Electronic Notes in Theoretical Computer Science 11 (2002)
URL: http://www.elsevier.nl/locate/entcs/volume11.html 27 pages

Factorization systems and fibrations: Toward a fibred Birkhoff variety theorem

Jesse Hughes and Bart Jacobs

Department of Computer Science, Univ. of Nijmegen
Toernooiveld 1, 6525 ED Nijmegen, The Netherlands
{*jesseh,bart*}*@cs.kun.nl*

Abstract

It is well-known that a factorization system on a category (with sufficient pullbacks) gives rise to a fibration. This paper characterizes the fibrations that arise in such a way, by making precise the logical structure that is given by factorization systems. The underlying motivation is to obtain general Birkhoff results in a fibred setting.

1 Introduction

Every function between sets can be factored as a surjection followed by an injection. Such a factorization is axiomatized in the notion of a *factorization system*: it consists of a pair of collections of maps $\langle H, S\rangle$ in a category satisfying familiar conditions (see Definition 2.1 below, or [Bor94, Volume I, Section 5.5] where factorization systems are studied in relation to reflections). In analogy with the situation for sets, the maps in S are usually called abstract monos, and the maps in H abstract epis. The properties that a factorization system satisfies can be understood logically, in tradition of categorical logic. More technically, they give rise to a fibration with certain structure. The aim of this paper is characterise this structure.

The original motivation for the investigations in this paper comes from Birkhoff's famous results about definability and deducibility for universal algebras [Bir35]. There has been a considerable amount of work over the last few years aimed at dualizing these results to the setting of co-algebras, including [Rut00], [GS01], [Gum01], [Kur01], [Kur00], [KR02], [AH00], [Hug01], [Hug02], [Roş00], [AP01]. Successful dualization often requires a reformulation at a suitable level of abstraction. Currently, the most abstract setting for Birkhoff's results is given by factorization systems on appropriate categories [BH76], [NS81].

Birkhoff's results make essential use of logical concepts. This motivated the authors to try and re-investigate these results in the context of fibred cat-

egories (also called fibrations). After all, these provide an abstract theory of indexing, suitable for capturing various logics. Already after the first attempt of this reinvestigation the question came up what exactly is the logical structure corresponding to factorization systems. This seemed to be unknown—although, as we shall see, [**?**] comes close to an answer. This issue turned out to be a non-entirely trivial research question on its own, to which the current paper is devoted. Only the first steps of the actual work on Birkhoff's results for fibrations are sketched in Section 4, and the further development of this approach is postponed to later publications.

The main question we address here is thus: to which kind of fibrations do factorization systems correspond? This matter is related to subobject fibrations. They come with a fair amount of logical structure, which can be described explicitly in the context of fibrations, see [Jac99, Section 4.6]. Our main result in this paper (Theorem 2.13) gives a similar correspondence for factorization systems. It describes the kind of logical structure one gets for free when working with factorization systems.

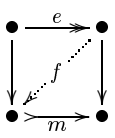
The organization of the paper is simple. The first section below describes how factorization systems give rise to bifibrations with certain logical properties (full subset types and associated strong coproducts). Subsequently, Section 3 describes how one can go in the reverse direction: from bifibrations with this structure to factorization systems. Finally, Section 4 sketches our motivation: the first steps of a fibred approach to Birkhoff. This only serves as illustration, and will be elaborated in a later publication.

2 Fibrations induced by factorization systems

We begin with a review of factorization systems and fibrations. For more details on the former, see [AHS90] or [Bor94] and for the latter, [Jac99].

Definition 2.1 *Let H, S be subclasses of the category $\mathbb{B}^{\to}$ of arrows in an arbitrary category $\mathbb{B}$. We say that $\langle H, S\rangle$ is a factorization system for $\mathbb{B}$ if the following hold.*

- $\mathsf{Iso} \subseteq H \cap S$ *(the first of many abuses of set notation for classes);*
- *H and S are closed under composition;*
- *H and S satisfy the* diagonal fill-in property*, namely, for every commutative square*



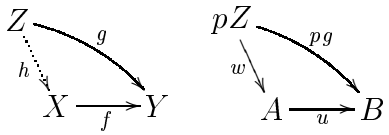
where $e \in H$ and $m \in S$, there is a unique arrow f, as shown, making each triangle commute;

- *every arrow f in $\mathbb{B}$ factors as $f = m \circ e$, where $e \in H$ and $m \in S$;*

Hereafter, we use $\twoheadrightarrow$ to denote arrows in H (the so-called "abstract epis") and $\rightarrowtail$ for S-morphisms (the "abstract monos").

The category **Set** has a factorization system $\langle \mathsf{Epi}, \mathsf{Mono} \rangle$ given by epi-mono factorizations. Any category $\mathbb{B}$ has the trivial factorization systems $\langle \mathsf{Iso}, \mathbb{B}^{\to} \rangle$ and $\langle \mathbb{B}^{\to}, \mathsf{Iso} \rangle$. Also, any category has a factorization system of extremal epis and monos, as well as the dual system. See [AHS90], Chapter 14.

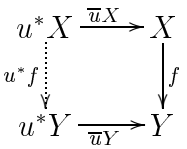
Definition 2.2 *Let $p : \mathbb{E} \to \mathbb{B}$ be a functor, and $f : X \to Y$ an arrow in $\mathbb{E}$, with $pf = u : A \to B$. We say that f is* Cartesian over u *just in case, for every $g : Z \to Y$ in $\mathbb{E}$ such that pg factors through u, $pg = u \circ w$, there is a unique $h : Z \to Y$ in $\mathbb{E}$ such that $g = f \circ h$ and $ph = w$.*



Definition 2.3 *A* fibration *is a functor $p : \mathbb{E} \to \mathbb{B}$ (often written $\begin{smallmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{smallmatrix}$) such that, for every $Y \in \mathbb{E}$ and $u : I \to pY$, there is a Cartesian $f : X \to Y$ over u (i.e., with $pf = u$).*

For a given fibration $\begin{smallmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{smallmatrix}$, and any $B \in \mathbb{B}$, let $\mathbb{E}_B$ denote the *fibre category* over B. That is, the objects of $\mathbb{E}_B$ are the objects $X \in \mathbb{E}$ such that $pX = B$. The morphisms of $\mathbb{E}_B$ are the morphisms $f : X \to Y$ of $\mathbb{E}$ such that $pf = \mathsf{id}_B$, the so-called *vertical morphisms*.

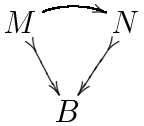
Let a fibration $\begin{smallmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{smallmatrix}$ be given and let $Y \in \mathbb{E}$ with $pY = A$. For each map $u : B \to A$ in $\mathbb{B}$, we fix a particular Cartesian map $\overline{u}Y$ above u and denote $\mathsf{dom}(\overline{u}Y) = u^*(Y)$. This gives the object part of a functor $u^* : \mathbb{E}_A \to \mathbb{E}_B$, called a *substitution functor*. The action of u^* on arrows $f : X \to Y$ in $\mathbb{E}_A$ is as follows. Since $pf = \mathsf{id}_A$, we see $p(f \circ \overline{u}X) = u \circ \mathsf{id}_B$. Hence, there is a unique vertical morphism $u^*f : u^*X \to Y^*$ making the square below commute.



