) \to Y$ as extension of the identity.

Conversely, given an algebra $\mathrm{ch}_X : \mathcal{U}(X) \to X$ one defines $U \subseteq X$ to be closed if for all $\mathcal{F} \in \mathcal{U}(X)$, $U \in \mathcal{F}$ implies $\mathrm{ch}(\mathcal{F}) \in U$. This yields a topology on X which is Hausdorff and compact. There can be at most one such algebra structure $\mathrm{ch}_X : \mathcal{U}(X) \to X$ on a set X corresponding to a compact Hausdorff topology, because of the following standard result.

Lemma 4 *Let X be a set with two topologies $\mathcal{O}_1(X), \mathcal{O}_2(X) \subseteq \mathcal{P}(X)$ with $\mathcal{O}_1(X) \subseteq \mathcal{O}_2(X)$, $\mathcal{O}_1(X)$ is Hausdorff and $\mathcal{O}_2(X)$ is compact, then $\mathcal{O}_1(X) = \mathcal{O}_2(X)$.* $\square$

Proof If U is closed in $\mathcal{O}_2(X)$, then it is compact, and, because $\mathcal{O}_1(X) \subseteq \mathcal{O}_2(X)$, also compact in $\mathcal{O}_1(X)$. Hence it is closed there. $\square$

We can apply this result to the space $\mathcal{U}(X)$ of ultrafilters: as described before Theorem 3, $\mathcal{U}(X)$ carries a compact Hausdorff topology with sets $D(U) = \{\mathcal{F} \in \mathcal{U}(X) \mid U \in \mathcal{F}\}$ as clopens. Also, it carries a compact Hausdorff topology via the (free) algebra $\mu_X : \mathcal{U}^2(X) \to \mathcal{U}(X)$. It is not hard to see that the subsets $D(U)$ are closed in the latter topology, so the two topologies on $\mathcal{U}(X)$ coincide by Lemma 4. Later we shall use a similar argument.

Example 5 *The unit interval* $[0,1] \subseteq \mathbb{R}$ *is a standard example of a compact Hausdorff space. Its Eilenberg-Moore algebra* $\mathrm{ch} : \mathcal{U}([0,1]) \to [0,1]$ *can be described concretely on* $\mathcal{F} \in \mathcal{U}([0,1])$ *as:*

$$\mathrm{ch}(\mathcal{F}) \;=\; \inf\{s \in [0,1] \mid [0,s] \in \mathcal{F}\}. \tag{5}$$

For the proof, recall that $\mathrm{ch}(\mathcal{F})$ *is the sole element of the intersection* $\bigcap\{\overline{V} \mid V \in \mathcal{F}\}$. *Hence if* $[0,s] \in \mathcal{F}$, *then* $\mathrm{ch}(\mathcal{F}) \in \overline{[0,s]} = [0,s]$, *so* $\mathrm{ch}(\mathcal{F}) \leq s$. *This establishes the* $(\leq)$*-part of* (5). *Assume next that* $\mathrm{ch}(\mathcal{F}) < \inf\{s \mid [0,s] \in \mathcal{F}\}$. *Then there is some* $r \in [0,1]$ *with* $\mathrm{ch}(\mathcal{F}) < r < \inf\{s \mid [0,s] \in \mathcal{F}\}$. *Then* $[0,r]$ *is not in* $\mathcal{F}$, *so that* $\neg[0,r] = (r,1] \in \mathcal{F}$. *But this means* $\mathrm{ch}(\mathcal{F}) \in \overline{(r,1)} = [r,1]$, *which is impossible.*

Notice that (5) *can be strengthened to:* $\mathrm{ch}(\mathcal{F}) = \inf\{s \in [0,1] \cap \mathbb{Q} \mid [0,s] \in \mathcal{F}\}$.

The second important result about compact Hausdorff spaces is as follows.

Theorem 6 (Gelfand) $\mathbf{CH} \simeq \mathbf{CCstar}^{\mathrm{op}}$, *i.e. the category* $\mathbf{CH}$ *of compact Hausdorff spaces is equivalent to the opposite of the category* $\mathbf{CCstar}$ *of commutative* C^**-algebras (with *-homomorphisms).* □

Later on we shall see probabilistic analogues of these two basic results (Theorems 3 and 6), involving *convex* compact Hausdorff spaces, see Theorems 24 and 25.

2.3 The continuation monad

The continuation monad is useful in the context of programming semantics, where it is employed for a particular style of evaluation. The monad starts from a fixed set C and takes the "double dual" of a set, where C is used as dualizing object. Hence we first form a functor $\mathcal{C} : \mathbf{Sets} \to \mathbf{Sets}$ by:

$$\mathcal{C}(X) \;=\; C^{(C^X)} \qquad \text{and} \qquad \mathcal{C}\Big(X \xrightarrow{f} Y\Big) \;=\; \lambda h \in C^{(C^X)}.\, \lambda g \in C^Y.\, h(g \circ f).$$

This functor $\mathcal{C}$ forms a monad via:

$$\begin{array}{ccc} X \xrightarrow{\;\eta\;} C^{(C^X)} & \qquad & C^{\left(C^{\left(C^{(C^X)}\right)}\right)} \xrightarrow{\qquad\mu\qquad} C^{(C^X)} \\ x \longmapsto \lambda g \in C^X.\, g(x) & & H \longmapsto \lambda g \in C^X.\, H\Big(\lambda k \in C^{(C^X)}.\, k(g)\Big). \end{array}$$

The following folklore result will be useful in the present context.

Lemma 7 *Let $T\colon \mathbf{Sets} \to \mathbf{Sets}$ be an arbitrary monad and $\mathcal{C}(X) = C^{(C^X)}$ be the continuation monad on a set C. Then there is a bijective correspondence between:*

$$\begin{array}{cl} T(C) \xrightarrow{\;a\;} C & \quad \textit{Eilenberg-Moore algebras} \\ \hline\hline T \underset{\sigma}{\Longrightarrow} \mathcal{C} & \quad \textit{maps of monads.} \end{array}$$

Proof First, given an algebra $a\colon T(C) \to C$ define $\sigma_X\colon T(X) \to C^{(C^X)}$ by:

$$\sigma_X(u)(g) \;=\; a\Big(T(g)(u)\Big).$$

Conversely, given a map of monads $\sigma\colon T \Rightarrow C^{(C^{(-)})}$, define as algebra $a\colon T(C) \to C$,

$$a(u) \;=\; \sigma_C(u)(\mathrm{id}_C). \qquad \square$$

Taking $C = 2 = \{0, 1\}$ to be the two-element set, yields as associated continuation monad $\mathcal{C}(X) = 2^{(2^X)} \cong \mathcal{P}(\mathcal{P}(X))$, the double-powerset monad. For a function $f\colon X \to Y$ we have a map $\mathcal{P}^2(X) \to \mathcal{P}^2(Y)$, by functoriality, given by double inverse image: $U \subseteq \mathcal{P}(X) \longmapsto (f^{-1})^{-1}(U) = \{V \subseteq Y \mid f^{-1}(V) \in U\}$. It is not hard to see that the inclusion maps:

$$\mathcal{U}(X) \underset{\cong}{\xrightarrow{(3)}} \mathbf{BA}(2^X, 2) \hookrightarrow 2^{(2^X)}$$

form a map of monads, from the ultrafilter monad to the continuation monad (with constant $C = 2$).

3 Effect modules

This section introduces the essentials of effect modules and refers to [20,25] for further details. Intuitively, effect modules are vector spaces, not with the real or complex numbers as scalars, but with scalars from the unit interval $[0, 1] \subseteq \mathbb{R}$. Also, the addition operation $+$ on vectors is only partial; it is written as $\ovee$. These effect modules occur in [37] under the name ‘convex effect algebras’.

More precisely, an effect module is an *effect algebra* E with an action $[0,1] \otimes E \to E$ for scalar multiplication. An effect algebra E carries both:

- a partial commutative monoid structure $(0, \ovee)$; this means that $\ovee$ is a partial operation $E \times E \to E$ which is both commutative and associative, taking suitably account of partiality, with 0 as neutral element;
- an orthocomplement $(-)^{\perp} \colon E \to E$. One writes $x \perp y$ if the sum $x \ovee y$ is defined; $x^{\perp}$ is then the unique element with $x \ovee x^{\perp} = 1$, where $1 = 0^{\perp}$; further $x \perp 1$ holds only for $x = 0$.

These effect algebras carry a partial order given by $x \leq y$ iff $x \ovee z = y$, for some element z. Then $x \perp y$ iff $x \leq y^{\perp}$ iff $y \leq x^{\perp}$. The unit interval $[0,1]$ is the prime example of an effect algebra with partial sum $r \ovee s = r + s$ if $r + s \leq 1$; then $r^{\perp} = 1 - r$.

A homomorphism $f \colon E \to D$ of effect algebras satisfies $f(1) = 1$ and: if $x \perp x'$ in E, then $f(x) \perp f(x')$ in D and $f(x \ovee x') = f(x) \ovee f(x')$. It is easy to deduce that $f(x^{\perp}) = f(x)^{\perp}$ and $f(0) = 0$. This yields a category, written as **EA**. It carries a symmetric monoidal structure $\otimes$ with the 2-element effect algebra $\{0, 1\}$ as tensor unit (which is at the same time the initial object), see [25]. The usual multiplication of real numbers (probabilities in this case) yields a monoid structure on $[0,1]$ in the category **EA**. An *effect module* is then an effect algebra with an $[0,1]$-action $[0,1] \otimes E \to E$. Explicitly, it can be described as a scalar multiplication $(r, x) \mapsto rx$ satisfying:

$$\begin{array}{rclcrcll} 1x & = & x & \qquad & (r+s)x & = & rx + sx & \quad \text{if } r+s \leq 1 \\ (rs)x & = & r(sx) & & r(x \ovee y) & = & rx \ovee ry & \quad \text{if } x \perp y. \end{array}$$

In particular, if $r + s \leq 1$, then a sum $rx \ovee sy$ always exists (see [37]).

Example 8 *The unit interval* $[0,1]$ *is again the prime example, this time for effect modules. But also, for an arbitrary set* X*, the set* $[0,1]^X$ *of all functions* $X \to [0,1]$ *is an effect module, with structure inherited pointwise from* $[0,1]$*. Another example, occurring in integration theory, is the set* $[X \to_s [0,1]]$ *of simple functions* $X \to [0,1]$*, having only finitely many output values (also known as 'step functions').*

A morphism $E \to D$ in the category **EMod** of such effect modules is a function $f \colon E \to D$ between the underlying sets satisfying:

$$f(rx) \;=\; rf(x) \qquad f(1) \;=\; 1 \qquad f(x \ovee y) \;=\; f(x) \ovee f(y) \quad \text{if } x \perp y.$$

We now come to the dual adjunction mentioned in the previous section (see [25] for more information).

Proposition 9 *For each effect module E the homset* $\mathbf{EMod}(E, [0,1])$ *is a convex set. In the other direction, each convex set X gives rise to an effect module* $\mathbf{Conv}(X, [0,1])$*. This gives the adjunction as below, with $[0,1]$ as dualizing object.*

$$\begin{tikzcd} \mathbf{EMod}^{\mathrm{op}} \arrow[r, bend right, "\mathbf{EMod}(-,[0,1])"'] \arrow[r, phantom, "\perp"] & \mathbf{Conv} = \mathcal{EM}(\mathcal{D}) \arrow[l, bend right, "\mathbf{Conv}(-,[0,1])"'] \end{tikzcd} \tag{6}$$

Proof The effect algebra structure on the set $\mathbf{Conv}(X, [0,1])$ of affine maps to $[0,1]$ is obtained pointwise: $f \ovee g$ is defined if $f(x)+g(x) \le 1$ for all $x \in X$, and in that case $f \ovee g$ at $x \in X$ is $f(x)+g(x)$. The orthocomplement is also obtained pointwise: $(f^{\perp})(x) = 1 - f(x)$. Scalar multiplication is done similarly $(rf)(x) = r(f(x))$. In the reverse direction, each effect module E gives rise to a convex set $\mathbf{EMod}(E, [0,1])$ of homomorphisms, with pointwise convex sums. The adjunction $\mathbf{Conv}(-, [0,1]) \dashv \mathbf{EMod}(-, [0,1])$ arises in the standard way, with unit and counit given by evaluation. $\square$

3.1 Totalization

In this section we prove that the category of effect modules is equivalent to the category of certain ordered vector spaces over the reals. For this we extend a result for effect algebras from [25]. We recall the basics below but for details and proofs we refer to that paper. The idea is that the partial operation $\ovee$ of effect algebras and effect modules is rather difficult to work with; therefore we develop an embedding into structures with total operations.

The first result we need is the following one from [25].

Proposition 10 *There is a coreflection*

$$\begin{tikzcd} \mathbf{EA} \arrow[r, bend left, "\mathcal{T}"] \arrow[r, phantom, "\perp"] & \mathbf{BCM} \arrow[l, bend left, "{[0,u]_{(-)}}"] \end{tikzcd} \tag{7}$$

where $\mathbf{BCM}$ *is the category of "barred commutative monoids": its objects are pairs (M, u), where M is a commutative monoid and $u \in M$ is a unit such that $x + y = 0$ implies $x = y = 0$ and $x + y = x + z = u$ implies $y = z$. The morphisms in* $\mathbf{BCM}$ *are monoid homomorphisms that preserve the unit. As this is a coreflection every effect algebra E is isomorphic to $[0,u]_{\mathcal{T}(E)}$.* $\square$

The partialization functor $[0,u]_{(-)}$ in (7) is defined by by the 'unit interval':

$$[0,u]_M \;=\; \{x \in M \mid x \preceq u\},$$

where $x \preceq y$ iff there exists a z such that $x + z = y$. The operation $\ovee$ is defined by $x \ovee y = x + y$ but this is only defined if $x + y \preceq u$, *i.e.* $x + y \in [0, u]_M$.

The totalization functor $\mathcal{T}$ in (7) is defined as:

$$\mathcal{T}(E) \;=\; (\,\mathcal{M}(E)/\!\sim,\; 1 \cdot 1_E\,),$$

where $\mathcal{M}(E)$ is the free commutative monoid on E, consisting of all finite formal sums $n_1 \cdot x_1 + \cdots + n_m \cdot x_m$, with $n_i \in \mathbb{N}$ and $x_i \in E$. Here we identify sums such as $1 \cdot x + 2 \cdot x$ with $3 \cdot x$. And $\sim$ is the smallest monoid congruence such that $1 \cdot x + 1 \cdot y \sim 1 \cdot (x \ovee y)$ whenever $x \ovee y$ is defined.

Example 11 *Totalization of the truth values* $\{0, 1\} \in$ **EA** *and of the probabilities* $[0, 1] \in$ **EA** *yields the natural numbers and the non-negative reals:*

$$\mathcal{T}(\{0,1\}) \;\cong\; \mathbb{N} \qquad \textit{and} \qquad \mathcal{T}([0,1]) \;\cong\; \mathbb{R}_{\geq 0} \;=\; \{x \in \mathbb{R} \mid x \geq 0\}.$$

Recall that an effect module E is just an effect algebra together with a scalar product $[0, 1] \otimes E \to E$. Now it turns out that $\mathcal{T}$ is a strong monoidal functor, and as a result $\mathcal{T}(E) \in$ **BCM** comes equipped with a scalar product $\mathbb{R}_{\geq 0} \otimes \mathcal{T}(E) \to \mathcal{T}(E)$. This gives the monoid $\mathcal{T}(E)$ the structure of a positive cone of some partially ordered vector space. To make this exact we give the following definition.

Construct a category **Coneu** as follows: its objects are pairs (M, u) where M is a commutative monoid equipped with a scalar product $\bullet : \mathbb{R}_{\geq 0} \times M \to M$ and $u \in M$ such that the following axioms hold.

$$\begin{array}{rclcrcl}
1 \bullet x & = & x & \qquad & (r + s) \bullet x & = & r \bullet x + s \bullet \\
(rs) \bullet x & = & r \bullet (s \bullet x) & & r \bullet (x + y) & = & r \bullet x + r \bullet y \\
x + y & = & 0 \;\text{ implies }\; x = y = 0 & & x + y \;=\; x + z & = & u \;\text{ implies }\; y = z,
\end{array}$$

and for all $x \in M$ there exists an $n \in \mathbb{N}$ such that $x \preceq n \bullet u$. Because of this last property we call u a *strong* unit. The morphisms of **Coneu** are monoid homomorphisms that respect both the scalar multiplication and the unit.

We can then extend the coreflection $\mathcal{T} \dashv [0, u]_{(-)}$ to the categories **EMod** and **Coneu**. This will actually be an equivalence of categories. To prove this we first need an auxiliary result.

Lemma 12 *If* $M \in$ **Coneu** *then the cancelation law holds in* M*.*

Proof Let $x, y, z \in M$ and suppose $x + y = x + z$. Since u is a strong unit we can find an n such that $x + y \preceq nu$. Therefore

$$\tfrac{1}{n} \bullet x + \tfrac{1}{n} \bullet y \;=\; \tfrac{1}{n} \bullet x + \tfrac{1}{n} \bullet z \;\preceq\; u.$$

Hence we can find an element $w \in M$ such that $\frac{1}{n} \bullet x + \frac{1}{n} \bullet y + w = \frac{1}{n} \bullet x + \frac{1}{n} \bullet z + w = u$. Then $\frac{1}{n} \bullet y = \frac{1}{n} \bullet z$. And thus $y = \sum_{i=1}^{n} \frac{1}{n} \bullet y = \sum_{i=1}^{n} \frac{1}{n} \bullet z = z$. $\square$

An immediate consequence is that the preorder $\preceq$ is a partial order; thus we shall write $\leq$ instead of $\preceq$ from now on.

Lemma 13 *The coreflection $\mathcal{T} \dashv [0, u]_{(-)}$ between* **EMod** *and* **Coneu** *is an equivalence of categories.*

Proof We only need to show that the counit of the adjunction $\mathcal{T} \dashv [0, u]_{(-)}$ is an isomorphism. So let $M \in$ **Coneu**; a typical element of $\mathcal{T}([0, u]_M)$ is an equivalence class of formal sums like $\sum n_i x_i$ where $n_i \in \mathbb{N}$ and $M \ni x_i \leq u$. The counit ε sends the class represented by this formal sum to its interpretation as an actual sum in M.

To show that ε is surjecive suppose $x \in M$. We can find a natural number n such that $x \leq nu$ so that $\frac{1}{n} \bullet x \leq u$. This gives us:

$$x \;=\; n \cdot (\tfrac{1}{n} \bullet x) \;=\; \varepsilon(n(\tfrac{1}{n} \bullet x)).$$

To prove injectivity suppose that $\varepsilon(\sum n_i x_i) = \varepsilon(\sum k_j y_j)$. Define $N = \sum n_i + \sum k_j$, so that:

$$\begin{aligned} \textstyle\sum n_i \cdot (\frac{1}{N} \bullet x_i) \;=\; \varepsilon(\sum n_i(\frac{1}{N} \bullet x_i)) \;&=\; \varepsilon(\tfrac{1}{N} \bullet (\textstyle\sum n_i x_i)) \\ &=\; \tfrac{1}{N} \bullet \varepsilon(\textstyle\sum n_i x_i) \\ &=\; \tfrac{1}{N} \bullet \varepsilon(\textstyle\sum k_j y_j) \;=\; \textstyle\sum k_j(\frac{1}{N} \bullet y_j). \end{aligned}$$

Because N is sufficiently large, the terms $\bigovoid_i n_i \cdot (\frac{1}{N} \bullet x_i)$ and $\bigovoid_j k_j \cdot (\frac{1}{N} \bullet y_j)$ are both defined in $[0, u]_M$ and by the previous calculation they are equal. This means that $\sum n_i(\frac{1}{N} \bullet x_i)$ and $\sum k_j(\frac{1}{N} \bullet y_k)$ represent equal elements of $\mathcal{T}([0, u]_M)$ and therefore the equation

$$\textstyle\sum n_i x_i \;=\; N \bullet (\sum n_i(\frac{1}{N} \bullet x_i)) \;=\; N \bullet (\sum k_j(\frac{1}{N} \bullet y_j)) \;=\; \sum k_j y_j.$$

holds in $\mathcal{T}([0, u]_M)$. $\square$

From positive cones it is but a small step to partially ordered vector spaces. Define a category **poVectu** as follows; the objects are partially ordered vector spaces over $\mathbb{R}$ with a strong order unit u, *i.e.* a positive element $u \in V$ such that for any $x \in V$ there is a natural number n with $x \leq nu$. The morphisms in **poVectu** are linear maps that preserve both the order and the unit.

Theorem 14 *The category* **EMod** *is equivalent to* **poVectu**.

Proof We will prove that **Coneu** is equivalent to **poVectu**; the result then follows from Lemma 13.

The functor $F\colon \mathbf{poVectu} \to \mathbf{Coneu}$ takes the positive cone of a partially ordered vector space. The contstruction of $G\colon \mathbf{Coneu} \to \mathbf{poVectu}$ is essentially just the usual construction of turning a cancellative monoid into a group.

In somewhat more detail: if $M \in \mathbf{Coneu}$ then define $G(M) = (M \times M)/\sim$ where $\sim$ is defined by $(x, y) \sim (x', y')$ iff $x + y' = y + x'$. We write $[x, y]$ for the equivalence class of $(x, y) \in M \times M$. Addition is defined by $[x, y] + [x', y'] = [x + x', y + y']$. If $\alpha \in \mathbb{R}$ we define $\alpha \bullet [x, y]$ as follows. If $\alpha \geq 0$ then $\alpha[x, y] = [\alpha \bullet x, \alpha \bullet y]$ and if $\alpha < 0$ then $\alpha[x, y] = [-\alpha \bullet y, -\alpha \bullet x]$. It's easy to check that $G(M)$ is indeed a vector space. Moreover, $G(M)$ is partially ordered by $[x, y] \leq [x', y']$ iff $x + y' \leq y + x'$, and $[u, 0]$ is its strong unit.

Both constructions can be made functorial and give an equivalence of categories. $\square$

We write $[0, u]_{(-)}\colon \mathbf{poVectu} \xrightarrow{\simeq} \mathbf{EMod}$ for this equivalence. For a partially ordered vector space V with a strong unit u the 'unit interval' effect module $[0, u]_V$ consists of all elements x such that $0 \leq x \leq u$. With this equivalence of categories in hands we can apply techniques from linear algebra to effect modules. Below we translate some properties of partially ordered vector spaces to the language of effect modules. We need these results later on.

If $V \in \mathbf{poVectu}$ and the unit u is Archimedean—in the sense that $x \leq ru$ for all $r > 0$ implies $x \leq 0$—then V is called an *order unit space*. The Archimedean property of the unit can be used to define a norm $\|x\| = \inf\{r \in \mathbb{R}_{\geq 0} \mid -ru \leq x \leq ru\}$. We denote by $\mathbf{OUS}$ the full subcategory of $\mathbf{poVectu}$ consisting of all order unit spaces. The full subcategory $\mathbf{BOUS} \hookrightarrow \mathbf{OUS}$ contains the "Banach" order unit spaces which are a complete metric (with distance defined via the norm $\| - \|$).

This Archimedean property can also be expressed on the effect module level but some caution is required as effect modules contain no elements less than 0 and sums might not be defined. The following formulation works: an effect module is said to be Archimedean if $x \leq y$ follows from $\frac{1}{2}x \leq \frac{1}{2}y \ovee \frac{r}{2}1$ for all $r \in (0, 1]$. Archimedean effect modules form a full subcategory $\mathbf{AEMod} \hookrightarrow \mathbf{EMod}$. Of course with this definition comes a theorem.

Proposition 15 *The equivalence* $[0, u]_{(-)}\colon \mathbf{poVectu} \xrightarrow{\simeq} \mathbf{EMod}$*, between partially ordered vector spaces with a strong unit and effect modules, restricts to an equivalence* $[0, u]_{(-)}\colon \mathbf{OUS} \xrightarrow{\simeq} \mathbf{AEMod}$*, between order unit spaces and Archimedean effect modules.*

Proof We only check that if $E \in \mathbf{AEMod}$ then its totalization satisfies $\mathcal{T}(E) \in \mathbf{OUS}$; the rest is left to the reader. Suppose $E \in \mathbf{AEMod}$ and $x \in \mathcal{T}(E)$ is such that $x \leq ru$ for all $r \in (0, 1]$. The trick is to transform x into an element in the unit interval $[0, u] \cong E$. Since u is a strong unit we can find a natural number n such that $x + nu \geq 0$, and again using the fact that u is a strong unit we can find a positive real number $s < 1$ such that $sx + nsu \leq u$. Hence $sx + nsu \in [0, u] \cong E$. Now, for $r \in (0, 1]$ we have $sx \leq x \leq ru$ and so $\frac{s}{2}x + \frac{ns}{2}u \leq \frac{ns}{2}u + \frac{r}{2}u$. Thus, by the Archimedean property of E, we get $sx + nsu \leq nsu$. Hence $sx \leq 0$ and therefore $x \leq 0$. $\square$

Since $E \in \mathbf{AEMod}$ is isomorphic to the unit interval of its totalization $\mathcal{T}(E)$, E inherits a metric from the normed space $\mathcal{T}(E)$. This metric can be described wholly in terms of E. However partiality of the sum $\ovee$ does force us into a somewhat awkward definition: for $x, y \in E$ their distance $d(x, y) \in [0, 1]$ can be defined as:

$$d(x,y) \;=\; \max\Big(\inf\{r \in (0,1] \mid \tfrac{1}{2}x \leq \tfrac{1}{2}y \ovee \tfrac{r}{2}1\}, \\ \inf\{r \in (0,1] \mid \tfrac{1}{2}y \leq \tfrac{1}{2}x \ovee \tfrac{r}{2}1\} \Big). \tag{8}$$

A trivial consequence is the following lemma.

Lemma 16 *A map of effect modules $f \colon M \to M'$ between Archimedean effect modules M, M' is automatically non-expansive: $d'(f(x), f(y)) \leq d(x, y)$, for all $x, y \in M$.* $\square$

Of particular interest later in this paper are Archimedean effect modules that are complete in their metric. We call these *Banach effect modules* and denote by **BEMod** the full subcategory of all Banach effect modules. The previous lemma implies that each map in **BEMod** is automatically continuous.

Since an order unit space is complete in its metric if and only if its unit interval is complete we get the following result.

Theorem 17 *The equivalences from Proposition 15 restrict further to an equivalence between Banach effect modules and Banach order unit spaces:*

$$\begin{tikzcd}
\mathbf{BOUS} \arrow[r, "\simeq"] \arrow[d, hook] & \mathbf{BEMod} \arrow[d, hook] \\
\mathbf{OUS} \arrow[r, "\simeq"] \arrow[d, hook] & \mathbf{AEMod} \arrow[d, hook] \\
\mathbf{poVectu} \arrow[r, "\simeq", "V \mapsto [0,u]_V"'] & \mathbf{EMod}
\end{tikzcd}$$

Proof Like in the proof of Proposition 15 one transforms a Cauchy sequence in $\mathcal{T}(E)$ into a sequence in $[0, u] \cong E$. $\square$

Example 18 *We review Example 8: both the effect modules $[0, 1]$ and $[0, 1]^X$ are Archimedean, and also Banach effect modules. Norms and distances in $[0, 1]$ are the usual ones, but limits in $[0, 1]^X$ are defined via the supremum (or* *uniform) norm: for $p \in [0,1]^X$, we have:*

$$\begin{aligned} \|p\| &= \inf\{r \in [0,1] \mid p \leq r \cdot u\} \quad \textit{where } u \textit{ is the constant function } \lambda x.\, 1 \\ &= \inf\{r \in [0,1] \mid \forall x \in X.\, p(x) \leq r\} \\ &= \sup\{p(x) \mid x \in X\} \\ &= \|p\|_\infty. \end{aligned}$$

The latter notation $\|p\|_\infty$ is common for this supremum norm. The associated metric on $[0,1]^X$ is according to (8):

$$\begin{aligned} d(p,q) &= \max\Big(\inf\{r \in (0,1] \mid \forall x \in X.\, \tfrac{1}{2}p(x) \leq \tfrac{1}{2}q(x) + \tfrac{r}{2}\}, \\ &\qquad\qquad \inf\{r \in (0,1] \mid \forall x \in X.\, \tfrac{1}{2}q(x) \leq \tfrac{1}{2}p(x) + \tfrac{r}{2}\} \Big). \\ &= \max\Big(\sup\{p(x) - q(x) \mid x \in X \textit{ with } p(x) \geq q(x)\}, \\ &\qquad\qquad \sup\{q(x) - p(x) \mid x \in X \textit{ with } p(x) \leq q(x)\} \Big) \\ &= \sup\{|p(x) - q(x)| \mid x \in X\} \\ &= \|p - q\|_\infty. \end{aligned}$$

Recall that the subset $[X \to_s [0,1]] \subseteq [0,1]^X$ of simple functions contains those $p \in [0,1]^X$ that take only finitely many values, i.e. for which the set $\{p(x) \mid x \in X\}$ is finite. If we write $\{p(x) \mid x \in X\} = \{r_1, \ldots, r_n\} \subseteq [0,1]$, then we obtain n disjoint non-empty sets $X_i = \{x \in X \mid p(x) = r_i\}$ covering X. For a subset $U \subseteq X$, let $\mathbf{1}_U \colon X \to [0,1]$ be the corresponding "characteristic" simple function, with $\mathbf{1}_U(x) = 1$ iff $x \in U$ and $\mathbf{1}_U(x) = 0$ iff $x \notin U$. Hence we can write such a simple function p in a normal form in the effect module $[X \to_s [0,1]]$ of simple functions, namely as finite sum of characteristic functions:

$$p = \bigovoid_i r_i \cdot \mathbf{1}_{X_i}. \tag{9}$$

Hence $\|p\| = \max\{r_1, \ldots, r_n\}$. These simple functions do not form a Banach (i.e. complete) effect module, since simple functions are not closed under countable suprema.

Lemma 19 *The inclusion of simple functions on a set X is dense in the Banach effect module of all fuzzy predicates on X:*

$$[X \to_s [0,1]] \rightarrowtail^{\textit{dense}} [0,1]^X$$

Explicitly, each predicate $p \in [0,1]^X$ can be written as limit $p = \lim\limits_{n\to\infty} p_n$ of simple functions $p_n \in [0,1]^X$ with $p_n \leq p$.