Let $\mathbb{B}$ have pullbacks along S-morphisms and let $\langle H, S \rangle$ be a factorization system for $\mathbb{B}$. Then, S is stable under these pullbacks (see, for instance, Proposition 14.16, [AHS90]). Thus, the codomain functor

$$\mathsf{cod} : S \longrightarrow \mathbb{B}$$

is a fibration. Given $B \in \mathbb{B}$, the fibre category S_B over B has, as objects, S-morphisms with codomain B, and, as arrows, commutative triangles as in the diagram below.



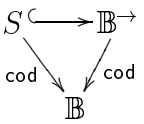
(In fact, the arrow $M \to N$ is necessarily an S-morphism as well [Bor94, Proposition 5.5.4, Volume 1].) We often identify S-morphisms $M \rightarrowtail B$ with their domains M. Given an arrow $f : A \to B$ in $\mathbb{B}$, the substitution functor

$$f^* : S_B \longrightarrow S_A$$

is defined by pullback along f, so that the inclusion

$$S \hookrightarrow \mathbb{B}^{\to}$$

is a fibred functor (commutes with substitution functors), as in the diagram below.

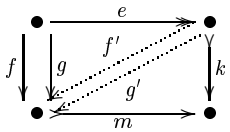


Throughout what follows, we assume that $\mathbb{B}$ has all pullbacks.

The next result is not relevant in what follows, but is worth mentioning. It states that the fibration resulting from a factorization system only has trivial maps in its fibres—and hence no "proof objects"—if and only if the maps in S are actual monos.

Lemma 2.4 *The fibration* $\mathsf{cod}\downarrow^{S}_{\mathbb{B}}$ *associated with a factorization system* $\langle H,\ S\rangle$ *is a fibred pre-order if and only if each map in* S *is a monomorphism in* $\mathbb{B}$.

Proof. The if-part of the statement is trivial, but the only-if-part requires some work. Consider a situation $m \circ f = m \circ g = h$, say, where $m \in S$. Factoring $h = k \circ e$ yields two commuting diagrams, with two diagonals f', g', as in:



This yields $f' = g'$, because we have a fibred pre-order, and thus $f = g$. □

As we will see, the image factorizations for a given factorization system $\langle H,\ S\rangle$ yields not merely a fibration, but a bifibration. That is, each substitution functor u^* has a left adjoint. We will make explicit use of this adjoint in the following, and so we introduce the appropriate terminology here.

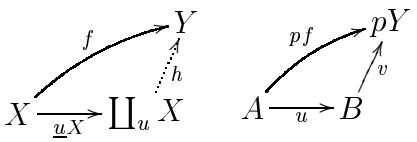
Definition 2.5 *Let* $\begin{smallmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{smallmatrix}$ *be given. We say that* p *is an* opfibration *if*

$$p^{\mathsf{op}}:\mathbb{E}^{\mathsf{op}}\longrightarrow\mathbb{B}^{\mathsf{op}}$$

is a fibration, that is, if above each morphism $pX\to B$ *in* $\mathbb{B}$*, there is an op-Cartesian morphism* $X\to Y$ *in* $\mathbb{E}$*.*

If p *is both a fibration and an opfibration, we say that* p *is a* bifibration.

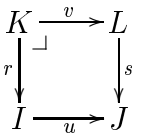
Let $\begin{smallmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{smallmatrix}$ be a bifibration. Let $u:A\to B$ be given. For $X\in\mathbb{E}_A$, we denote by $\underline{u}X:X\to\coprod_u X$ the op-Cartesian morphism above u with domain X. We have the following characterization of $\underline{u}X$: For any object Y and arrow $f:X\to Y$ in $\mathbb{E}$ such that pf factors through u, say via v, there is a unique $h:\coprod_u X\to Y$ over v such that $h\circ\underline{u}X=f$.



Recall Lemma 9.1.2 in [Jac99].

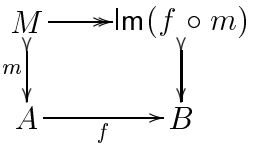
Lemma 2.6 *A fibration* $\begin{smallmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{smallmatrix}$ *is a bifibration just in case each substitution functor* $u^*:\mathbb{E}_B\to\mathbb{E}_A$ *has a left adjoint, denoted* $\coprod_u:\mathbb{E}_A\to\mathbb{E}_B$*.*

Remark 2.7 A bifibration $p:\mathbb{E}\to\mathbb{B}$ is said to satisfy Beck-Chevalley (for coproducts) just in case, for every pullback square in $\mathbb{B}$,



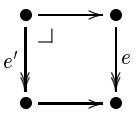
the canonical natural transformation $\coprod_v r^*\Rightarrow s^*\coprod_u$ is an isomorphism. We say in this case that p *has coproducts*. The distinction is non-trivial: not all bifibrations satisfy Beck-Chevalley condition, as we see in Lemma 2.8.

Given a factorization system $\langle H,\,S\rangle$ for $\mathbb{B}$, the codomain fibration $\begin{smallmatrix}S\\ {\scriptstyle \mathsf{cod}}\downarrow\\ \mathbb{B}\end{smallmatrix}$ is a bifibration. Indeed, for any $f:A\to B$ in $\mathbb{B}$, let $\mathsf{Im}(f\circ -):S_A\to S_B$ be the functor taking $m:M\rightarrowtail A$ to the image of $f\circ m$, as shown below.



It's easy to check that $\mathsf{Im}(f \circ -) \dashv f^*$. Moreover, the induced bifibration satisfies Beck-Chevalley just in case the factorization system is stable.

A factorization system $\langle H, S\rangle$ is *stable* just in case H is stable under pullbacks, i.e., for every pullback like so,



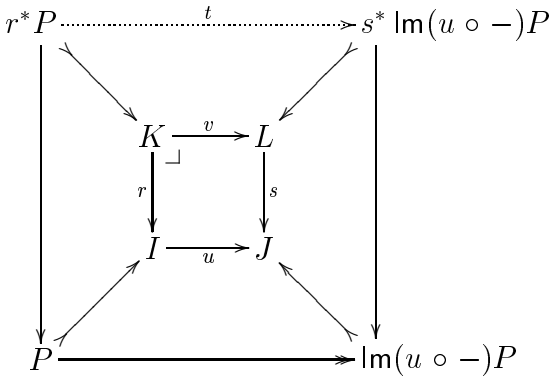
if $e \in H$, then $e' \in H$.

Lemma 2.8 *The bifibration* $\begin{smallmatrix} S \\ \mathsf{cod}\downarrow \\ \mathbb{B} \end{smallmatrix}$ *induced by a factorization system* $\langle H, S\rangle$ *has coproducts (i.e., satisfies the Beck-Chevalley condition) iff* $\langle H, S\rangle$ *is stable.*

Proof. Suppose that $\langle H, S\rangle$ is stable. Given a pullback diagram in $\mathbb{B}$,

$$\begin{array}{ccc} K & \xrightarrow{v} & L \\ {\scriptstyle r}\downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ I & \xrightarrow[u]{} & J \end{array}$$

we wish to show that $\mathsf{Im}(v \circ -)r^* \cong s^* \mathsf{Im}(u \circ -)$. Let $P \twoheadrightarrow I$ be given and consider the diagram below, where $t : r^*P \to s^* \mathsf{Im}(u \circ -)P$ is the mediating morphism for the pullback on the right.