Proof Define for instance:

$$p_n(x) = 0.d_1 d_2 \cdots d_n \qquad \text{where} \qquad d_i = \text{the } i\text{-th decimal of } p(x) \in [0,1].$$

Clearly, p_n is simple, because it can take at most 10^n different values, since $d_i \in \{0, 1, \ldots, 9\}$. Also, by construction, $p_n \leq p$. For each $\epsilon > 0$, take $N \in \mathbb{N}$ such that for all decimals d_i we have:

$$0.\underbrace{00\cdots00}_{N \text{ times}} d_1 d_2 d_3 \cdots \; < \; \epsilon.$$

Then for each $n \geq N$ we have $p(x) - p_n(x) < \epsilon$, for all $x \in X$, and thus $d(p, p_n) \leq \epsilon$. $\square$

We conclude with some basic observations about Banach order unit spaces. For such a space V we write $\mathit{Stat}(V) = \mathbf{BOUS}(V, \mathbb{R})$ of linear, monontone, unit-preserving maps $V \to \mathbb{R}$. Such maps are also called *states*. This mapping $V \mapsto \mathit{Stat}(V)$ yields a functor $\mathit{Stat}\colon \mathbf{BOUS}^{\mathrm{op}} \to \mathbf{Sets}$.

There is also a functor in the other direction: given a set X, we write $\ell^\infty(X)$ for the set of functions $\phi\colon X \to \mathbb{R}$ which are bounded: there is an $N \in \mathbb{N}$ with $|\phi(x)| \leq N$ for all $x \in X$. These functions form an ordered vector space, via pointwise operations and order. The function $u\colon X \to \mathbb{R}$ with $u(x) = 1$ is a strong unit that is Archimedean. The induced norm is the *uniform* or *supremum* norm $\|\phi\|_\infty = \sup\{|\phi(x)| \mid x \in X\}$. It is not hard to see that $\ell^\infty(X)$ is complete in this norm, and thus a Banach order unit space.

If X is a compact Hausdorff space we write $C(X)$ for the set of continuous functions $X \to \mathbb{R}$. Such functions are automatically bounded. They form a Banach order unit space, like before.

Proposition 20 *There are adjunctions*

$$\begin{tikzcd}
\mathbf{BOUS}^{\mathrm{op}} \arrow[d, bend left, "\mathit{Stat}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Sets} \arrow[u, bend left, "\ell^\infty"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{BOUS}^{\mathrm{op}} \arrow[d, bend left, "\mathit{Stat}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{CH} \arrow[u, bend left, "C"]
\end{tikzcd}$$

Via a combination with the adjunction $\mathbf{CH} \rightleftarrows \mathbf{Sets}$ *from Theorem 3 we get a natural isomorphism* $C \circ \mathcal{U} \cong \ell^\infty$ *in:*

$$\begin{tikzcd}
\mathbf{BOUS}^{\mathrm{op}} \arrow[d, bend left, "\mathit{Stat}"] \arrow[d, phantom, "\dashv"] \arrow[dd, bend left=90, "\mathit{Stat}"] \\
\mathbf{CH} \arrow[u, bend left, "C"] \arrow[d, bend left, "\mathit{forget}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Sets} \arrow[u, bend left, "\mathcal{U}"] \arrow[uu, bend left=90, "\ell^\infty"]
\end{tikzcd}$$

where $\mathcal{U}$ *is the ultrafilter functor, and* $\mathcal{E}$ *is the expectation functor.*

Proof The adjunctions are based on "swapping arguments", as in the bijective correspondence, for $X \in \mathbf{Sets}$ and $V \in \mathbf{BOUS}$,

$$\frac{V \xrightarrow{f} \ell^\infty(X) \qquad \text{in } \mathbf{BOUS}}{X \xrightarrow[g]{} Stat(V) \qquad \text{in } \mathbf{Sets}}$$

given by $\overline{f}(x)(v) = f(v)(x)$ and $\overline{g}(v)(x) = g(x)(v)$.

This works in the same way for the adjunction $C \dashv Stat$ between Banach order unit spaces and compact Hausdorff spaces. Here we need that the set of states $Stat(V)$ carries a compact Hausdorff topology, via the Banach-Alaoglu theorem. This is the weak *-topology, with subbasic opens $\{\omega \in Stat(V) \mid \omega(v) \in U\}$ for $v \in V$ and $U \subseteq \mathbb{R}$ open.

The natural isomorphism $C \circ \mathcal{U} \cong \ell^\infty$ follows from two basic facts: (a) adjunctions are closed under composition, and (b) adjoints are unique up-to-isomorphism, see also Lemma 2. $\square$

3.2 Convex compact Hausdorff spaces and Kadison duality

So far we have seen the categories $\mathbf{CH}$ of compact Hausdorff spaces and $\mathbf{Conv} = \mathcal{EM}(\mathcal{D})$ of convex sets. We now wish to consider both convex and compact Hausdorff spaces. An example where this combination occurs is the unit interval $[0, 1]$. But also state spaces $Stat(V)$ of (Banach) order unit spaces V are convex compact Hausdorff. This state space construction is part of the 'Kadison' duality that we describe below.

We describe two ways of describing convex compact Hausdorff and show them to be equivalent. The first way uses a separation property.

Definition 21 *We write $\mathbf{CCH}_{\mathrm{sep}}$ for the category whose objects X are both convex sets and compact Hausdorff spaces, and satisfy the following separation condition: for $x, x' \in X$ with $x \neq x'$ there is an affine continuous function $q\colon X \to [0, 1]$ with $q(x) \neq q(x')$. Maps in the category $\mathbf{CCH}_{\mathrm{sep}}$ are functions which are both affine and continuous.*

Often we write $AC(X, Y)$ for the set of affine and continuous maps between convex spaces X, Y — recall that affine means preserving convex combinations. By definition, $[0, 1]$ is a cogenerator in the category $\mathbf{CCH}_{\mathrm{sep}}$. Also, by definition, there are forgetful functors $\mathbf{Conv} \leftarrow \mathbf{CCH}_{\mathrm{sep}} \rightarrow \mathbf{CH}$.

Definition 22 *We also have a category $\mathbf{CCLcvx}$ whose objects are pairs (X, E), where E is a (Hausdorff) locally convex space, and $X \rightarrowtail E$ is a compact convex subset of E. A morphism $(X, E) \to (Y, F)$ is a continuous affine map $X \to Y$, with no condition on E and F.*

We shortly show that the categories $\mathbf{CCH}_{\mathrm{sep}}$ and $\mathbf{CCLcvx}$ are equivalent. This is a bit surprising since they are formulated differently: separation in $\mathbf{CCH}_{\mathrm{sep}}$ is clearly a property, whereas an embedding in a locally convex space looks like structure.

Proposition 23 *There is an equivalence of categories* $\mathbf{CCLcvx} \simeq \mathbf{CCH}_{\mathrm{sep}}$.

Proof If $(X, E) \in \mathbf{CCLcvx}$, where the subspace $X \rightarrowtail E$ is convex compact, then points in X are separated Hahn-Banach, see [16, Cor 5.10 (iv)]. Hence the mapping $(X, E) \mapsto X$ yields a full and faithful functor $\mathbf{CCLcvx} \to \mathbf{CCH}_{\mathrm{sep}}$. We show that it is essentially surjective on objects. For $X \in \mathbf{CCH}_{\mathrm{sep}}$ consider the function:

$$X \xrightarrow{\xi} \mathcal{A}C(X,\mathbb{R})^* = \mathit{Lin}\Big(\mathcal{A}C(X,\mathbb{R}),\mathbb{R}\Big) \qquad \text{with} \qquad \xi(x)(\phi) \;=\; \phi(x).$$

where the algebraic dual $\mathcal{A}C(X,\mathbb{R})^*$ is equipped with the weak *-topology, which is locally convex. This topology generated by subbasic opens $\{\omega \mid \omega(\phi) \in U\}$, for $\phi \colon X \to \mathbb{R}$ affine continuous and $U \subseteq \mathbb{R}$ open. This map ξ is continuous, affine, and injective.

- It is continuous, since for a subbasic open $W = \{\omega \mid \omega(\phi) \in U\}$ the inverse image is open:

$$\xi^{-1}(W) \;=\; \{x \mid \xi(x)(\phi) \in U\} \;=\; \{x \mid \phi(x) \in U\} \;=\; \phi^{-1}(U).$$

- This ξ is affine: for a convex sum $\sum_i r_i x_i$ in X,

$$\xi\Big(\textstyle\sum_i r_i x_i\Big)(\phi) \;=\; \phi(\sum_i r_i x_i) \;=\; \sum_i r_i \phi(x_i) \;=\; \sum_i r_i \xi(x_i)(\phi).$$

  It is also injective: if $x \neq x'$ in X, then there is, by separation, an affine conintuous $q \colon X \to [0,1]$ with $q(x) \neq q(x')$. But then $q \in \mathcal{A}C(X,\mathbb{R})$ and $\xi(x)(q) = q(x) \neq q(x') = \xi(x')(q)$, so that $\xi(x) \neq \xi(x')$.

Thus, X with $\mathcal{A}C(X,\mathbb{R})^*$ is in $\mathbf{CCLcvx}$. $\square$

For the category $\mathbf{CH}$ we have seen a monadicity and a duality result in Subsection 2.2. There are similar results for the categories $\mathbf{CCH}_{\mathrm{sep}} \simeq \mathbf{CCLcvx}$, due to Świrszcz and to Kadison. We briefly discuss these results.

Theorem 24 (Świrszcz) *The equivalent categories* $\mathbf{CCLcvx} \simeq \mathbf{CCH}_{\mathrm{sep}}$ *of convex compact Hausdorff spaces are monadic over* $\mathbf{Sets}$*, where the left adjoint to the forgetful functor* $\mathbf{CCH}_{\mathrm{sep}} \to \mathbf{Sets}$ *is the following composite.*

$$\mathcal{S} = \Big(\mathbf{Sets} \xrightarrow{\mathcal{U}} \mathbf{CH} \xrightarrow{C} \mathbf{BOUS}^{\mathrm{op}} \xrightarrow{\mathit{Stat}} \mathbf{CCH}_{\mathrm{sep}}\Big). \qquad \square$$

The proof in [40] uses Linton's monadicity theorem. A more elementary proof (of monadicity over $\mathbf{CH}$) can be found in [39].

Theorem 25 (Kadison) *There is an equivalence of categories between Banach order unit spaces and convex compact Hausdorff spaces:*

$$\begin{tikzcd} \mathbf{BOUS}^{\mathrm{op}} \arrow[r, bend right, "Stat"'] \arrow[r, phantom, "\simeq"] & \mathbf{CCH}_{\mathrm{sep}} \arrow[l, bend right, "AC(-,\mathbb{R})"'] \end{tikzcd} \qquad \square$$

This result goes back to [28], see also [3, Thm. II.1.8]; details are scattered around in the literature. Unfortunately, there is no modern (categorical) version of the proof[1].

With these two results in place we can relate some of the functors that play a role in this paper.

Corollary 26 *There are a natural isomorphism and an adjunction*

$$Stat \circ \ell^{\infty} \;\cong\; \mathcal{S} \qquad\qquad Stat \circ C \dashv U$$

where $\mathcal{S}\colon \mathbf{Sets} \to \mathbf{CCH}_{\mathrm{sep}}$ *is as in Theorem 24, and* $U \circ Stat \circ C\colon \mathbf{CH} \to \mathbf{CH}$ *is the Radon monad* $\mathcal{R}$ *on* $\mathbf{CH}$ *from [15], in a situation:*

$$\begin{tikzcd} \mathbf{CCH}_{\mathrm{sep}} \arrow[d, bend left, "AC(-,\mathbb{R})"] \arrow[dd, bend left=60, "U"] \arrow[ddd, bend left=80, "V"] \\ \mathbf{BOUS}^{\mathrm{op}} \arrow[u, bend left, "Stat"] \arrow[u, phantom, "\simeq"] \arrow[d, bend left, "UStat"] \\ \mathbf{CH} \arrow[u, bend left, "C"] \arrow[u, phantom, "\dashv"] \arrow[d, bend left] \\ \mathbf{Sets} \arrow[u, bend left, "\mathcal{U}"] \arrow[u, phantom, "\dashv"] \arrow[uu, bend left=50, "\ell^{\infty}"] \arrow[uuu, bend left=80, "\mathcal{S}"] \end{tikzcd} \tag{10}$$

Proof Again we use that adjoints compose and are unique up-to-isomorphism. Thus we get $Stat \circ \ell^{\infty} \dashv V$ and since $\mathcal{S} \dashv V$ by Theorem 24 we get $Stat \circ \ell^{\infty} \cong \mathcal{S}$. Similarly, $Stat \circ C \dashv UStat \circ AC(-,\mathbb{R}) \cong U$, by Theorem 25. $\square$

Remark 27 *Because the functor* $Stat \circ \ell^{\infty}\colon \mathbf{Sets} \to \mathbf{CCH}_{\mathrm{sep}}$ *in diagram* (10) *is left adjoint to the forgetful functor, it induces a monad on* $\mathbf{Sets}$. *For the record we note that its unit and multiplication are given by:*

$$\begin{array}{ccc} X \xrightarrow{\;\eta\;} Stat(\ell^{\infty}(X)) & \qquad & Stat(\ell^{\infty}(Stat(\ell^{\infty}(X)))) \xrightarrow{\;\mu\;} Stat(\ell^{\infty}(X)) \\ x \longmapsto \lambda\phi.\, \phi(x) & & \omega \longmapsto \lambda\phi.\, \omega(\lambda\rho.\, \rho(\phi)). \end{array}$$

[1] The slides `www.cs.ru.nl/B.Jacobs/TALKS/kadison-lectures-6up.pdf` describe the main points.

By Lemma 2 the isomorphism $Stat \circ \ell^\infty \cong \mathcal{S}$ *from Corollary 26 is an isomorphism of monads.*

4 The expectation monad

This section introduces the main object of study in this paper, namely the expectation monad. We give several isomorphic presentations.

Recall that for an arbitrary set X the set $[0,1]^X$ of fuzzy predicates on X is an effect module. Hence we can describe the expectation monad $\mathcal{E}\colon \mathbf{Sets} \to \mathbf{Sets}$ as a homset:

$$\begin{array}{rcl} \mathcal{E}(X) & = & \mathbf{EMod}\Big([0,1]^X, [0,1]\Big) \\ \mathcal{E}\Big(X \xrightarrow{f} Y\Big) & = & \lambda h \in \mathcal{E}(X).\, \lambda p \in [0,1]^Y.\, h(p \circ f). \end{array} \tag{11}$$

Lemma 28 *The definition of* $\mathcal{E}$ *in* (11) *yields a monad on* **Sets**, *with unit* $\eta_X\colon X \to \mathcal{E}(X)$ *given by:*

$$\eta_X(x) \;=\; \lambda p \in [0,1]^X.\, p(x)$$

and multiplication $\mu_X\colon \mathcal{E}^2(X) \to \mathcal{E}(X)$ *given on* $h\colon [0,1]^{\mathcal{E}(X)} \to [0,1]$ *in* **EMod** *by:*

$$\mu_X(h) \;=\; \lambda p \in [0,1]^X.\, h\Big(\lambda k \in \mathcal{E}(X).\, k(p)\Big).$$

Proof It is not hard to see that $\eta(x)$ and $\mu(h)$ are homomorphisms of effect modules. We check explicitly that the μ-η laws hold and leave the remaining verifications to the reader. For $h \in \mathcal{E}(X)$,

$$\begin{aligned}
\big(\mu_X \circ \eta_{\mathcal{E}(X)}\big)(h) &= \mu_X\big(\eta_{\mathcal{E}(X)}(h)\big) \\
&= \lambda p \in [0,1]^X.\, \eta_{\mathcal{E}(X)}(h)\big(\lambda k \in \mathcal{E}(X).\, k(p)\big) \\
&= \lambda p \in [0,1]^X.\, \big(\lambda k \in \mathcal{E}(X).\, k(p)\big)(h) \\
&= \lambda p \in [0,1]^X.\, h(p) \\
&= h \\
\big(\mu_X \circ \mathcal{E}(\eta_X)\big)(h) &= \mu_X\big(\mathcal{E}(\eta_X)(h)\big) \\
&= \lambda p \in [0,1]^X.\, \mathcal{E}(\eta_X)(h)\big(\lambda k \in \mathcal{E}(X).\, k(p)\big) \\
&= \lambda p \in [0,1]^X.\, h\big((\lambda k \in \mathcal{E}(X).\, k(p)) \circ \eta_X\big) \\
&= \lambda p \in [0,1]^X.\, h\big(\lambda x \in X.\, \eta_X(x)(p)\big) \\
&= \lambda p \in [0,1]^X.\, h\big(\lambda x \in X.\, p(x)\big) \\
&= \lambda p \in [0,1]^X.\, h(p) \\
&= h.
\end{aligned}$$

□

Remark 29 *(1) We think of elements $h \in \mathcal{E}(X)$ as measures. Below, in Proposition 33, it will be proven that $\mathcal{E}(X)$ is isomorphic to the set of finitely additive measures $\mathcal{P}(X) \to [0,1]$ on X. The application $h(p)$ of $h \in \mathcal{E}(X)$ to a function $p \in [0,1]^X$ may then be understood as integration $\int p\, dh$, giving the expected value of the stochastic variable/predicate p for the measure h.*

(2) The description $\mathcal{E}(X) = \mathbf{EMod}\big([0,1]^X, [0,1]\big)$ of the expectation monad in (11) *bears a certain formal resemblance to the ultrafilter monad $\mathcal{U}$ from Subsection 2.2. Recall from* (3) *that:*

$$\mathcal{U}(X) \;\cong\; \mathbf{BA}\big(\{0,1\}^X,\ \{0,1\}\big).$$

Thus, the expectation monad $\mathcal{E}$ can be seen as a "fuzzy" or "probabilistic" version of the ultrafilter monad $\mathcal{U}$, in which the set of Booleans $\{0,1\}$ is replaced by the set $[0,1]$ of probabilities. The relation between the two monads is further investigated in Section 5.

The following result is not a surprise, given the resemblance between the unit and multiplication for the expectation monad and the ones for the continuation monad (see Subsection 2.3).

Lemma 30 *The inclusion maps:*

$$\mathcal{E}(X) \;=\; \mathbf{EMod}\big([0,1]^X, [0,1]\big) \hookrightarrow [0,1]^{([0,1]^X)}$$

form a map of monads, from the expectation monad to the continuation monad (with the set $[0,1]$ as constant). □

There are several alternative descriptions of the sets $\mathcal{E}(X)$. They arise via the relation between effect modules and order unit spaces. But the expectation monad can also be described in terms of finitely additive measures, described as effect algebra homomorphisms. like in [19, Cor. 4.3].

Proposition 31 *There are natural isomorphisms:*

$$\mathcal{E} \;\cong\; \mathit{Stat} \circ \ell^{\infty} \;\overset{\text{Cor 26}}{\cong}\; \mathcal{S}.$$

Moreover, the expectation monad $\mathcal{E}$ arises from the composition of the following adjunctions.

$$\begin{tikzcd}
\mathbf{EMod}^{\mathrm{op}} \arrow[d, bend left, "{\mathbf{EMod}(-,[0,1]}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Conv} \arrow[u, bend left, "{\mathbf{Conv}(-,[0,1])}"] \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\
\mathbf{Sets} \arrow[u, bend left, "\mathcal{D}"] \arrow[uu, bend left=80, "{[0,1]^{(-)}}"]
\end{tikzcd}$$

Proof We use that the 'unit' functor $[0,u]_{(-)} \colon \mathbf{poVectu} \to \mathbf{EMod}$ is an equivalence, see Theorem 14, and thus full and faithful. Hence:

$$\begin{aligned}
\mathcal{E}(X) \;=\; \mathbf{EMod}([0,1]^X, [0,1]) \;&=\; \mathbf{EMod}([0,u]_{\ell^\infty(X)}, [0,u]_{\mathbb{R}}) \\
&\cong\; \mathbf{poVectu}(\ell^\infty(X), \mathbb{R}) \\
&=\; \mathbf{BOUS}(\ell^\infty(X), \mathbb{R}) \\
&=\; \mathit{Stat}(\ell^\infty(X)).
\end{aligned}$$

For the second part, recall that $\mathcal{D}(X)$ is the free convex set on X. Hence $\mathbf{Conv}(\mathcal{D}(X), [0,1]) = \mathbf{Sets}(X, [0,1]) = [0,1]^X$. □

Corollary 32 *The category of Eilenberg-Moore algebras of the expectation monad can be described as: $\mathcal{EM}(\mathcal{E}) \simeq \mathbf{CCLcvx} \simeq \mathbf{CCH}_{\mathrm{sep}}$.*

Proof We know from Remark 27 that we have an isomorphism of monads $\mathit{Stat} \circ \ell^\infty \cong \mathcal{S}$. We need to check that the restriction map $\rho_X \colon \mathit{Stat}(\ell^\infty(X)) \overset{\cong}{\Rightarrow} \mathcal{E}(X)$ is also a map of monads. But this is easy: the descriptions of the units and multiplications in Lemma 28 and Remark 27 coincide, when suitably restricted. But then $\mathcal{EM}(\mathcal{E}) \simeq \mathcal{EM}(\mathcal{S}) \simeq \mathbf{CCLcvx} \simeq \mathbf{CCH}_{\mathrm{sep}}$ by Theorem 24. □

Proposition 33 *For each set X there is a bijection:*

$$\mathcal{E}(X) = \mathbf{EMod}\Big([0,1]^X,\ [0,1]\Big) \xrightarrow[\cong]{\Phi} \mathbf{EA}\Big(\mathcal{P}(X), [0,1]\Big)$$

given by $\Phi(h)(U) = h(\mathbf{1}_U)$.

Proof We first check that Φ is well-defined, in the sense that $\Phi(h)\colon \mathcal{P}(X) \to [0,1]$ is a map of effect algebras. Clearly, $\Phi(h)(X) = h(\mathbf{1}_X) = 1$, and for $U, V \in \mathcal{P}(X)$ with $U \perp V$, that is $U \cap V = \emptyset$, we have $\Phi(h)(U \ovee V) = h(\mathbf{1}_{U\cup V}) = h(\mathbf{1}_U \ovee \mathbf{1}_V) = h(\mathbf{1}_U) + h(\mathbf{1}_V) = \Phi(h)(U) \ovee \phi(h)(V)$.

For injectivity of Φ, assume $\Phi(h) = \Phi(h')$, for $h, h' \in \mathcal{E}(X)$. We need to show $h(p) = h'(p)$ for an arbitrary $p \in [0,1]^X$. We first prove $h(q) = h'(q)$ for a simple function $q \in [0,1]^X$. Recall that such a simple q can be written as $q = \ovee_i r_i \mathbf{1}_{X_i}$, like in (9), where the (disjoint) subsets $X_i \subseteq X$ cover X. Since $h, h' \in \mathcal{E}(X)$ are maps of effect modules we get:

$$\begin{aligned} h(q) \;&=\; \textstyle\sum_i r_i h(\mathbf{1}_{X_i}) \;=\; \sum_i r_i \Phi(h)(X_i) \\ &\;=\; \textstyle\sum_i r_i \Phi(h')(X_i) \;=\; \sum_i r_i h'(\mathbf{1}_{X_i}) \;=\; h'(q). \end{aligned}$$

For an arbitrary $p \in [0,1]^X$ we first write $p = \lim_n p_n$ as limit of simple functions p_n like in Lemma 19. Lemma 16 implies that h, h' are continuous, and so we get $h = h'$ from:

$$h(p) \;=\; \lim_n h(p_n) \;=\; \lim_n h'(p_n) \;=\; h'(p).$$

For surjectivity of Φ, assume a finitely additive measure $m\colon \mathcal{P}(X) \to [0,1]$. We need to define a function $h \in \mathcal{E}(X)$ with $\Phi(h) = m$. We define such a h first on a simple function $q = \ovee_i r_i \mathbf{1}_{X_i}$ as $h(q) = \sum_i r_i m(X_i)$. For an arbitrary $p \in [0,1]^X$, written as $p = \lim_n p_n$, like in Lemma 19, we define $h(p) = \lim_n h(p_n)$. Then $\Phi(h) = m$, since for $U \subseteq X$ we have:

$$\Phi(h)(U) \;=\; h(\mathbf{1}_U) \;=\; m(U). \qquad \square$$

The inverse $h = \Phi^{-1}(m)$ that is constructed in this proof may be understood as an integral $h(p) = \int p dm$. The precise nature of the bijection Φ remains unclear at this stage since we have not yet identified the (algebraic) structure of the sets $\mathcal{E}(X)$. But via this bijection we can understand mapping a set to its finitely additive measures, *i.e.* $X \mapsto \mathbf{EA}(\mathcal{P}(X), [0,1])$, as a monad.

Yet another perspective is useful in this context. The characteristic function mapping:

$$[0,1] \times \mathcal{P}(X) \longrightarrow [0,1]^X \qquad \text{given by} \qquad (r, U) \longmapsto r \cdot \mathbf{1}_U$$

is a bihomomorphism of effect modules. Hence it gives rise to a map of effect modules $[0,1] \otimes \mathcal{P}(X) \to [0,1]^X$, where the tensor product $[0,1] \otimes \mathcal{P}(X)$ forms a more abstract description of the effect module of simple (step) functions $[X \to_s [0,1]]$ from Lemma 19 (see also [19, Thm. 5.6]). Lemma 19 says that this map is dense in $[0,1]^X$. This gives a quick proof of Proposition 33:

$$\begin{aligned}
\mathcal{E}(X) \;&=\; \mathbf{EMod}\Big([0,1]^X,\, [0,1]\Big) \\
&\cong\; \mathbf{EMod}\Big([0,1] \otimes \mathcal{P}(X),\, [0,1]\Big) \qquad \text{by denseness} \\
&\cong\; \mathbf{EA}\Big(\mathcal{P}(X), [0,1]\Big).
\end{aligned}$$

This last isomorphism is standard, because $[0,1] \otimes \mathcal{P}(X)$ is the free effect module on $\mathcal{P}(X)$.

5 The expectation and ultrafilter monads

In Corollary 32 we have seen a description $\mathcal{EM}(\mathcal{E}) \simeq \mathbf{CCLcvx} \simeq \mathbf{CCH}_{\text{sep}}$ of the category of Eilenberg-Moore algebras of the expectation monad $\mathcal{E}$. In particular, each free algebra $\mathcal{E}(X) = \mathbf{EMod}([0,1]^X, [0,1])$ is a convex compact Hausdorff space. In this section we investigate this topological structure categorically, via a map of monads $\mathcal{U} \Rightarrow \mathcal{E}$ from the ultrafilter monad to the expectation monad.

The unit interval $[0,1]$ plays an important role. It is a compact Hausdorff space, which means that it carries an algebra of the ultrafilter monad, see Subsection 2.2. We shall write this algebra as $\mathrm{ch} = \mathrm{ch}_{[0,1]} \colon \mathcal{U}([0,1]) \to [0,1]$. What this map precisely does is described in Example 5; but mostly we use it abstractly, as an $\mathcal{U}$-algebra. The technique that we use to define the following map of monads is copied from Lemma 7.

Proposition 34 *There is a map of monads $\tau \colon \mathcal{U} \Longrightarrow \mathcal{E}$, given on an ultrafilter $\mathcal{F} \in \mathcal{U}(X)$ and $p \in [0,1]^X$ by:*

$$\begin{aligned}
\tau_X(\mathcal{F})(p) \;&=\; \mathrm{ch}\Big(\mathcal{U}(p)(\mathcal{F})\Big) \\
&=\; \inf\{s \in [0,1] \mid [0,s] \in \mathcal{U}(p)(\mathcal{F})\} \qquad \textit{by } (5) \\
&=\; \inf\{s \in [0,1] \mid \{x \in X \mid p(x) \leq s\} \in \mathcal{F}\}.
\end{aligned}$$

In this description the functor $\mathcal{U}$ is applied to p, as function $X \to [0,1]$, giving $\mathcal{U}(p) \colon \mathcal{U}(X) \to \mathcal{U}([0,1])$.

Proof We first have to check that τ is well-defined, *i.e.* that $\tau_X(\mathcal{F}) \colon [0,1]^X \to [0,1]$ is a morphism of effect modules.

- Preservation $\tau_X(\mathcal{F})(r \cdot p) = r \cdot p\tau_X(\mathcal{F})$ of multiplication with scalar $r \in [0,1]$. This follows by observing that multiplication $r \cdot (-) \colon [0,1] \to [0,1]$ is a continuous function, and thus a morphism of algebras in the square below.

$$\begin{tikzcd}
\mathcal{U}([0,1]) \arrow[r, "\mathcal{U}(r\cdot(-))"] \arrow[d, "\mathrm{ch}"'] & \mathcal{U}([0,1]) \arrow[d, "\mathrm{ch}"] \\
{[0,1]} \arrow[r, "r\cdot(-)"'] & {[0,1]}
\end{tikzcd}$$

Thus:

$$\begin{aligned}
\tau(\mathcal{F})(r\cdot p) &= \Big(\mathrm{ch}\circ\mathcal{U}(r\cdot(-)\circ p)\Big)(\mathcal{F}) \\
&= \Big(r\cdot(-)\circ\mathrm{ch}\circ\mathcal{U}(p)\Big)(\mathcal{F}) \;=\; r\cdot\tau(\mathcal{F})(p).
\end{aligned}$$

- Preservation of $\owedge$, is obtained in the same manner, using that addition $+\colon [0,1]\times[0,1]\to[0,1]$ is continuous.
- Constant functions $\lambda x.\,a \in [0,1]^X$, including 0 and 1, are preserved:

$$\begin{aligned}
\tau_X(\mathcal{F})(\lambda x.\,a) &= \mathrm{ch}\Big(\mathcal{U}(\lambda x.\,a)(\mathcal{F})\Big) \\
&= \mathrm{ch}\Big(\{U\in\mathcal{P}([0,1]) \mid (\lambda x.\,a)^{-1}(U)\in\mathcal{F}\}\Big) \\
&= \mathrm{ch}\Big(\{U\in\mathcal{P}([0,1]) \mid \{x\in X \mid a\in U\}\in\mathcal{F}\}\Big) \\
&= \mathrm{ch}\Big(\{U\in\mathcal{P}([0,1]) \mid a\in U\}\Big) \qquad \text{since } \emptyset\notin\mathcal{F} \\
&= \mathrm{ch}(\eta(a)) \\
&= a.
\end{aligned}$$