Since the center and left faces are pullbacks, the square

$$\begin{array}{ccc} r^*P & \longrightarrow & L \\ \downarrow & \lrcorner & \downarrow \\ P & \longrightarrow & J \end{array}$$

is also a pullback. Since the right face is a pullback as well, then the outermost square is also one. Hence, by stability, t is an H-morphism. This yields a factorization of the composite

$$r^*P \rightarrowtail K \xrightarrow{v} L$$

into an H-morphism followed by a S-morphism. Since such factorizations are unique up to isomorphism, we see that $\mathsf{Im}(v \circ -)r^* \cong s^* \,\mathsf{Im}(u \circ -)$.

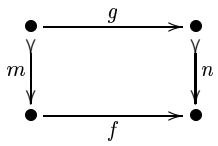
Conversely, suppose that $\mathsf{cod} : S \to \mathbb{B}$ has coproducts and the arrow $u : I \twoheadrightarrow J$ is an H-morphism. We wish to show that the pullback v of u along s is also an H-morphism. By assumption, $\mathsf{Im}(v \circ -)r^* \cong s^* \,\mathsf{Im}(u \circ -)$. Thus,

$$\mathsf{Im}\, v \cong \mathsf{Im}(v \circ -)\,\mathsf{id}_A \cong \mathsf{Im}(v \circ -)r^*\,\mathsf{id}_B \cong s^*\,\mathsf{Im}(u \circ -)\,\mathsf{id}_B \cong s^*\,\mathsf{id}_D \cong \mathsf{id}_C,$$

and so $v \in H$. □

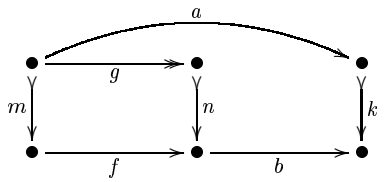
The following lemma gives a characterization of H-morphisms in a factorization system $\langle H,\, S\rangle$ in terms of the codomain bifibration. Explicitly, the H-morphisms are those maps $g \in \mathbb{B}$ such that, for every $f \in \mathbb{B}$ such that (g, f) forms a morphism in S, we have (g, f) is op-Cartesian (over f).

Lemma 2.9 *Consider the codomain bifibration* $\begin{array}{c} S \\ \downarrow \\ \mathbb{B} \end{array}$*, induced by a factorization system* $\langle H,\, S\rangle$*, and an arbitrary map* g *in* $\mathbb{B}$*. Then:* $g \in H$ *if and only if each morphism in the total category* S *of the form*

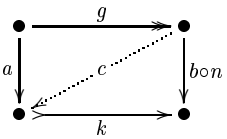


is op-Cartesian—so that $n \cong \coprod_f m$.

Proof. First, suppose $g \in H$, so that we have to prove that the above square is op-Cartesian. Consider therefore the following situation in the category S:

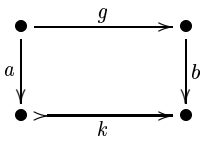


It gives rise to the square:

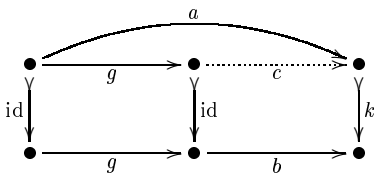


The resulting diagonal c forms an appropriate mediating map (c, b) from n to k in S.

Conversely, assume for a given map g in $\mathbb{B}$ that each square as in the lemma is op-Cartesian. For an arbitrary square,



we have to find a diagonal c. It is obtained by considering the pair (g, g) as an op-Cartesian map between identities:



□

2.1 Subset types

Given a fibration $\begin{matrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{matrix}$, we view the category $\mathbb{E}$ as providing predicates over the types in $\mathbb{B}$, where the functor p takes a predicate to the type of its free variable. If p has a right adjoint $\top\colon\mathbb{B}\to\mathbb{E}$ such that $p\top = \mathsf{id}$, then this adjoint picks out the maximal or "true" predicate for each type. That is, for each $X \in \mathbb{E}_B$, there is a unique vertical map $X \to \top_B$. A right adjoint $\{-\}\colon\mathbb{E}\to\mathbb{B}$ to $\top$ is interpreted as mapping a predicate to its extension in $\mathbb{B}$, i.e.,

$$X \mapsto \{b\colon B \mid X(b)\}.$$

Definition 2.10 *Let* $\begin{matrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{matrix}$ *be a given fibration. We say that* p has subset types, *if* p *has a right adjoint* $\top\colon\mathbb{B}\to\mathbb{E}$, *where* $p \circ \top = \mathsf{id}_{\mathbb{B}}$, *and* $\top$ *has a further right adjoint* $\{-\}\colon\mathbb{E}\to\mathbb{B}$.

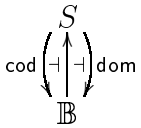
For $X \in \mathbb{E}$, define the *projection* $\pi_X\colon\{X\}\to pX$ to be $p\varepsilon_X$, where

$$\varepsilon\colon\top\{-\}\Longrightarrow\mathsf{id}_{\mathbb{E}}$$

is the counit of the adjunction $\top \dashv \{-\}$. In case that the functor $X \mapsto \pi_X$, from $\mathbb{E}$ to $\mathbb{B}^{\to}$, is full (and faithful), we say that p has *full* subset types.

For a given factorization system $\langle H,\ S\rangle$, the codomain fibration $\begin{matrix}S\\ \downarrow\\ \mathbb{B}\end{matrix}$ has full subset types. The "truth" functor $\mathbb{B}\to S$ is given by $A \mapsto (\mathsf{id}_A\colon A\to A)$, and its right adjoint $S\to\mathbb{B}$ is the domain functor $m \mapsto \mathsf{dom}(m)$. Thus, we have

the following situation, where the middle functor $\mathbb{B} \to S$ is id.



The following observations will be useful later on.

Lemma 2.11 *Suppose the fibration p as above is given with full subset types.*

(i) *If the projection π_X of $X \in \mathbb{E}$ is an isomorphism, then the unique vertical map $!: X \to \top_{pX}$ is an also an isomorphism;*

(ii) *For each X, the counit $\varepsilon_X : \top_{\{X\}} \to X$ is op-Cartesian over π_X.*

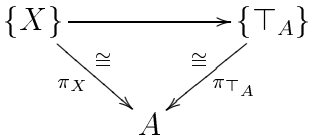
(iii) *The functor $\pi : \mathbb{E} \to \mathbb{B}^{\to}$ preserves and reflects Cartesian morphisms.*

Proof.

(i) We first note that the terminal object functor $\top$ is full and faithful (since $p \circ \top = 1$). Hence the unit η of the adjunction $\top \dashv \{-\}$ is an isomorphism (see [ML71, IV, §3, Theorem 1]. The inverse of $\eta_A : \{\top_A\} \to A$ is $\pi_{\top_A}$ using a triangular identity:

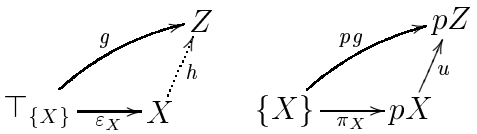
$$\pi_{\top_A} \circ \eta_A = p\big(\varepsilon_{\top_A} \circ \top(\eta_A)\big) = \mathsf{id}\,.$$

Now if $\pi_X : \{X\} \to pX$, for $X \in \mathbb{E}$ over $A \in \mathbb{B}$ is an isomorphism, we get a diagram:



Because subset types are full, we have a unique vertical isomorphism $f : X \to \top_A$ with $\{f\} = (\pi_{\top_A})^{-1} \circ \pi_X$.