We leave naturality of τ and commutation with units to the reader and check that τ commutes with multiplications $\mu^{\mathcal{E}}$ and $\mu^{\mathcal{U}}$ of the expectation and ultrafilter monads. Thus, for $\mathcal{A}\in\mathcal{U}^2(X)$ and $p\in[0,1]^X$, we calculate:

$$\begin{aligned}
\Big(\mu^{\mathcal{E}}\circ\tau\circ\mathcal{U}(\tau)\Big)(\mathcal{A})(p) &= \mu\Big(\tau\Big(\mathcal{U}(\tau)(\mathcal{A})\Big)\Big)(p) \\
&= \tau\Big(\mathcal{U}(\tau)(\mathcal{A})\Big)\Big(\lambda k.\,k(p)\Big) \\
&= \mathrm{ch}\Big(\mathcal{U}(\lambda k.\,k(p))\Big(\mathcal{U}(\tau)(\mathcal{A})\Big)\Big) \\
&= \mathrm{ch}\Big(\mathcal{U}(\lambda\mathcal{F}.\,\tau(\mathcal{F})(p))(\mathcal{A})\Big) \\
&= \mathrm{ch}\Big(\mathcal{U}(\lambda\mathcal{F}.\,\mathrm{ch}(\mathcal{U}(p)(\mathcal{F})))(\mathcal{A})\Big) \\
&= \mathrm{ch}\Big(\mathcal{U}(\mathrm{ch}\circ\mathcal{U}(p))(\mathcal{A})\Big) \\
&= \Big(\mathrm{ch}\circ\mathcal{U}(\mathrm{ch}\circ\mathcal{U}(p))\Big)(\mathcal{A}) \\
&= \Big(\mathrm{ch}\circ\mathcal{U}(\mathrm{ch})\circ\mathcal{U}^2(p)\Big)(\mathcal{A}) \\
&= \Big(\mathrm{ch}\circ\mu^{\mathcal{U}}\circ\mathcal{U}^2(p)\Big)(\mathcal{A}) \\
&= \Big(\mathrm{ch}\circ\mathcal{U}(p)\circ\mu^{\mathcal{U}}\Big)(\mathcal{A}) \\
&= \mathrm{ch}\Big(\mathcal{U}(p)(\mu^{\mathcal{U}}(\mathcal{A}))\Big) \\
&= \Big(\tau\circ\mu^{\mathcal{U}}\Big)(\mathcal{A})(p).
\end{aligned}$$

□

Corollary 35 *There is a functor* $\mathcal{EM}(\mathcal{E}) \to \mathcal{EM}(\mathcal{U}) = \mathbf{CH}$*, by pre-composition with the map of monads* τ*, as in:* $\Big(\mathcal{E}(X) \overset{\alpha}{\to} X\Big) \longmapsto \Big(\mathcal{U}(X) \xrightarrow{\alpha\circ\tau} X\Big)$*. This functor has a left adjoint by Lemma 1, which is the Radon functor* $\mathcal{R} \cong \mathit{Stat} \circ C$ *from Corollary 26.*

In particular, the underlying set X *of each* $\mathcal{E}$*-algebra* $\alpha\colon \mathcal{E}(X) \to X$ *carries a compact Hausdorff topology, with* $U \subseteq X$ *closed iff for each* $\mathcal{F} \in \mathcal{U}(X)$ *with* $U \in \mathcal{F}$ *one has* $\alpha(\tau(\mathcal{F})) \in U$*, as described in Subsection 2.2.* □

With respect to this topology on $\mathcal{E}(X)$, several maps are continuous.

Lemma 36 *The following maps are continuous functions.*

$$\mathcal{U}(X) \xrightarrow{\tau_X} \mathcal{E}(X) \qquad \mathcal{E}(X) \xrightarrow[\text{algebra}]{\alpha} X \qquad \mathcal{E}(X) \xrightarrow{\mathcal{E}(f)} \mathcal{E}(Y) \qquad \mathcal{E}(X) \xrightarrow[\lambda h.\, h(p)]{\mathrm{ev}_p\,=} [0,1].$$

Proof One shows that these maps are morphisms of $\mathcal{U}$-algebras. For instance, τ_X is continuous because it is a map of monads: commutation with multiplications, as required in (2), precisely says that it is a map of algebras, in the square on the left below.

$$\begin{tikzcd}
\mathcal{U}^2(X) \arrow[r, "\mathcal{U}(\tau_X)"] \arrow[d, "\mu_X"'] & \mathcal{U}(\mathcal{E}(X)) \arrow[d, "\mu_X\circ\tau_{\mathcal{E}(X)}"] \\
\mathcal{U}(X) \arrow[r, "\tau_X"'] & \mathcal{E}(X)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{U}(\mathcal{E}(X)) \arrow[r, "\mathcal{U}(\alpha)"] \arrow[d, "\mu_X\circ\tau_{\mathcal{E}(X)}"'] & \mathcal{U}(X) \arrow[d, "\alpha\circ\tau_X"] \\
\mathcal{E}(X) \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

The rectangle on the right expresses that an Eilenberg-Moore algebra $\alpha\colon \mathcal{E}(X) \to X$ is a continuous function. It commutes by naturality of τ:

$$\alpha \circ \tau_X \circ \mathcal{U}(\alpha) \;=\; \alpha \circ \mathcal{E}(\alpha) \circ \tau_{\mathcal{E}(X)} \;=\; \alpha \circ \mu_X \circ \tau_{\mathcal{E}(X)}.$$

For $f\colon X \to Y$, continuity of $\mathcal{E}(f)\colon \mathcal{E}(X) \to \mathcal{E}(Y)$ follows directly from naturality of τ. Finally, for $p \in [0,1]^X$ the map $\mathrm{ev}_p = \lambda h.\, h(p)\colon \mathcal{E}(X) \to [0,1]$ is continuous because for $\mathcal{F} \in \mathcal{U}(\mathcal{E}(X))$,

$$\begin{aligned}
\Big(\mathrm{ev}_p \circ \mu_X \circ \tau_{\mathcal{E}(X)}\Big)(\mathcal{F}) \;&=\; \mu_X\Big(\tau_{\mathcal{E}(X)}(\mathcal{F})\Big)(p) \\
&=\; \tau_{\mathcal{E}(X)}(\mathcal{F})(\lambda k.\, k(p)) \\
&=\; \tau_{\mathcal{E}(X)}(\mathcal{F})(\mathrm{ev}_p) \\
&=\; \mathrm{ch}\Big(\mathcal{U}(\mathrm{ev}_p)(\mathcal{F})\Big) \\
&=\; \Big(\mathrm{ch} \circ \mathcal{U}(\mathrm{ev}_p)\Big)(\mathcal{F}).
\end{aligned}$$ □

The next step is to give a concrete description of this compact Hausdorff topology on sets $\mathcal{E}(X)$, as induced by the algebra $\mathcal{U}(\mathcal{E}(X)) \to \mathcal{E}(X)$.

Proposition 37 *Fix a set X. For a predicate $p \in [0,1]^X$ and a rational number $s \in [0,1] \cap \mathbb{Q}$ write:*

$$\Box_s(p) \;=\; \{h \in \mathcal{E}(X) \mid h(p) > s\}.$$

These sets $\Box_s(p) \subseteq \mathcal{E}(X)$ form a subbasis for the topology on $\mathcal{E}(X)$.

Proof We reason as follows. The subsets $\Box_s(p)$ are open in the compact Hausdorff topology induced on $\mathcal{E}(X)$ by the algebra structure $\mathcal{U}(\mathcal{E}(X)) \to \mathcal{E}(X)$. They form a subbasis for a Hausdorff topology on $\mathcal{E}(X)$. Hence by Lemma 4 this topology is the induced one. We now elaborate these steps.

The Eilenberg-Moore algebra $\mathcal{U}(\mathcal{E}(X)) \to \mathcal{E}(X)$ is given by $\mu_X \circ \tau_{\mathcal{E}(X)}$. Hence the associated closed sets $U \subseteq \mathcal{E}(X)$ are those satisfying $U \in \mathcal{F} \Rightarrow \mu_X(\tau_{\mathcal{E}(X)}(\mathcal{F})) \in U$, for each $\mathcal{F} \in \mathcal{U}(\mathcal{E}(X))$, see Subsection 2.2. We wish to show that $\neg\Box_s(p) = \{h \mid h(p) \leq s\} \subseteq \mathcal{E}(X)$ is closed. We reason backwards, starting with the required conclusion.

$$\begin{array}{ll}
\multicolumn{2}{l}{\mu(\tau(\mathcal{F})) \in \neg\Box_s(p)} \\
\iff & \mu(\tau(\mathcal{F}))(p) \leq s \\
\iff & \mathrm{ch}\Big(\mathcal{U}(\lambda k.\, k(p))(\mathcal{F})\Big) \in [0,s] \quad \text{since} \\
& \quad \mu(\tau(\mathcal{F}))(p) = \tau(\mathcal{F})(\lambda k.\, k(p)) = \mathrm{ch}\Big(\mathcal{U}(\lambda k.\, k(p))(\mathcal{F})\Big) \\
\Longleftarrow & [0,s] \in \mathcal{U}(\lambda k.\, k(p))(\mathcal{F}) \\
& \quad \text{since } [0,s] \subseteq [0,1] \text{ is closed} \\
\iff & (\lambda k.\, k(p))^{-1}([0,s]) \in \mathcal{F} \\
\iff & \{h \in \mathcal{E}(X) \mid h(p) \in [0,s]\} = \neg\Box_s(p) \in \mathcal{F}.
\end{array}$$

Hence $\neg\Box_s(p) \subseteq \mathcal{E}(X)$ is closed, making $\Box_s(p)$ open.

Next we need to show that these $\Box_s(p)$'s give rise to a Hausdorff topology. So assume $h \neq h' \in \mathcal{E}(X)$. Then there must be a $p \in [0,1]^X$ with $h(p) \neq h'(p)$. Without loss of generality we assume $h(p) < h'(p)$. Find an $s \in [0,1] \cap \mathbb{Q}$ with $h(p) < s < h'(p)$. Then $h' \in \Box_s(p)$. Also:

$$h(p^{\perp}) \;=\; 1 - h(p) \;>\; 1 - s \;>\; 1 - h'(p) \;=\; h'(p^{\perp}).$$

Hence $h \in \Box_{1-s}(p^{\perp})$. These sets $\Box_s(p)$ and $\Box_{1-s}(p^{\perp})$ are disjoint, since: $k \in \Box_s(p) \cap \Box_{1-s}(p^{\perp})$ iff both $k(p) > s$ and $1 - k(p) > 1 - s$, which is impossible. $\square$

As is well-known, ultrafilters on a set X can also be understood as finitely additive measures $\mathcal{P}(X) \to \{0,1\}$. Using Proposition 33 we can express more precisely how the expectation monad $\mathcal{E}$ is a probabilistic version of the ultrafilter monad $\mathcal{U}$, namely via the descriptions:

$$\mathcal{E}(X) \cong \mathbf{EA}\Big(\mathcal{P}(X), [0,1]\Big) \quad \text{and} \quad \mathcal{U}(X) \cong \mathbf{EA}\Big(\mathcal{P}(X), \{0,1\}\Big).$$

We have $\mathbf{EA}\Big(\mathcal{P}(X), \{0,1\}\Big) = \mathbf{BA}\Big(\mathcal{P}(X), \{0,1\}\Big)$ because in general, for Boolean algebras B, B' a homomorphism of Boolean algebras $B \to B'$ is the same as an effect algebra homomorphism $B \to B'$.

Lemma 38 *The components $\tau_X \colon \mathcal{U}(X) \to \mathcal{E}(X)$ are injections.*

Proof Because there are isomorphisms:

$$\begin{tikzcd} \mathcal{U}(X) \arrow[rr, "\tau_X"] \arrow[d, phantom, "\wr\|"] & & \mathcal{E}(X) \arrow[d, phantom, "\wr\|"] \\ \mathbf{EA}\Big(\mathcal{P}(X), \{0,1\}\Big) \arrow[rr, tail] & & \mathbf{EA}\Big(\mathcal{P}(X), [0,1]\Big) \end{tikzcd}$$

$\square$

6 The expectation and distribution monads

This section is very similar to the previous one: it establishes a map of monads $\mathcal{D} \Rightarrow \mathcal{E}$, from the distribution monad to the expectation monad. It gives a concrete description of the convex structure on free algebras $\mathcal{E}(X)$.

Lemma 39 *There is a map of monads:*

$$\sigma \colon \mathcal{D} \Longrightarrow \mathcal{E} \quad \textit{given by} \quad \sigma_X(\varphi) = \lambda p \in [0,1]^X.\, \textstyle\sum_x \varphi(x) \cdot p(x), \tag{12}$$

where the dot $\cdot$ describes multiplication in $[0,1]$.

All components $\sigma_X \colon \mathcal{D}(X) \to \mathcal{E}(X)$ are injections. And for finite *sets X the component at X is an isomorphism $\mathcal{D}(X) \xrightarrow{\cong} \mathcal{E}(X)$.*

With this result we have completed the positioning of the expectation monad in Diagram (1), in between the distribution and ultrafilter monad on the hand, and the continuation monad on the other.

Proof It is laborious but straightforward to check that $\sigma \colon \mathcal{D} \Rightarrow \mathcal{E}$ is a map of monads. Next, assume X is finite, say $X = \{x_1, \ldots, x_n\}$. Each $p \in [0,1]^X$ is determined by the values $p(x_i) \in [0,1]$. Using the effect module structure of $[0,1]^X$, this p can be written as sum of scalar multiplications:

$$p = p(x_1) \cdot \mathbf{1}_{x_1} \ovee \cdots \ovee p(x_n) \cdot \mathbf{1}_{x_n},$$

where $\mathbf{1}_{x_i} \colon X \to [0,1]$ is the characteristic function of the singleton $\{x_i\} \subseteq X$. A map of effect modules $h \in \mathcal{E}(X) = \mathbf{EMod}([0,1]^X, [0,1])$ will thus send such a predicate p to:

$$\begin{aligned} h(p) &= h\Big(p(x_1)\cdot \mathbf{1}_{x_1} \ovee \cdots \ovee p(x_n)\cdot \mathbf{1}_{x_n}\Big) \\ &= p(x_1)\cdot h(\mathbf{1}_{x_1}) + \cdots + p(x_n)\cdot h(\mathbf{1}_{x_n}), \end{aligned}$$

since $\ovee$ is $+$ in $[0,1]$. Hence h is completely determined by these values $h(\mathbf{1}_{x_i}) \in [0,1]$. But since $\ovee_i \mathbf{1}_{x_i} = 1$ in $[0,1]^X$ we also have $\sum_i h(\mathbf{1}_{x_i}) = 1$. Hence h can be described by the convex sum $\varphi \in \mathcal{D}(X)$ given by $\varphi(x) = h(\mathbf{1}_x)$. Thus we have a bijection $\mathcal{E}(X) \cong \mathcal{D}(X)$. In fact, σ_X describes (the inverse of) this bijection, since:

$$\begin{aligned} \sigma_X(\varphi)(p) &= \textstyle\sum_i \varphi(x_i)\cdot p(x_i) \\ &= \textstyle\sum_i p(x_i)\cdot h(\mathbf{1}_{x_i}) \\ &= h\Big(\ovee_i\, p(x_i)\cdot \mathbf{1}_{x_i}\Big) \\ &= h(p). \end{aligned}$$

□

Corollary 40 *There is a functor $\mathcal{EM}(\mathcal{E}) \to \mathcal{EM}(\mathcal{D}) = \mathbf{Conv}$, by pre-composition: $\Big(\mathcal{E}(X) \xrightarrow{\alpha} X\Big) \longmapsto \Big(\mathcal{D}(X) \xrightarrow{\alpha\circ\sigma} X\Big)$. It has a left adjoint by Lemma 1.* □

Explicitly, for each $\mathcal{E}$-algebra $\alpha\colon \mathcal{E}(X) \to X$, the set X is a convex set, with sum of a formal convex combination $\sum_i r_i x_i$ given by the element:

$$\alpha\Big(\sigma_X(\textstyle\sum_i r_i x_i)\Big) \;=\; \alpha\Big(\lambda p \in [0,1]^X.\, \textstyle\sum_i r_i\cdot p(x_i)\Big) \;\in\; X.$$

Lemma 39 implies that if the carrier X is finite, the algebra structure α corresponds precisely to such convex structure on X. If X is non-finite we still have to find out what α involves.

Here is another (easy) consequence of Lemma 39.