(ii) Let g, u be given so that the diagram on the right commutes.



Let $\widetilde{g} : \{X\} \to \{Z\}$ be the adjoint transpose of g. Then

$$u \circ \pi_X = pg = p(\varepsilon_Z \circ \top\widetilde{g}) = \pi_Z \circ \widetilde{g}.$$

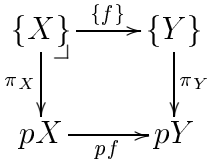
Hence, by fullness, there is a unique $h : X \to Z$ such that $\{h\} = \widetilde{g}$ and $ph = u$. One calculates

$$h \circ \varepsilon_X = \varepsilon_Z \circ \top\{h\} = \varepsilon_Z \circ \top\widetilde{g} = g.$$

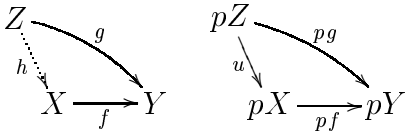
Uniqueness is left to the reader..

(iii) The functor π preserves Cartesian morphisms ([Jac99, Lemma 4.6.2(iii)], where the assumption that p is a pre-order fibration is not necessary), so we concentrate on reflection.

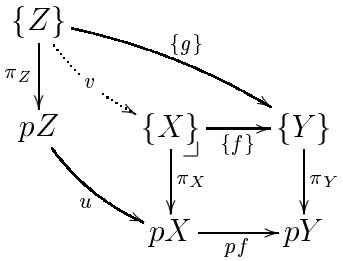
Suppose that the square below is a pullback in $\mathbb{B}$.



Our aim is to show that f is Cartesian. Let $g : Z \to Y$ and $u : pZ \to pX$ be given, as in the diagrams below.



We must show there is a unique h over u, as shown, making the left hand diagram commute. Consider the diagram below.



The map $v : \{Z\} \to \{X\}$ is the mediating morphism for the pullback. By fullness, there is a (unique) $h : Z \to X$ such that $\{h\} = v$ and $ph = u$. Faithfulness ensures that $f \circ h = g$, as desired.

□

Let $\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{array}$ be a bifibration with subset types. Let $u : A \to B$ be given and let $\delta : \mathsf{id} \Rightarrow u^* \coprod_u$ be the (vertical) unit of the adjunction $\coprod_u \dashv u^*$. For $X \in \mathbb{E}_A$, one can see that $\underline{u}X$ is given by the composite

$$X \xrightarrow{\delta_X} u^* \coprod_u X \xrightarrow{\overline{u}\coprod_u X} \coprod_u X.$$

For each $A \in \mathbb{B}$, we define

$$\kappa_u : A \longrightarrow \{\coprod_u \top_A\}$$

to denote the adjoint transpose of $\underline{u}\top_A : \top_A \to \coprod_u \top_A$, i.e., $\kappa_u = \{\underline{u}\top_A\} \circ \eta_A$.

We come now to the final definition of the section. It appears as Definition 10.5.2 in [Jac99]. The terminology is not ideal in this setting, since we assume only that p is a bifibration, and not that it has coproducts.

Definition 2.12 *Let $p:\mathbb{E}\to\mathbb{B}$ be a bifibration with full subset types. We say that p* admits strong coproducts along subset projections *just in case, for every $X \in \mathbb{E}$, $Y \in \mathbb{E}_{\{X\}}$, the canonical arrow $\{\underline{\pi_X} Y\}$ is an isomorphism.*

$$\begin{array}{ccc} \{Y\} & \xrightarrow{\{\underline{\pi_X} Y\}} & \{\coprod_{\pi_X} Y\} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \{X\} & \xrightarrow[\pi_X]{} & pX \end{array}$$

Basically, this says that the subset projections are closed under composition. The terminology "strong coproducts" comes from dependent type theory [**?**]. There, dependent sums $\Sigma_{x:\,A}B(x)$ of one type $B(x)$ indexed over inhabitants $x\colon A$ of another type exist. Inhabitants of such a sum type are (dependent) pairs $\langle a, b\rangle\colon \Sigma_{x:\,A}B(x)$, where $a\colon A$ and $b\colon B[a/x]$. These sum types standardly have a first projection map $\pi_1:(\Sigma_{x:\,A}B(x))\to A$. There are two natural elimination rules, a so-called 'weak' and a 'strong' one. Characteristic for the strong version is the existence of a second project $\pi_2 z\colon B[(\pi_1 z)/x]$ for a pair $z\colon \Sigma_{x:\,A}B(x)$. This is what is captured by the above definition. See [Jac99, Chapter 10] for the details.

Note that for any factorization system $\langle H,\, S\rangle$, the bifibration cod admits strong coproducts with respect to projections. Indeed, let $m:M\rightarrowtail B$ and $n:B\rightarrowtail C$ be given. Then, the relevant diagram for strong coproducts along subset projections is the diagram below, where the top arrow is an isomorphism just because S is closed under compositions.

$$\begin{array}{ccc} M & \xrightarrow{\cong} & \mathsf{Im}(n\circ m) \\ {\scriptstyle m=\pi_m}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \rightarrowtail_{n=\pi_n} & C \end{array}$$

Thus, we have proved the following theorem.

Theorem 2.13 *Let $\mathbb{B}$ have a factorization system $\langle H,\, S\rangle$. Then the bifibration $\mathsf{cod}\downarrow^{S}_{\mathbb{B}}$ has full subset types and admits strong coproducts along subset projections. Additionally, H is stable if and only if $\downarrow^{S}_{\mathbb{B}}$ has coproducts.*

This completes the first part of our task. We have shown that factorization systems induce bifibrations with full subset types and strong coproducts along subset projections. In the next section, we will see how to construct a factorization system from such a bifibration.

3 From bifibrations to factorization systems

We begin by making explicit the relevant features of the fibration constructed in Section 2.

Definition 3.1 *Let* $\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{array}$ *satisfy the following conditions.*

(i) *p is a bifibration;*

(ii) *p has full subset types;*

(iii) *p has strong coproducts along subset projections;*

We call such p a factorization fibration.

Throughout this section, we work in a factorization fibration $\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{array}$.

We will construct an associated factorization system $\langle H,\, S\rangle$. The abstract epis H will consist of composites

$$A \xrightarrow{u} \{X\} \xrightarrow{\cong} B$$

where the adjoint transpose $\widetilde{u}\colon \top_A \to X$ is op-Cartesian (over $\pi_X \circ u$). The abstract monos S consist of composites

$$B \xrightarrow{\cong} \{X\} \xrightarrow{\pi_X} pX.$$

Lemma 3.2 *Any map $f\colon A \to B$ in $\mathbb{B}$ can be factored as*

$$A \xrightarrow{u} \{X\} \xrightarrow{\pi_X} B,$$

for some u such that $\widetilde{u}\colon \top_A \to X$ is op-Cartesian, $X \in \mathbb{E}_B$.

Proof. Take the factorization

$$A \xrightarrow{\kappa_f} \{\textstyle\coprod_f \top_A\} \xrightarrow{\pi} B,$$

where, recall, κ_f is the transpose of the op-Cartesian map $\underline{f}\top_A\colon \top_A \to \coprod_f \top_A$. This works, since:

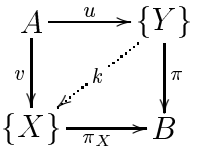
$$\begin{aligned}\pi \circ \kappa_f &= p\big(\varepsilon_{\coprod_f \top_A} \circ \top\kappa_f\big)\\ &= p(\underline{f}\top_A)\\ &= f.\end{aligned}$$

□

The next lemma is the first step toward the diagonal fill-in property. It says that, for any $f\colon A \to B$, the factorization from Lemma 3.2 is the "minimal" factorization through a projection $\pi\colon \{X\} \to B$. In other words, this

lemma establishes the diagonal fill-in property restricted to the inclusion of (extensions of) predicates into B.

Lemma 3.3 *In a factorization fibration, suppose that the outer square in the diagram below commutes, where the adjoint transpose $\widetilde{u}:\top_A \to Y$ of u is op-Cartesian. Then there is a unique morphism $k:\{Y\}\to\{X\}$ making each triangle commute.*

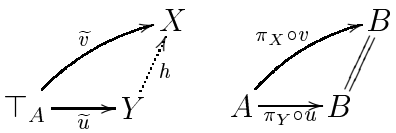


Proof. The proof is a modification of [Jac99, Example 4.6.5], to accommodate the relaxation of the assumption that $\mathbb{E}$ is a pre-order fibration.

Let $\widetilde{u}$ and $\widetilde{v}$ represent the adjoint transposes of u, v, respectively. Note that

$$p(\widetilde{u}) = p(\varepsilon_Y \circ \top u) = \pi_Y \circ u = \pi_X \circ v = p(\widetilde{v}),$$

and, by assumption, $\widetilde{u}$ is op-Cartesian. Let h be the vertical map shown in the diagram below.



Since h is vertical, it is clear that the lower triangle commutes for $k = \{h\}$. To see also that $v = \{h\} \circ u$, we take transposes on both sides and calculate:

$$\varepsilon_X \circ \top v = \widetilde{v} = h \circ \widetilde{u} = h \circ \varepsilon_Y \circ \top u = \varepsilon_X \circ \top(\{h\} \circ u).$$

Uniqueness of k follows from fullness of subset types. □

The next result says that twice taking images is the same as doing it once. It is the key lemma for the construction of our factorization system.

Lemma 3.4 *Let $f:A\to C$ be given, and consider the canonical factorization of f,*

$$A \xrightarrow{\kappa_f} \{\textstyle\coprod_f \top_A\} \xrightarrow{\pi} C.$$

Then the projection

$$\pi:\{\textstyle\coprod_{\kappa_f} \top_A\} \longrightarrow \{\textstyle\coprod_f \top_A\}$$

is an isomorphism—and thus, by Lemma 2.11(i), the unique map

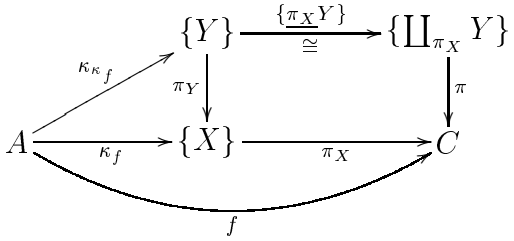
$$!:\textstyle\coprod_{\kappa_f} \top_A \longrightarrow \top_{\{\coprod_f \top_A\}}$$

is an isomorphism.

Proof. Let's use $X, Y \in \mathbb{E}$ as abbreviations:

$$\top_A \xrightarrow{\underline{f}\top_A} \textstyle\coprod_f \top_A \stackrel{\text{def}}{=} X \qquad\qquad \top_A \xrightarrow{\underline{\kappa_f}\top_A} \textstyle\coprod_{\kappa_f} \top_A \stackrel{\text{def}}{=} Y$$

where, recall, $\kappa_f : A \to \{X\}$ is the transpose of $\underline{f}\top_A$, so $\kappa_f = \{\underline{f}\top_A\} \circ \eta_A$. This yields a diagram:



We thus have to prove that π_Y is an isomorphism.

The composite $\top_A \xrightarrow{\underline{\kappa_f}\top_A} Y \xrightarrow{\underline{\pi_X} Y} \coprod_{\pi_X} Y$ is op-Cartesian over $\pi_X \circ \kappa_f = f$. Since $\underline{f}\top_A$ is also op-Cartesian over f, there is a unique vertical isomorphism $h : \coprod_{\pi_X} Y \to X$ with

$$h \circ \underline{\pi_X} Y \circ \underline{\kappa_f}\top_A = \underline{f}\top_A.$$

We claim that

$$\{h\} \circ \{\underline{\pi_X} Y\} = \pi_Y$$

which yields that π_Y is an isomorphism, as required.

Because we have full subset types there is a unique vertical isomorphism

$$g : \coprod_{\pi_X} Y \longrightarrow X$$

with $\{g\} = \pi_Y \circ \{\underline{\pi_X} Y\}^{-1}$. We are done if we can prove $g = h$. But this follows from $g \circ \underline{\pi_X} Y \circ \underline{\kappa_f}\top_A = \underline{f}\top_A$, which follows if we take transposes on both sides:

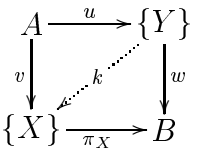
$$\begin{aligned}
\{g \circ \underline{\pi_X} Y \circ \underline{\kappa_f}\top_A\} \circ \eta_A &= \pi_Y \circ \{\underline{\pi_X} Y\}^{-1} \circ \{\underline{\pi_X} Y\} \circ \{\underline{\kappa_f}\top_A\} \circ \eta_A \\
&= \pi_Y \circ \kappa_{\kappa_f} \\
&= \kappa_f \\
&= \{\underline{f}\top_A\} \circ \eta_A.
\end{aligned}$$

□

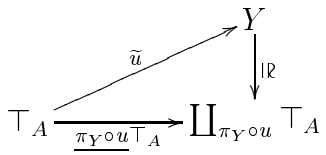
With the previous result in hand, we may strengthen our diagonal fill-in property from Lemma 3.3 and remove the assumption that both maps into B are projections.

Lemma 3.5 *Let $u : A \to \{Y\}$ be given such that its transpose, $\widetilde{u} : \top_A \to Y$, is op-Cartesian. Then there is a unique morphism $k : \{Y\} \to \{X\}$ making each*

triangle commute.

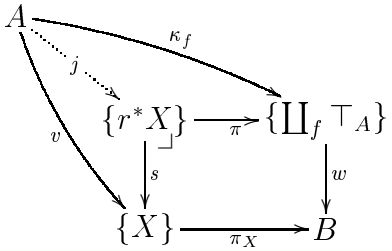


Proof. Whenever $\widetilde{u}:\top_A \to Y$ is op-Cartesian, there is an evident vertical isomorphism, shown below.

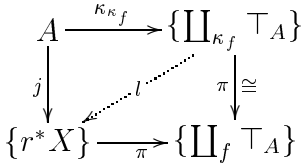


Thus, without loss of generality, we may take $u:A \to \{Y\}$ to be of the form $\kappa_f:A \to \{\coprod_f \top_A\}$, for $f = \pi_Y \circ u:A \to pY$.

Take the pullback shown below, and let j be the evident mediating map $A \to \{r^*X\}$.



This yields the commutative square below.



The projection on the right is an isomorphism by Lemma 3.4 and the diagonal l is given by Lemma 3.3. Take $k = s \circ l \circ \pi^{-1}$. □

Finally, we are in a position to construct the factorization system $\langle H,\ S\rangle$, using the previous lemmas. This is the key result of our analysis of factorization systems and bifibrations.