Corollary 41 *On the first few finite sets: empty 0, singleton 1, and two-element 2 one has:*

$$\mathcal{E}(0) \;\cong\; 0 \qquad\qquad \mathcal{E}(1) \;\cong\; 1 \qquad\qquad \mathcal{E}(2) \;\cong\; [0,1].$$

The isomorphism in the middle says that $\mathcal{E}$ is an affine *functor.*

Proof The isomorphisms follow easily from $\mathcal{E}(X) \cong \mathcal{D}(X)$ for finite X. □

Remark 42 *(1) The natural transformation $\sigma\colon \mathcal{D} \Rightarrow \mathcal{E}$ from* (12) *implicitly uses that the unit interval $[0,1]$ is convex. This can be made explicit in the following way. Describe this convexity via an algebra* $\mathrm{cv}\colon \mathcal{D}([0,1]) \to [0,1]$*. Then we can equivalently describe σ as:*

$$\sigma_X(\varphi)(p) \;=\; \mathrm{cv}\Big(\mathcal{D}(p)(\varphi)\Big).$$

This alternative description is similar to the construction in Proposition 34, for a natural transformation $\mathcal{U} \Rightarrow \mathcal{E}$ (see also Lemma 7).

(2) From Corollaries 35 and 40 we know that the sets $\mathcal{E}(X)$ are both compact Hausdorff and convex. This means that we can take free extensions of the maps $\tau\colon \mathcal{U}(X) \to \mathcal{E}(X)$ and $\sigma\colon \mathcal{D}(X) \to \mathcal{E}(X)$, giving maps $\mathcal{D}(\mathcal{U}(X)) \to \mathcal{E}(X)$ and $\mathcal{U}(\mathcal{D}(X)) \to \mathcal{E}(X)$, etc. The latter map is the composite:

$$\mathcal{U}(\mathcal{D}(X)) \xrightarrow{\mathcal{U}(\sigma)} \mathcal{U}(\mathcal{E}(X)) \xrightarrow{\tau} \mathcal{E}^2(X) \xrightarrow{\mu} \mathcal{E}(X).$$

Using Example 5, it can be described more concretely on $\mathcal{F} \in \mathcal{U}(\mathcal{D}(X))$ and $p \in [0,1]^X$ as:

$$\inf\{s \in [0,1] \mid \{\varphi \in \mathcal{D}(X) \mid \textstyle\sum_x \varphi(x) \cdot p(x) \leq s\} \in \mathcal{F}\}.$$

The next result is the affine analogue of Lemma 36.

Lemma 43 *The following maps are affine functions.*

$$\mathcal{D}(X) \xrightarrow{\sigma_X} \mathcal{E}(X) \qquad \mathcal{E}(X) \xrightarrow[\text{algebra}]{\alpha} X \qquad \mathcal{E}(X) \xrightarrow{\mathcal{E}(f)} \mathcal{E}(Y) \qquad \mathcal{E}(X) \xrightarrow[\lambda h.\, h(p)]{\mathrm{ev}_p =} [0,1].$$

Proof Verifications are done like in the proof of Lemma 36. We only do the last one. We need to prove that the following diagram commutes,

$$\begin{tikzcd}
\mathcal{D}(\mathcal{E}(X)) \arrow[r, "\mathcal{D}(\mathrm{ev}_p)"] \arrow[d, "\mu_X \circ \sigma_X"'] & \mathcal{D}([0,1]) \arrow[d, "\mathrm{cv}"] \\
\mathcal{E}(X) \arrow[r, "\mathrm{ev}_p"'] & {[0,1]}
\end{tikzcd}$$

where the algebra cv interprets formal convex combinations as actual combinations. For a distribution $\Phi = \sum_i r_i h_i \in \mathcal{D}(\mathcal{E}(X))$ we have:

$$\begin{aligned}
\big(\mathrm{ev}_p \circ \mu \circ \sigma\big)(\Phi) &= \mu\big(\sigma(\Phi)\big)(p) \\
&= \sigma(\Phi)(\mathrm{ev}_p) \\
&= \textstyle\sum_i r_i \cdot \mathrm{ev}_p(h_i) \\
&= \mathrm{cv}\big(\textstyle\sum_i r_i \mathrm{ev}_p(h_i)\big) \\
&= \mathrm{cv}\big(\mathcal{D}(\mathrm{ev}_p)(\textstyle\sum_i r_i h_i)\big) \\
&= \big(\mathrm{cv} \circ \mathcal{D}(\mathrm{ev}_p)\big)(\Phi).
\end{aligned}$$

□

The $\mathcal{D}$-algebras obtained from $\mathcal{E}$-algebras turn out to be continuous functions. This connects the convex and topological structures in such algebras.

Lemma 44 *The maps $\sigma_X\colon \mathcal{D}(X) \rightarrowtail \mathcal{E}(X)$ are (trivially) continuous when we provide $\mathcal{D}(X)$ with the subspace topology with basic opens $\Box_s(p) \subseteq \mathcal{D}(X)$*

given by restriction: $\Box_s(p) = \{\varphi \in \mathcal{D}(X) \mid \sum_x \varphi(x) \cdot p(x) > s\}$, *for* $p \in [0,1]^X$ *and* $s \in [0,1] \cap \mathbb{Q}$.

For each $\mathcal{E}$*-algebra* $\alpha \colon \mathcal{E}(X) \to X$ *the associated* $\mathcal{D}$*-algebra* $\alpha \circ \sigma \colon \mathcal{D}(X) \to X$ *is then also continuous.*

Proof Lemma 36 states that $\mathcal{E}$-algebras $\alpha \colon \mathcal{E}(X) \to X$ are continuous. Hence $\alpha \circ \sigma \colon \mathcal{D}(X) \to X$, as composition of continuous maps, is also continuous. $\Box$

The following property of the map of monads $\mathcal{D} \Rightarrow \mathcal{E}$ will play a crucial role.

Proposition 45 *The inclusions* $\sigma_X \colon \mathcal{D}(X) \rightarrowtail \mathcal{E}(X)$ *are dense: the topological closure of* $\mathcal{D}(X)$ *is the whole of* $\mathcal{E}(X)$.

Proof We need to show that for each non-empty open $U \subseteq \mathcal{E}(X)$ there is a distribution $\varphi \in \mathcal{D}(X)$ with $\sigma(\varphi) \in U$. By Proposition 37 we may assume U is of the form $U = \Box_{s_1}(p_1) \cap \cdots \cap \Box_{s_m}(p_m)$, for certain $s_i \in [0,1] \cap \mathbb{Q}$ and $p_i \in [0,1]^X$. For convenience we do the proof for $m = 2$. Since U is non-empty there is some inhabitant $h \in \Box_{s_1}(p_2) \cap \Box_{s_2}(p_2)$. Thus $h(p_i) > s_i$. We claim there are simple functions $q_i \leq p_i$ with $h(q_i) > s_i$.

In general, this works as follows. If $h(p) > s$, write $p = \lim_n p_n$ for simple functions $p_n \leq p$, like in Lemma 19. Then $h(p) = \lim_n h(p_n) > s$. Hence $h(p_n) > s$ for some simple $p_n \leq p$.

In a next step we write the simple functions as weighted sum of characteristic functions, like in (9). Thus, let

$$q_1 \;=\; \varovee_j r_j \mathbf{1}_{U_j} \qquad \text{and} \qquad q_2 \;=\; \varovee_k t_k \mathbf{1}_{V_k},$$

where these $U_j \subseteq X$ and $V_k \subseteq X$ form non-empty partitions, each covering X. We form a new, refined partition $(W_\ell \subseteq X)_{\ell \in L}$ consisting of the non-empty intersections $U_j \cap V_j$, and choose $x_\ell \in W_\ell$. Then:

- $\sum_\ell h(\mathbf{1}_{W_\ell}) \;=\; h(\varovee_\ell \mathbf{1}_{W_\ell}) \;=\; h(\mathbf{1}_X) \;=\; 1$.
- There are subsets $L_j \subseteq L$ so that each $U_j \subseteq X$ can be written as disjoint union $U_j = \bigcup_{\ell \in L_j} W_\ell$.
- Similarly, $V_k = \bigcup_{\ell \in L_k} W_\ell$ for subsets $L_k \subseteq L$.

We take as distribution $\varphi = \sum_{\ell \in L} h(\mathbf{1}_{W_\ell})\, x_\ell \in \mathcal{D}(X)$. Then $\sigma(\varphi) \in \Box_{s_i}(p_i)$.

We do the proof for $i = 1$.

$$\begin{array}{rcl}
\sigma(\varphi)(p_1) & = & \sum_{\ell \in L} \varphi(x_\ell) \cdot p_1(x_\ell) \\
& \geq & \sum_{\ell \in L} h(\mathbf{1}_{W_\ell}) \cdot q_1(x_\ell) \\
& = & \sum_j \sum_{\ell \in L_j} h(\mathbf{1}_{W_\ell}) \cdot q_1(x_\ell) \\
& = & \sum_j \sum_{\ell \in L_j} h(\mathbf{1}_{W_\ell}) \cdot r_j \\
& = & \sum_j h(\varovee_{\ell \in L_j} \mathbf{1}_{W_\ell}) \cdot r_j \\
& = & \sum_j h(\mathbf{1}_{U_j}) \cdot r_j \\
& = & h(\varovee_j r_j \cdot \mathbf{1}_{U_j}) \\
& = & h(p_1) \\
& > & s_1.
\end{array}$$

□

Corollary 46 *Each map $\mathcal{U}(\mathcal{D}(X)) \to \mathcal{E}(X)$, described in Example 42.(3), is onto (surjective).*

Proof Since $\mathcal{D}(X) \rightarrowtail \mathcal{E}(X)$ is dense, each $h \in \mathcal{E}(X)$ is a limit of elements in $\mathcal{D}(X)$. Such limits can be described for instance via nets or via ultrafilters. In the present context we choose the latter approach. Thus there is an ultrafilter $\mathcal{F} \in \mathcal{U}(\mathcal{D}(X))$ such that h is the limit of this ultrafilter $\mathcal{U}(\sigma)(\mathcal{F}) \in \mathcal{U}(\mathcal{E}(X))$, when mapped to $\mathcal{E}(X)$. The limit is expressed via the ultrafilter algebra $\mu \circ \tau \colon \mathcal{U}(\mathcal{E}(X)) \to \mathcal{E}(X)$. This means that $(\mu \circ \tau \circ \mathcal{U}(\sigma))(\mathcal{F}) = h$. □

We have now seen that free algebras $\mathcal{E}(X)$ are convex sets and compact Hausdorff spaces. It is easy to see that they also satisfy the separation condition from Definition 21: if $h \neq h'$ in $\mathcal{E}(X)$, then $h(q) \neq h'(q)$ for some fuzzy predicate $q \in [0,1]^X$. This predicate q gives rise to an evaluation map $\mathrm{ev}_q \colon \mathcal{E}(X) \to [0,1]$ which is continuous and affine by Lemmas 36 and 43. It satisfies $\mathrm{ev}_q(h) = h(q) \neq h'(q) = \mathrm{ev}_q(h')$, so it separates $h, h' \in \mathcal{E}(X)$. Hence $\mathcal{E}(X)$ is an object in the category $\mathbf{CCH}_{\mathrm{sep}}$, a fact that we already knew from Corollary 32.

7 Algebras of the expectation monad

Corollary 32 abstractly characterizes the category $\mathcal{EM}(\mathcal{E})$ of Eilenberg-Moore algebras of the expectation monad via the equivalences $\mathcal{EM}(\mathcal{E}) \simeq \mathbf{CCLcvx} \simeq \mathbf{CCH}_{\mathrm{sep}}$. We like to better understand the structure involved. This section explicitly describes algebras of the expectation monad via barycenters of measures.

We start with the unit interval $[0,1]$. It is both compact Hausdorff and convex. Hence it carries algebras $\mathcal{U}([0,1]) \to [0,1]$ and $\mathcal{D}([0,1]) \to [0,1]$. This interval also carries an algebra structure for the expectation monad.

Lemma 47 *The unit interval* $[0,1]$ *carries an* $\mathcal{E}$*-algebra structure:*

$$\mathcal{E}([0,1]) \xrightarrow{\mathrm{ev}_{\mathrm{id}}} [0,1] \qquad \textit{by} \qquad h \longmapsto h(\mathrm{id}_{[0,1]}).$$

More generally, for an arbitrary set A *the set of (all) functions* $[0,1]^A$ *carries an* $\mathcal{E}$*-algebra structure:*

$$\mathcal{E}([0,1]^A) \longrightarrow [0,1]^A \qquad \textit{namely} \qquad h \longmapsto \lambda a \in A.\, h\Big(\lambda f \in [0,1]^A.\, f(a)\Big).$$

Proof It is easy to see that the evaluation-at-identity map $\mathrm{ev}_{\mathrm{id}} \colon \mathcal{E}([0,1]) \to [0,1]$ is an algebra. We explicitly check the details:

$$\begin{aligned}
&\Big(\mathrm{ev}_{\mathrm{id}} \circ \eta\Big)(x) \\
&= \mathrm{ev}_{\mathrm{id}}\Big(\eta(x)\Big) \\
&= \eta(x)(\mathrm{id}) \\
&= \mathrm{id}(x) \\
&= x
\end{aligned}$$

$$\begin{aligned}
\Big(\mathrm{ev}_{\mathrm{id}} \circ \mathcal{E}(\mathrm{ev}_{\mathrm{id}})\Big)(H) &= \mathrm{ev}_{\mathrm{id}}\Big(\mathcal{E}(\mathrm{ev}_{\mathrm{id}})(H)\Big) \\
&= \mathcal{E}(\mathrm{ev}_{\mathrm{id}})(H)(\mathrm{id}) \\
&= H(\mathrm{id} \circ \mathrm{ev}_{\mathrm{id}}) \\
&= H\Big(\lambda k \in \mathcal{E}([0,1]).\, k(\mathrm{id})\Big) \\
&= \mu(H)(\mathrm{id}) \\
&= \mathrm{ev}_{\mathrm{id}}\Big(\mu(H)\Big) \\
&= \Big(\mathrm{ev}_{\mathrm{id}} \circ \mu\Big)(H).
\end{aligned}$$

Since Eilenberg-Moore algebras are closed under products, there is also an $\mathcal{E}$-algebra on $[0,1]^A$. □

From Corollaries 35 and 40 we know that the underlying set X of an algebra $\mathcal{E}(X) \to X$ is both compact Hausdorff and convex. Additionally, Lemma 44 says that the algebra $\mathcal{D}(X) \to X$ is continuous.

We first characterize homomorphisms of algebras.

Lemma 48 *Consider Eilenberg-Moore algebras* $(\mathcal{E}(X) \stackrel{\alpha}{\to} X)$ *and* $(\mathcal{E}(Y) \stackrel{\beta}{\to} Y)$*. A function* $f \colon X \to Y$ *is an algebra homomorphism if and only if it is both continuous and affine, that is, iff the following two diagrams commute.*

$$\begin{tikzcd}
\mathcal{U}(X) \arrow[r, "\mathcal{U}(f)"] \arrow[d, "\alpha\circ\tau"'] & \mathcal{U}(Y) \arrow[d, "\beta\circ\tau"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}(X) \arrow[r, "\mathcal{D}(f)"] \arrow[d, "\alpha\circ\sigma"'] & \mathcal{D}(Y) \arrow[d, "\beta\circ\sigma"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Thus, the functor $\mathcal{EM}(\mathcal{E}) \to \mathbf{CCH}$ *is full and faithful.*

Proof If f is an algebra homomorphism, then $f \circ \alpha = \beta \circ \mathcal{E}(f)$. Hence the two rectangles above commute by naturality of τ and σ.