Theorem 3.6 *Let* $\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{array}$ *be a factorization fibration. The fibration* p *induces a factorization system on* $\mathbb{B}$.

Proof. Define

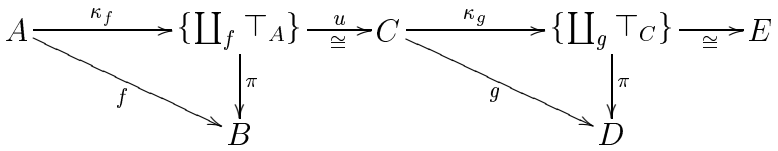
$$H = \{\, A \xrightarrow{u} \{X\} \xrightarrow{\cong} B \;\mid \widetilde{u}:\top_A \longrightarrow X \text{ is op-Cartesian}\},$$
$$S = \{\, B \xrightarrow{\cong} \{X\} \xrightarrow{\pi} pX \;\mid X \in \mathbb{E}\}.$$

Equivalently, we may take H to be the collection of all composites

$$A \xrightarrow{\kappa} \{\coprod \top_A\} \xrightarrow{\cong} B\,,$$

as in the preceding proof. Thus, H is the closure of the collection of "coprojections" $A \rightarrowtail \{\coprod \top_A\}$ under isomorphisms, and S the corresponding closure of the projections $\{X\} \rightarrowtail pX$.

We first check that H and S are closed under composition. For H, consider a situation:



Above the composition $h = g \circ u \circ \kappa_f$, we find the following op-Cartesian map—using Lemma 3.4:

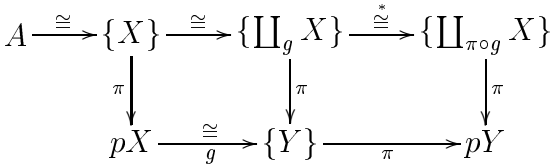
$$\top_A \xrightarrow{\underline{\kappa_f}\top_A} \coprod_{\kappa_f} \top_A \xrightarrow{\cong} \top_{\{\coprod_f \top_A\}} \xrightarrow[\cong]{\top u} \top_C \xrightarrow{\underline{g}\top_C} \coprod_g \top_C$$

This yields an isomorphism

$$\{\coprod_h \top_A\} \cong \{\coprod_g \coprod_u \coprod_{\kappa_f} \top_A\} \cong \{\coprod_g \coprod_u \top_{\coprod_{\kappa_f} \top_A}\} \cong \{\coprod_g \top_C\} \cong E,$$

as required.

Similarly, the composite of two S morphisms $A \rightarrowtail pX$ and $pX \rightarrowtail pY$ is given in the following diagram,



where the marked isomorphism $\overset{*}{\cong}$ is justified by strong coproducts along subset projections.

By Lemma 3.2, any map f in $\mathbb{B}$ can be factored into an H-morphism followed by an S-morphism.

The diagonal fill-in property was proved as Lemma 3.5. □

The next two theorems show that this construction is coherent, in a sense. In particular, if we begin with a factorization system $\langle H, S\rangle$, and construct the system associated with the codomain fibration $\overset{S}{\downarrow}{}_{\mathbb{B}}$, we get $\langle H, S\rangle$ again. On the other hand, if we begin with a suitable fibration $p\colon \mathbb{E} \rightarrow \mathbb{B}$, construct the associated factorization system $\langle H, S\rangle$ and consider the codomain fibration

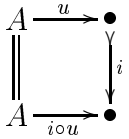
$S \to \mathbb{B}$, we do not get $\mathbb{E} \to \mathbb{B}$ again, but an equivalent fibration. As a corollary, one notes that the fibrational construction is idempotent. Do it a second time, and we get $S \to \mathbb{B}$ again.

Theorem 3.7 *Let $\langle H, S\rangle$ be a factorization system on $\mathbb{B}$. Let $\langle H', S'\rangle$ be the factorization system constructed via the codomain bifibration $\begin{array}{c} S \\ \downarrow \\ \mathbb{B} \end{array}$, as in Theorem 3.6. Then $H' = H$ and $S' = S$.*

Proof. By the proof of Theorem 3.6, we have

$$\begin{aligned} H' &= \{\, A \xrightarrow{u} \mathsf{dom}(i) \xrightarrow{\cong} B \;\mid \widetilde{u} : \mathsf{id}_A \longrightarrow i \text{ is op-Cartesian}, i \in S\} \\ &= \{\, A \xrightarrow{u} \mathsf{dom}(i) \;\mid \widetilde{u} : \mathsf{id}_A \longrightarrow i \text{ is op-Cartesian}, i \in S\}, \\ S' &= \{\, B \xrightarrow{\cong} \mathsf{dom}(i) \xrightarrow{i} \mathsf{cod}(i) \;\mid i \in S\} = S. \end{aligned}$$

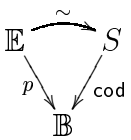
Let $i \in S$ and $u : A \to \mathsf{dom}\, i$ be given, where $u \in H$. Then, applying Lemma 2.9, the morphism



is op-Cartesian. But, this morphism is the transpose of u, and thus, $u \in H'$.

Conversely, suppose we have $u : A \to \mathsf{dom}(i)$, with $\widetilde{u} : \mathsf{id}_A \to i$ op-Cartesian (over $i \circ u$). Let $i \circ u = m \circ e$ be the $\langle H, S\rangle$ factorization of $i \circ u$, so that e is also op-Cartesian over $i \circ u$. Thus, we have an isomorphism f such that $u = f \circ e$ and hence $u \in H$. □

Theorem 3.8 *Let $\begin{array}{c} \mathbb{E} \\ {\scriptstyle p}\downarrow \\ \mathbb{B} \end{array}$ be a factorization fibration and let $\langle H, S\rangle$ be the corresponding factorization system, constructed via Theorem 3.6. Then we have an equivalence:*



Proof. Let $\pi : \mathbb{E} \to S$ be the evident functor taking $X \in \mathbb{E}$ to $\{X\} \xrightarrow{\pi_X} pX$ in S, where

$$S = \{\, B \xrightarrow{\cong} \{Y\} \xrightarrow{\pi} pY \,\}.$$

We will show that π is a "bifibred equivalence". Since it also commutes with the truth and comprehension functors, we may conclude that π is an equivalence of factorization fibrations.

Clearly, π is full, faithful and essentially surjective on objects, so it is an equivalence of categories [ML71, Theorem IV.4.1]. Furthermore, given

$$B \xrightarrow{\cong} \{Y\} \xrightarrow{\pi} pY$$

in S, the isomorphism $(B \to pY) \cong (\{Y\} \to pY)$ is vertical and π preserves and reflects Cartesian morphisms by Lemma 2.11(iii). We must show that π preserves and reflects op-Cartesian morphisms.

Suppose that $f : X \to Y$ is op-Cartesian in $p : \mathbb{E} \to \mathbb{B}$. We must show that the square

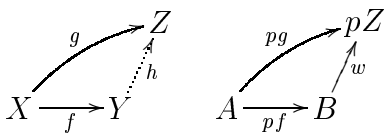
$$\begin{array}{ccc} \{X\} & \xrightarrow{\{f\}} & \{Y\} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ A & \xrightarrow[pf]{} & B \end{array}$$

is op-Cartesian in $\mathsf{cod} : S \to \mathbb{B}$. For this, it suffices (by Lemma 2.9) to show that $\{f\} \in H$. The adjoint transpose of $\{f\}$ is given by

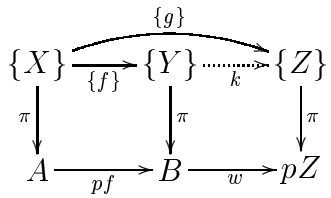
$$\varepsilon_Y \circ \top\{f\} = f \circ \varepsilon_X,$$

which is op-Cartesian in $\mathbb{E}$ by Lemma 2.11(ii). Hence, the functor π preserves op-Cartesian morphisms.