For the (if) part we use the property from Proposition 45 that the maps $\sigma_X : \mathcal{D}(X) \rightarrowtail \mathcal{E}(X)$ are dense monos. This means that for each map $g \colon \mathcal{D}(X) \to Z$ into a Hausdorff space Z there is at most one continuous $h \colon \mathcal{E}(X) \to Z$ with $h \circ \sigma = g$. We use this property as follows.

$$\begin{tikzcd}
\mathcal{D}(X) \arrow[r, tail, "\sigma", "\text{dense}"'] \arrow[dr] & \mathcal{E}(X) \arrow[d, shift right, "f\circ\alpha"'] \arrow[d, shift left, "\beta\circ\mathcal{E}(f)"] \\
& Y
\end{tikzcd}$$

The triangle commutes for both maps since f is affine:

$$f \circ \alpha \circ \sigma \;=\; \beta \circ \sigma \circ \mathcal{D}(f) \;=\; \beta \circ \mathcal{E}(f) \circ \sigma.$$

Also, both vertical maps are continuous, by Lemma 36. Hence $f \circ \alpha = \beta \circ \mathcal{E}(f)$, so that f is an algebra homomorphism. $\square$

Recall that for convex compact Hausdorff spaces X, Y we write $AC(X, Y)$ for the homset of affine continuous functions $X \to Y$. In light of the previous result, we shall also use this notation $AC(X, Y)$ when X, Y are carriers of $\mathcal{E}$-algebras, in case the algebra structure is clear from the context.

The next result gives a better understanding of $\mathcal{E}$-algebras: it shows that such algebras send measures to barycenters (like for instance in [29]).

Proposition 49 *Let $\mathcal{E}(X) \stackrel{\alpha}{\to} X$ be an $\mathcal{E}$-algebra. For each (algebra) map $q \in AC(X, [0, 1])$ the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{E}(X) \arrow[d, "\alpha"'] \arrow[dr, "\mathrm{ev}_q = \lambda h.\, h(q)"] & \\
X \arrow[r, "q"'] & {[0,1]}
\end{tikzcd}$$

This says that $x = \alpha(h) \in X$ is a barycenter *for $h \in \mathcal{E}(X)$, in the sense that $q(x) = h(q)$ for all affine continuous $q \colon X \to [0, 1]$.*

Proof Since $\mathrm{ev}_q = \mathrm{ev}_{\mathrm{id}} \circ \mathcal{E}(q)$ the above triangle can be morphed into a rectangle expressing that q is a map of algebras:

$$\begin{tikzcd}
\mathcal{E}(X) \arrow[r, "\mathcal{E}(q)"] \arrow[d, "\alpha"'] \arrow[dr, "\mathrm{ev}_q"] & \mathcal{E}({[0,1]}) \arrow[d, "\mathrm{ev}_{\mathrm{id}}"] \\
X \arrow[r, "q"'] & {[0,1]}
\end{tikzcd}$$

where $\mathrm{ev}_{\mathrm{id}}$ is the $\mathcal{E}$-algebra on $[0, 1]$ from Lemma 47. $\square$

Now that we have a reasonable grasp of $\mathcal{E}$-algebras, namely as convex compact Hausdorff spaces with a barycentric operation, we wish to comprehend how such algebras arise. We first observe that measures in $\mathcal{E}(X)$ in the images of $\mathcal{D}(X) \rightarrowtail \mathcal{E}(X)$ and $\mathcal{U}(X) \rightarrowtail \mathcal{E}(X)$ have barycenters, if X carries appropriate structure.

Lemma 50 *Let X be a convex compact Hausdorff space, described via $\mathcal{D}$- and $\mathcal{U}$-algebra structures* $\mathrm{cv}\colon \mathcal{D}(X) \to X$ *and* $\mathrm{ch}\colon \mathcal{U}(X) \to X$*. Then:*

(1) $\mathrm{cv}(\varphi) \in X$ *is a barycenter of* $\sigma(\varphi) \in \mathcal{E}(X)$*, for* $\varphi \in \mathcal{D}(X)$*;*
(2) $\mathrm{ch}(\mathcal{F}) \in X$ *is a barycenter of* $\tau(\mathcal{F}) \in \mathcal{E}(X)$*, for* $\mathcal{F} \in \mathcal{U}(X)$*.*

Proof We write $\mathrm{cv}_{[0,1]}\colon \mathcal{D}([0,1]) \to [0,1]$ and $\mathrm{ch}_{[0,1]}\colon \mathcal{U}([0,1]) \to [0,1]$ for the convex and compact Hausdorff structure on the unit interval. Then for $q \in AC(X,[0,1])$,

$$\begin{array}{rcll}
q\Big(\mathrm{cv}(\varphi)\Big) & = & \mathrm{cv}_{[0,1]}\Big(\mathcal{D}(q)(\varphi)\Big) & \text{since } q \text{ is affine} \\
& = & \mathrm{cv}_{[0,1]}\Big(\sum_i r_i q(x_i)\Big) & \text{if } \varphi = \sum_i r_i x_i \\
& = & \sum_i r_i \cdot q(x_i) & \\
& = & \sigma(\varphi)(q) & \\
q\Big(\mathrm{ch}(\mathcal{F})\Big) & = & \mathrm{ch}_{[0,1]}\Big(\mathcal{U}(q)(\mathcal{F})\Big) & \text{since } q \text{ is continuous} \\
& = & \tau(\mathcal{F})(q). &
\end{array}$$

□

We now deal with the general case, using the separation condition from Definition 21.

Proposition 51 *Let X be a convex compact Hausdorff space, described via $\mathcal{D}$- and $\mathcal{U}$-algebra structures* $\mathrm{cv}\colon \mathcal{D}(X) \to X$ *and* $\mathrm{ch}\colon \mathcal{U}(X) \to X$*.*

(1) Via the Axiom of Choice one obtains a function $\alpha\colon \mathcal{E}(X) \to X$ such that $\alpha(h) \in X$ is a barycenter for $h \in \mathcal{E}(X)$; that is, $q(\alpha(h)) = h(q)$ for each $q \in AC(X,[0,1])$.
(2) If $X \in \mathbf{CCH}_{\mathrm{sep}}$, i.e. points in X can be separated, then there is precisely one such $\alpha\colon \mathcal{E}(X) \to X$; moreover, it is an $\mathcal{E}$-algebra; and its induced convex and topological structures are the original ones on X, as expressed via the commuting triangles:

$$\begin{tikzcd}
\mathcal{D}(X) \arrow[r, tail, "\sigma"] \arrow[dr, "\mathrm{cv}"'] & \mathcal{E}(X) \arrow[d, "\alpha"] & \mathcal{U}(X) \arrow[l, tail, "\tau"'] \arrow[dl, "\mathrm{ch}"] \\
& X &
\end{tikzcd}$$

This yields a functor $\mathbf{CCH}_{\mathrm{sep}} \to \mathcal{EM}(\mathcal{E})$.

Proof Recall from Corollary 46 that the function $\mu \circ \tau \circ \mathcal{U}(\sigma)\colon \mathcal{U}(\mathcal{D}(X)) \to \mathcal{E}(X)$ is surjective. Using the Axiom of Choice we choose a section $s\colon \mathcal{E}(X) \to \mathcal{U}(\mathcal{D}(X))$ with $\mu \circ \tau \circ \mathcal{U}(\sigma) \circ s = \mathrm{id}_{\mathcal{E}(X)}$. We now obtain, via the choice of s, a map $\alpha\colon \mathcal{E}(X) \to X$ in:

$$\begin{tikzcd}
\mathcal{U}(\mathcal{D}(X)) \arrow[rr, "\mu\circ\tau\circ\mathcal{U}(\sigma)"'] \arrow[drr, "\mathrm{ch}\circ\mathcal{U}(\mathrm{cv})"'] & & \mathcal{E}(X) \arrow[ll, bend right, "s"'] \arrow[d, "\alpha=\mathrm{ch}\circ\mathcal{U}(\mathrm{cv})\circ s"] \\
& & X
\end{tikzcd}$$

We show that $\alpha(h) \in X$ is a barycenter for the measure $h \in \mathcal{E}(X)$. For each $q \in AC(X, [0,1])$ one has:

$$\begin{aligned}
h(q) &= \Big(\mu \circ \tau \circ \mathcal{U}(\sigma) \circ s\Big)(h)(q) \\
&= \mu\Big((\tau \circ \mathcal{U}(\sigma) \circ s)(h)\Big)(q) \\
&= \Big(\tau \circ \mathcal{U}(\sigma) \circ s\Big)(h)(\mathrm{ev}_q) \\
&= \Big(\mathrm{ch}_{[0,1]} \circ \mathcal{U}(\mathrm{ev}_q) \circ \mathcal{U}(\sigma) \circ s\Big)(h) \\
&= \Big(\mathrm{ch}_{[0,1]} \circ \mathcal{U}(\lambda\varphi.\, \mathrm{ev}_q(\sigma(\varphi))) \circ s\Big)(h) \\
&= \Big(\mathrm{ch}_{[0,1]} \circ \mathcal{U}(\lambda\varphi.\, \mathrm{cv}_{[0,1]}(\mathcal{D}(q)(\varphi))) \circ s\Big)(h) \qquad \text{see Remark 42.(1)} \\
&= \Big(\mathrm{ch}_{[0,1]} \circ \mathcal{U}(\mathrm{cv}_{[0,1]} \circ \mathcal{D}(q)) \circ s\Big)(h) \\
&= \Big(\mathrm{ch}_{[0,1]} \circ \mathcal{U}(q \circ \mathrm{cv}) \circ s\Big)(h) \qquad \text{since } q \text{ is affine} \\
&= \Big(q \circ \mathrm{ch} \circ \mathcal{U}(\mathrm{cv}) \circ s\Big)(h) \qquad \text{since } q \text{ is continuous} \\
&= \Big(q \circ \alpha\Big)(h) \\
&= q(\alpha(h)).
\end{aligned}$$

For (2), assume points in X can be separated, or equivalently, the collection of maps $q \in AC(X, [0,1])$ is jointly monic. Barycenters are then unique, since if both $x, x' \in X$ satisfy $q(x) = h(q) = q(x')$ for all $q \in AC(X, [0,1])$, then $x = x'$. Hence the function $\alpha\colon \mathcal{E}(X) \to X$ picks barycenters, in a unique manner. We need to prove the algebra equations (see the beginning of Section 2). They are obtained via the barycentric property $q(\alpha(h)) = h(q)$ and separability. First, the equation $\alpha \circ \eta = \mathrm{id}$ holds, since for each $x \in X$ and $q \in AC(X, [0,1])$,

$$q\Big((\alpha \circ \eta)(x))\Big) \;=\; q\Big(\alpha(\eta(x))\Big) \;=\; \eta(x)(q) \;=\; q(x) \;=\; q\Big(\mathrm{id}(x)\Big).$$

In the same way we obtain the equation $\alpha \circ \mu = \alpha \circ \mathcal{E}(\alpha)$. For $H \in \mathcal{E}^2(X)$

we have:

$$\begin{aligned}
\Big(q \circ \alpha \circ \mu\Big)(H) &= q\Big(\alpha(\mu(H))\Big) \\
&= \mu(H)(q) \\
&= H\Big(\lambda k \in \mathcal{E}(X).\, k(q)\Big) \\
&= H\Big(\lambda k \in \mathcal{E}(X).\, q(\alpha(k))\Big) \\
&= H\Big(q \circ \alpha\Big) \\
&= \mathcal{E}(\alpha)(H)(q) \\
&= q\Big(\alpha(\mathcal{E}(\alpha)(H))\Big) \\
&= \Big(q \circ \alpha \circ \mathcal{E}(\alpha)\Big)(H).
\end{aligned}$$

We still need to show that α induces the original convexity and topological structures. Since barycenters are unique, the equations $\alpha(\sigma(\varphi)) = \mathrm{cv}(\varphi)$ and $\alpha(\tau(\mathcal{F})) = \mathrm{ch}(\mathcal{F})$ follow directly from Lemma 48.

Finally, we need to check functoriality. So assume $f\colon X \to Y$ is a map in $\mathbf{CCH}_{\mathrm{sep}}$, and let $\alpha\colon \mathcal{E}(X) \to X$ and $\beta\colon \mathcal{E}(Y) \to Y$ be the induced algebras obtained by picking barycenters. We need to prove $\beta \circ \mathcal{E}(f) = f \circ \alpha$. Of course we use that points in Y can be separated. For $h \in \mathcal{E}(X)$, one has for all $q \in AC(Y, [0,1])$,

$$\begin{aligned}
q\Big(\beta(\mathcal{E}(f)(h))\Big) &= \mathcal{E}(f)(h)(q) \\
&= h\Big(q \circ f\Big) \\
&= (q \circ f)(\alpha(h)) \\
&= q\Big(f(\alpha(h))\Big). \qquad \square
\end{aligned}$$

In the approach followed above barycenters are obtained via the Axiom of Choice. Alternatively, they can be obtained via the Hahn-Banach theorem, see for instance [3, Prop. I.2.1].

8 A new formulation of Gleason's theorem

Gleason's theorem in quantum mechanics says that every state on a Hilbert space of dimension three or greater corresponds to a density matrix [18]. In this section we introduce a reformulation of Gleason's theorem, and prove the equivalence via Banach effect modules. This reformulation says that effects are the free effect module on projections. In formulas: $\mathrm{Ef}(\mathcal{H}) \cong [0,1] \otimes \mathrm{Pr}(\mathcal{H})$, for a Hilbert space $\mathcal{H}$.

Gleason's theorem is not easy to prove (see *e.g.* [13]). Even proofs using elementary methods are quite involved [11]. A state on a Hilbert space $\mathcal{H}$ is a certain probability distribution on the projections $\mathrm{Pr}(\mathcal{H})$ of $\mathcal{H}$. These projections $\mathrm{Pr}(\mathcal{H})$ form an orthomodular lattice, and thus an effect algebra [14,25]. In our current context these are exactly the effect algebra maps $\mathrm{Pr}(\mathcal{H}) \to [0,1]$. So Gleason's (original) theorem states:

$$\mathbf{EA}\Big(\mathrm{Pr}(\mathcal{H}), [0,1]\Big) \;\cong\; \mathrm{DM}(\mathcal{H}). \tag{13}$$

This isomorphism, from right to left, sends a density matrix M to the map $p \mapsto \mathrm{tr}(Mp)$—where tr is the trace map acting on operators.

Recall that $\mathrm{Ef}(\mathcal{H})$ is the set of positive operators on $\mathcal{H}$ below the identity. It is a Banach effect module. One can also consider the effect module maps $\mathrm{Ef}(\mathcal{H}) \to [0,1]$. For these maps there is a "lightweight" version of Gleason's theorem:

$$\mathbf{EMod}\Big(\mathrm{Ef}(\mathcal{H}), [0,1]\Big) \;\cong\; \mathrm{DM}(\mathcal{H}). \tag{14}$$

This isomorphism involves the same trace computation as (13). This statement is significantly easier to prove than Gleason's theorem itself, see [10].

Because Gleason's original theorem (13) is so much harder to prove than the lightweight version (14) one could wonder what Gleason's theorem states that Gleason light doesn't. In Theorem 53 we will show that the difference amounts exactly to the statement:

$$[0,1] \otimes \mathrm{Pr}(\mathcal{H}) \;\cong\; \mathrm{Ef}(\mathcal{H}), \tag{15}$$

where $\otimes$ is the tensor of effect algebras (see [25]). A general result, see [34, VII,§4], says that the tensor product $[0,1] \otimes \mathrm{Pr}(\mathcal{H})$ is the free effect module on $\mathrm{Pr}(\mathcal{H})$, see also (16) below.

The following table gives an overview of the various formulations of Gleason's theorem.

Description	**Formulation**	**Label**
original Gleason, for projections	$\mathbf{EA}\Big(\mathrm{Pr}(\mathcal{H}), [0,1]\Big) \cong \mathrm{DM}(\mathcal{H})$	(13)
lightweight version, for effects	$\mathbf{EMod}\Big(\mathrm{Ef}(\mathcal{H}), [0,1]\Big) \cong \mathrm{DM}(\mathcal{H})$	(14)
effects as free module on projections	$[0,1] \otimes \mathrm{Pr}(\mathcal{H}) \cong \mathrm{Ef}(\mathcal{H})$	(15)

In this section we shall prove (13) $\Longleftrightarrow$ (15), in presence of (14), see Theorem 53. Since (13) is true, for dimension ≥ 3, the same then holds for (15).

But before we can start with the proof we need to collect some basic results. In Theorem 25 we have seen the 'Kadison' duality $\mathbf{BOUS}^{\mathrm{op}} \simeq \mathbf{CCH}_{\mathrm{sep}}$ between Banach (complete) order unit spaces and convex compact Hausdorff spaces. The following two points are important steps in the proof. For an order unit space V,

(1) the evaluation map $\mathrm{ev}\colon V \to AC(\mathit{Stat}(V), \mathbb{R})$ is a dense embedding;
(2) this map ev is an isomorphism if and only if V is complete.

We have also seen the equivalences $\mathbf{EMod} \simeq \mathbf{poVectu}$ and $\mathbf{BEMod} \simeq \mathbf{BOUS}$ between effect modules and ordered vector spaces, see Theorem 17. In combination with the above two points we get the following result that will play an important role below.

Lemma 52 *For an arbitrary effect module E there is an injective map of effect modules:*

$$E \overset{\mathrm{ev}}{\rightarrowtail} AC\Big(\mathbf{EMod}(E, [0,1]),\ [0,1]\Big) \quad \textit{where} \quad \mathrm{ev}(x)(f) \ = \ f(x). \qquad \square$$

As observed in Section 3 the unit interval $[0, 1]$ is a monoid in the category **EA** of effect algebras, and **EMod** is the category of associated actions. This means that for general reasons, see [34, VII,§4], tensoring with $[0, 1]$ yields a left adjoint to the forgetful functor in:

$$\begin{tikzcd} \mathbf{EMod} \arrow[d, ""{name=U}] \\ \mathbf{EA} \arrow[u, bend left=60, "{[0,1]\otimes(-)}"] \end{tikzcd} \qquad \dashv \tag{16}$$

Now we come to the main result of this section.

Theorem 53 (13) $\Longleftrightarrow$ (15)*, in presence of* (14)*.*

That is, using Gleason light (14) *the following statements are equivalent, for a Hilbert space $\mathcal{H}$ with finite dimension ≥ 3.*

(13)**:** $\mathbf{EA}(\mathrm{Pr}(\mathcal{H}), [0,1]) \cong \mathrm{DM}(\mathcal{H})$*, i.e. Gleason's original theorem;*
(15)**:** *The canonical map* $[0,1] \otimes \mathrm{Pr}(\mathcal{H}) \to \mathrm{Ef}(\mathcal{H})$ *is an isomorphism.*

Proof Assuming $[0,1] \otimes \mathrm{Pr}(\mathcal{H}) \xrightarrow{\cong} \mathrm{Ef}(\mathcal{H})$ we get Gleason's theorem:

$$\begin{array}{rcll} \mathbf{EA}\Big(\mathrm{Pr}(\mathcal{H}), [0,1]\Big) & \cong & \mathbf{EMod}\Big([0,1] \otimes \mathrm{Pr}(\mathcal{H}), [0,1]\Big) & \text{by freeness (16)} \\ & \cong & \mathbf{EMod}\Big(\mathrm{Ef}(\mathcal{H}), [0,1]\Big) & \text{by assumption} \\ & \cong & \mathrm{DM}(\mathcal{H}) & \text{by Gleason light (14).} \end{array}$$

In the other direction, applying Lemma 52 to the free effect module $[0,1] \otimes \Pr(\mathcal{H})$ yields an injection $[0,1] \otimes \Pr(\mathcal{H}) \rightarrowtail \mathrm{Ef}(\mathcal{H})$, namely:

$$\begin{tikzcd}[row sep=small]
{[0,1] \otimes \Pr(\mathcal{H})} \arrow[r, tail, "\mathrm{ev}"] & AC\Big(\mathbf{EMod}\big([0,1] \otimes \Pr(\mathcal{H}), [0,1]\big),\ [0,1]\Big) \arrow[d, phantom, "\wr\| \quad \text{by (52)}"] \\
& AC\Big(\mathbf{EA}\big(\Pr(\mathcal{H}), [0,1]\big),\ [0,1]\Big) \arrow[d, phantom, "\wr\| \quad \text{by assumption}"] \\
& AC\Big(\mathrm{DM}(\mathcal{H}),\ [0,1]\Big) \arrow[d, phantom, "\wr\| \quad \text{by (14)}"] \\
& \mathrm{Ef}(\mathcal{H})
\end{tikzcd}$$

This injection is given by $\ovee_i r_i \otimes p_i \mapsto \sum_i r_i \cdot p_i$. It is surjective since the spectral composition $A = \sum_i r_i \cdot p_i$ of an effect $A \in \mathrm{Ef}(\mathcal{H})$ yields an element $\ovee_i r_i \otimes p_i$ as required. □

9 The expectation monad for program semantics

In this final section we put some earlier result together in a 'state-and-effect' triangle that captures essential ingredients of program semantics and logic, like in [22,23,24]. We subsequently elaborate on this semantics.

Theorem 54 *The Kleisli category $\mathcal{K}\ell(\mathcal{E})$ and Eilenberg-Moore category $\mathcal{EM}(\mathcal{E})$ of the expectation monad fit in a diagram:*

$$\begin{tikzcd}
\mathbf{BEMod}^{\mathrm{op}} \arrow[rr, bend left=10, "Hom(-,[0,1])"] \arrow[rr, phantom, "\simeq"] & & \mathcal{EM}(\mathcal{E}) = \mathbf{CCH}_{\mathrm{sep}} \arrow[ll, bend left=10, "Hom(-,[0,1])"] \\
& \mathcal{K}\ell(\mathcal{E}) \arrow[ul, "Pred={[0,1]}^{(-)}"] \arrow[ur, "Stat"'] &
\end{tikzcd} \tag{17}$$

where:

- *$Stat\colon \mathcal{K}\ell(\mathcal{E}) \to \mathcal{EM}(\mathcal{E})$ is the full and faithful 'comparison' functor from the Kleisli category to the category of algebra (which exists for any monad);*
- *$[0,1]^{(-)}\colon \mathcal{K}\ell(\mathcal{E}) \to \mathbf{EMod}^{\mathrm{op}}$ is the 'predicate' functor like in [24], which is full and faithful in this situation;*
- *the equivalence $\mathbf{BEMod}^{\mathrm{op}} \simeq \mathcal{EM}(\mathcal{E})$ is obtained via totalization, Kadison duality and Świrszcz's monadicity result (Theorems 17, 25 and 24):*

$$\mathbf{BEMod}^{\mathrm{op}} \simeq \mathbf{BOUS}^{\mathrm{op}} \simeq \mathbf{CCH}_{\mathrm{sep}} \simeq \mathcal{EM}(\mathcal{E}).$$

As elaborated in [24] the predicate and state functors can be described as the *homsets* $\mathit{Pred} \cong \mathit{Hom}(-, 1+1)$ *and* $\mathit{Stat} \cong \mathit{Hom}(1, -)$ *in the Kleisli category* $\mathcal{K}\ell(\mathcal{E})$.

Proof The assignment $X \mapsto [0,1]^X$ yields a full and faithful functor $\mathcal{K}\ell(\mathcal{E}) \to \mathbf{EMod}^{\mathrm{op}}$ since there is a bijective correspondence between:

$$\begin{array}{cl}
X \xrightarrow{f} Y & \text{in } \mathcal{K}\ell(\mathcal{E}) \\
\hline\hline
X \xrightarrow{f} \mathcal{E}(Y) & \text{in } \mathbf{Sets} \\
\hline\hline
[0,1]^Y \xrightarrow[g]{} [0,1]^X & \text{in } \mathbf{EMod}
\end{array}$$

It sends f to the associated 'substitution' or 'weakest precondition' map $f^*\colon [0,1]^Y \to [0,1]^X$ given by $f^*(q)(x) = f(x)(q)$. This swapping of arguments is clearly bijective.

The two triangles commute, up-to-isomorphism. In one direction, we have $\mathit{Hom}([0,1]^X, [0,1]) = \mathcal{E}(X) = K(X)$, and in the other direction we use that $\mathcal{E}(X)$ is the free algebra to get: $\mathit{Hom}(K(X), [0,1]) = \mathcal{EM}(\mathcal{E})(\mathcal{E}(X), [0,1]) \cong \mathbf{Sets}(X, [0,1]) = [0,1]^X$. $\square$

This paper uses the expectation monad $\mathcal{E}(X) = \mathbf{EMod}([0,1]^X, [0,1])$ and relates it to characterization and duality results for convex compact Hausdorff spaces. Elements of $\mathcal{E}(X)$ are characterized as states (see Proposition 31) and as (finitely additive) measures (see esp. Proposition 33). Measures have been captured via monads before, first by Giry [17] following ideas of Lawvere. Such a description in terms of monads is useful to provide semantics for probabilistic programs [31,27,35,36]. The term 'expectation monad' seems to occur first in [38], where it is formalized in Haskell. Such a formalization in a functional language is only partial, because the relevant equations and restrictions are omitted, so that there is not really a difference with the continuation monad $X \mapsto [0,1]^{([0,1]^X)}$. A formalization of what is also called 'expectation monad' in the theorem prover Coq occurs in [4] and is more informative. It involves maps $h\colon [0,1]^X \to [0,1]$ which are required to be monotone, continuous, linear (preserving partial sum $\varovee$ and scalar multiplication) and compatible with inverses—meaning $h(1-p) \leq 1 - h(p)$. This comes very close to the notion of homomorphism of effect module that is used here, but effect modules themselves are not mentioned in [4]. This Coq formalization is used for instance in the semantics of game-based programs for the certification of cryptographic proofs in [8] (see [41] for an overview of this line of work). Finally, in [30] a monad is used of maps $h\colon [0,1]^X \to [0,1]$ that are (Scott) continuous and sublinear—*i.e.* $h(p \varovee q) \leq h(p) \varovee h(q)$, and $h(r \cdot p) = r \cdot h(p)$.

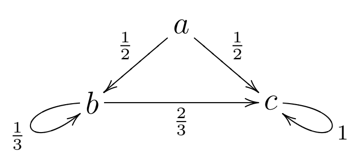
The definition $\mathcal{E}(X) = \mathbf{EMod}([0,1]^X, [0,1])$ of the expectation monad that is used here has good credentials to be the right definition, because:

- The sets $\mathcal{E}(X)$ as defined here form a stable collection, in the sense that

its elements can be characterized in several other ways, namely as states on certain order unit spaces (Proposition 31) or as finitely additive measures (Proposition 33).
- The assignment $X \mapsto \mathcal{E}(X)$ has nice categorical properties: it is a left adjoint, giving the free convex compact Hausdorff space, and thus also a monad.
- The monad $\mathcal{E}$ gives rise to a state-and-effect triangle (17) that can be exploited for program logics, see [22,12].

It is thus worthwhile to systematically develop a program semantics and logic based on the expectation monad and its duality. This is a project on its own. We conclude by sketching some ingredients, focusing on the program constructs that can be used.

First we include a small example. Suppose we have a set of states $S = \{a, b, c\}$ with probabilistic transitions between them as described on the left below.



$$\begin{array}{rcl} S & \longrightarrow & \mathcal{D}(S) \\ a & \longmapsto & \frac{1}{2}b + \frac{1}{2}c \\ b & \longmapsto & \frac{1}{3}b + \frac{2}{3}c \\ c & \longmapsto & 1c \end{array}$$

On the right is the same system described as a function, namely as coalgebra of the distribution monad $\mathcal{D}$. It maps each state to the corresponding discrete probability distribution. We can also describe the same system as coalgebra $S \to \mathcal{E}(S)$ of the expectation monad, via the map $\mathcal{D} \rightarrowtail \mathcal{E}$. Then it looks as follows:

$$\begin{array}{rcl} S & \longrightarrow & \mathcal{E}(S) \\ a & \longmapsto & \lambda q \in [0,1]^S.\, \frac{1}{2}q(b) + \frac{1}{2}q(c) \\ b & \longmapsto & \lambda q \in [0,1]^S.\, \frac{1}{3}q(b) + \frac{2}{3}q(c) \\ c & \longmapsto & \lambda q \in [0,1]^S.\, q(c) \end{array}$$

Thus, via the $\mathcal{E}$-monad we obtain a probabilistic continuation style semantics.

Let's consider this from a more general perspective. Let S now be an arbitrary, unspecified set of states, for which we consider programs as functions $S \to \mathcal{E}(S)$, *i.e.* as Kleisli endomaps in the base category in (17), or as $\mathcal{E}$-coalgebras. In a standard way the monad structure provides a monoid structure on these maps $S \to \mathcal{E}(S)$ for sequential composition, with the unit $S \to \mathcal{E}(S)$ as neutral element 'skip'. We briefly sketch some other algebraic structure on such programs (coalgebras), see also [36].

Programs $S \to \mathcal{E}(S)$ are closed under convex combinations: if we have programs $P_1, \ldots, P_n \colon S \to \mathcal{E}(S)$ and probabilities $r_i \in [0,1]$ with $\sum_i r_i = 1$, then we can form a new program $P = \sum_i r_i P_i \colon S \to \mathcal{E}(S)$. For $q \in [0,1]^S$,

$$P(s)(q) \;=\; \textstyle\sum_i r_i \cdot P_i(s)(q).$$

In this way we get an interpretation for probabilistic guarded commands.

Since the set $S \to \mathcal{E}(S)$ carries a pointwise order with suprema of ω-chains we can also give meaning to iteration constructs like 'while' and 'for ... do'.

Further we can also do "probabilistic assignment", written for instance as $n := \varphi$, where n is a variable, say of integer type int, and φ is a distribution of type $\mathcal{D}(\mathsf{int})$. The intended meaning of such an assignment $n := \varphi$ is that afterwards the variable n has value $m \colon \mathsf{int}$ with probability $\varphi(m) \in [0,1]$. In order to model this we assume an update function $\mathsf{upd}_n \colon S \times \mathsf{int} \to S$, which we leave unspecified (similar functions exist for other variables). The interpretation $[\![\, n := \varphi \,]\!]$ of the probabilistic assignment is a function $S \to \mathcal{E}(S)$, defined as follows.

$$\begin{array}{rcl} [\![\, n := \varphi \,]\!](s) & = & \mathcal{E}\Big(\mathsf{upd}_n(s,-)\Big)\Big(\sigma(\varphi)\Big) \\ & = & \lambda q \in [0,1]^S.\, \sum_i r_i \cdot q(\mathsf{upd}_n(m_i)), \qquad \text{if } \varphi = \sum_i r_i m_i. \end{array}$$

It applies the functor $\mathcal{E}$ to the function $\mathsf{upd}_n(s,-) \colon \mathsf{int} \to S$ and uses the natural transformation $\sigma \colon \mathcal{D} \Rightarrow \mathcal{E}$ from (12).

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