To see that π also reflects op-Cartesian maps, suppose that $f : X \to Y$ is given such that the square above is op-Cartesian in $\mathsf{cod} : S \to \mathbb{B}$. Let $g : X \to Z$ and $w : B \to pZ$ be given as in the diagrams below.



Since $\pi_f = (\{f\}, pf)$ is op-Cartesian in cod, there is a map $k : \{Y\} \to \{Z\}$ making the diagram below commute.

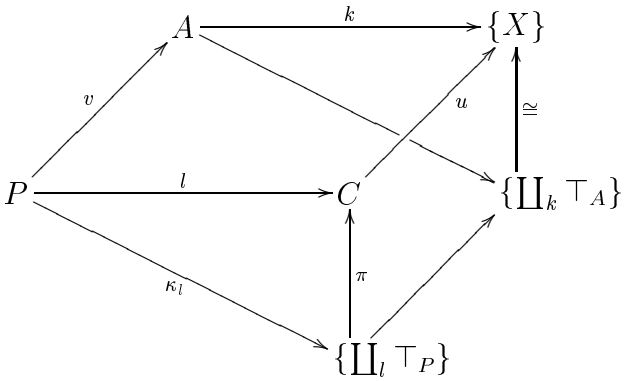


By fullness, there is a unique map $h : Y \to Z$ such that $\{h\} = k$ and $ph = w$. By faithfulness, $h \circ f = g$. □

Corollary 3.9 *Let* $\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{array}$ *be a factorization fibration. The induced factorization system* $\langle H, S\rangle$ *in Theorem 3.6 is stable if and only if* p *has coproducts (i.e., satisfies Beck-Chevalley).*

Proof. Suppose that the induced factorization system $\langle H, S\rangle$ is stable. Then the bifibration $\begin{array}{c}S\\ {\scriptstyle \mathsf{cod}}\downarrow\\ \mathbb{B}\end{array}$ satisfies Beck-Chevalley (Lemma 2.8) and is equivalent to p (Theorem 3.8).

Conversely, suppose that p satisfies Beck-Chevalley; we will prove that the induced factorization system $\langle H,\, S\rangle$ is stable. Since isomorphisms are stable under pullback, it suffices to show that the pullback of a map $k : A \to \{X\}$, where $\widetilde{k} : \top_A \to X$ is op-Cartesian, is again an H-morphism. Let $u : C \to \{X\}$ be given and consider the diagram below, where P is the pullback of k along u.



By the Beck-Chevalley condition, we have that

$$u^* \coprod_k \top_A \cong \coprod_l v^* \top_A \cong \coprod_l \top_P,$$

and hence, $\pi : \{\coprod_l \top_P\} \to C$ is an isomorphism. Thus, the composite $l = \pi \circ \kappa_l$ is an H-morphism. □

3.1 Fibred diagonal fill-in property

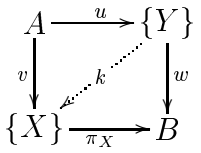
In [**?**, end of §§3.1] there is a remark (without proof) that is highly relevant in our setting. It relates "regular fibrations" with an additional property to stable factorization systems. The property used there is repeated in the next definition. It is a translation into the language of fibrations of the familiar diagonal fill-in property. Below we will show that it is equivalent to our logical formulation using strong coproducts.

Definition 3.10 *Let* $\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{array}$ *have subset types. We say that* p *has* fibred diagonal fill-ins *just in case, if* $f : X \to Y$ *is op-Cartesian, then so is* $\top\{f\} : \top_{\{X\}} \to \top_{\{Y\}}$.

The formulation used in this definition comes from [**?**], but the terminology 'fibred diagonal fill-ins' is ours. The following proposition motivates the terminology.

Proposition 3.11 *A fibration* $\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{array}$ *with subset types and fibred diagonal fill-ins also has (ordinary) diagonal fill-ins in the sense of Lemma 3.5, namely, for*

every commutative square

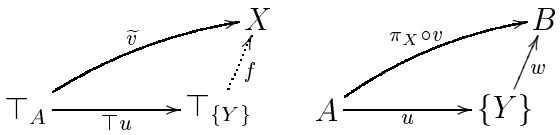


where the adjoint transpose $\widetilde{u}:\top_A \to Y$ *of* u *is op-Cartesian, there is a unique diagonal* $k:\{Y\}\to\{X\}$*, as shown, making each triangle commute. Moreover, if* p *has ordinary diagonal fill-ins and subset types are full, then* p *has fibred diagonal fill-ins.*

Proof. Suppose that p has the fibred diagonal fill-in property, and let the diagram above be given, where $\widetilde{u}:\top_A \to Y$ is op-Cartesian. We wish to show that there is a unique arrow $k:\{Y\}\to\{X\}$, as shown, making the diagram commute.

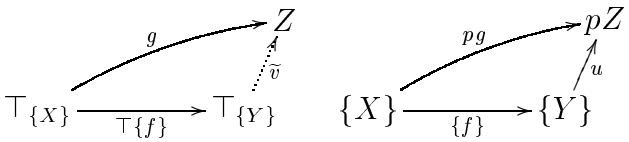
By assumption, $\top\{\widetilde{u}\}$ is op-Cartesian, and because $\top$ is full and faithful, η_A is an isomorphism (as noted in Lemma 2.11(i)). Hence the composite $\top\{\widetilde{u}\} \circ \top\eta_A = \top u$ is op-Cartesian over u.

Thus, there is a unique morphism $f:\top_{\{Y\}}\to X$ over w such that the left hand diagram commutes.



Let $k:\{Y\}\to\{X\}$ be the adjoint transpose of f. Then, $\varepsilon_X \circ \top(k \circ u) = f \circ \top u = \widetilde{v} = \varepsilon_X \circ \top v$, and so the upper triangle commutes. Also, $\pi_X \circ k = p(\varepsilon_X \circ \top k) = pf = w$, and so the bottom triangle commutes, too. Uniqueness follows from the fact that the morphism $\top u$ is op-Cartesian.

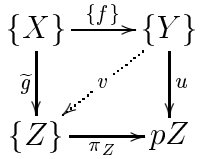
For the converse, assume that p has ordinary diagonal fill-ins and suppose that $f:X\to Y$ is op-Cartesian. In order to see that $\top\{f\}$ is also op-Cartesian, let $g:\top_{\{X\}}\to Z$ and $u:\{Y\}\to pZ$ be given as in the diagrams below.



We will show that there is a unique $\widetilde{v}:\top_{\{Y\}}\to Z$ over u, as shown, making the left hand diagram commute.

The adjoint transpose of $\{f\}$ is given by $\varepsilon_Y \circ \top\{f\} = f \circ \varepsilon_X$ and hence is op-Cartesian by Lemma 2.11(ii). Hence, we apply the ordinary diagonal fill-in property of Lemma 3.5 to the commutative square below, yielding

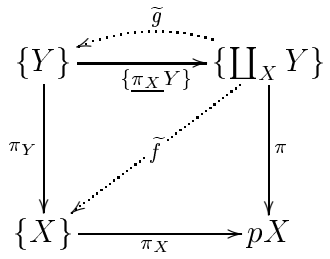
$v : \{Y\} \to \{Z\}$, as shown.



Then $\widetilde{v} \circ \top\{f\} = \varepsilon_Z \circ \top(v \circ \{f\}) = \varepsilon_Z \circ \top\widetilde{g} = g$ and $p\widetilde{v} = p(\varepsilon_Z \circ \top v) = \pi_Z \circ v = u$. Again, uniqueness is left to the reader. □

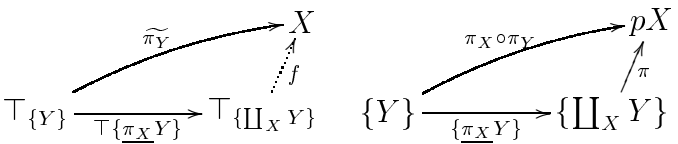
Proposition 3.12 *Let* $\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{array}$ *be a bifibration with full subset types. Then* p *has fibred diagonal fill-ins iff* p *has strong coproducts along subset projections.*

Proof. Suppose that p has fibred diagonal fill-ins and let $X \in \mathbb{E}$ and $Y \in \mathbb{E}_{\{X\}}$ be given, as in the diagram below.



We will show that there is a $\widetilde{g}$, as shown, which is inverse to $\{\underline{\pi_X}Y\}$. To do this, we first show there is an $\widetilde{f}$, also shown, making both triangles commute.

By assumption, $\top\{\underline{\pi_X}Y\} : \top_{\{Y\}} \to \top_{\{\coprod_X Y\}}$ is op-Cartesian over $\{\underline{\pi_X}Y\}$. Let $\widetilde{\pi_Y} : \top_{\{Y\}} \to X$ be the adjoint transpose of π_Y, i.e., $\widetilde{\pi_Y} = \varepsilon_X \circ \top\pi_Y$. Since $\widetilde{\pi_Y}$ is over $\pi_X \circ \pi_Y = \pi \circ \{\underline{\pi_X}Y\}$, there is a unique map $f : \top_{\{\coprod_X Y\}} \to X$ such that $pf = \pi$ and the diagram on the left commutes.



Let $\widetilde{f} : \{\coprod_X Y\} \to \{X\}$ be the adjoint transpose of f.

We calculate

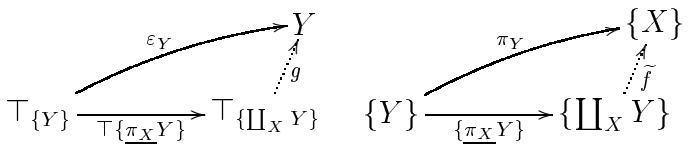
$$\varepsilon_X \circ \top(\widetilde{f} \circ \{\underline{\pi_X}Y\}) = f \circ \top\{\underline{\pi_X}Y\} = \widetilde{\pi_Y} = \varepsilon_X \circ \top\pi_Y,$$

and so the upper triangle commutes. Also,

$$\pi_X \circ \widetilde{f} = p(\varepsilon_X \circ \top\widetilde{f}) = pf = \pi,$$

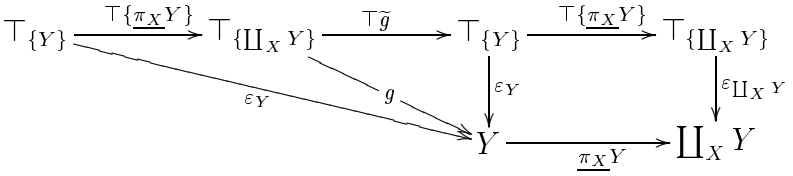
and so the lower triangle commutes as well.

Because the upper triangle commutes, and $\top\{\underline{\pi_X} Y\}$ is op-Cartesian, there is a unique $g : \top_{\{\coprod_X Y\}} \to Y$ such that $pg = \widetilde{f}$ and the left hand diagram commutes.



Let $\widetilde{g} : \{\coprod_X Y\} \to \{Y\}$ be its adjoint transpose.

Since $\varepsilon_Y \circ \top(\widetilde{g} \circ \{\underline{\pi_X} Y\}) = g \circ \top\{\underline{\pi_X} Y\} = \varepsilon_Y$, we see that $\widetilde{g} \circ \{\underline{\pi_X} Y\} = \mathsf{id}_{\{Y\}}$. For the other composite, consider the commutative diagram below.



We see that

$$(\varepsilon_{\coprod_X Y} \circ \top(\underline{\pi_X} Y \circ \widetilde{g})) \circ \top\{\underline{\pi_X} Y\} = \underline{\pi_X} Y \circ \varepsilon_Y = \varepsilon_{\coprod_X Y} \circ \top\{\underline{\pi_X} Y\}.$$

Since $\top\{\underline{\pi_X} Y\}$ is op-Cartesian, and since also

$$\begin{aligned} p(\varepsilon_{\coprod_X Y} \circ \top(\{\underline{\pi_X} Y\} \circ \widetilde{g})) &= p(\underline{\pi_X} Y \circ g) \\ &= \pi_X \circ \widetilde{f} \\ &= p(\varepsilon_X \circ \top \widetilde{f}) \\ &= pf \\ &= p\varepsilon_{\coprod_X Y}, \end{aligned}$$

we conclude that $\varepsilon_{\coprod_X Y} \circ \top(\{\underline{\pi_X} Y\} \circ \widetilde{g}) = \varepsilon_{\coprod_X Y} \circ \top \mathsf{id}$. Hence, $\{\underline{\pi_X} Y\} \circ \widetilde{g} = \mathsf{id}_{\{\coprod_X Y\}}$. This completes the proof that fibred diagonal fill-ins implies strong coproducts along subset projections.

For the converse, suppose that $\begin{array}{c}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{array}$ is a bifibration with full subset types and strong coproducts along subset projections, i.e., a factorization fibration. Apply Lemma 3.5 and Proposition 3.11 to conclude that p has fibred diagonal fill-ins. □

4 (Co-)equational logic in a fibred setting

In this section, we give some motivation for our analysis of factorization systems and associated fibrations. Birkhoff's variety and completeness theorems

and related theorems for quasi-varieties, etc., are essentially logical theorems. That is, the variety theorem states an equivalence between models of an appropriate logical theory and closure conditions on collections of objects, while the completeness theorem states the analogous result for the logic at hand (equational, in the case of [Bir35]). Our aim is to reinterpret these results in a fibred setting, where the predicates in a fibration give the logic and the types provide the semantics for the language.

We present an interpretation of the basic notions of Birkhoff's variety theorem in a fibred setting and give a preliminary result (the "fibred quasi-variety theorem"). It is more natural to inherit the notions of satisfaction not from Birkhoff's classical work, but rather from the dual work in categories of coalgebras. Consequently, our interpretations of the relation $\models$ is inherited from the relevant definition in terms of coequations, found in [Rut00], [GS01], [Kur01], [**?**] and elsewhere. Theorem 4.2 can be considered a generalization of the quasi-covariety theorem, presented as Theorem 3.6.3 in [**?**].

Consider a fibration $\begin{smallmatrix}\mathbb{E}\\ {\scriptstyle p}\downarrow\\ \mathbb{B}\end{smallmatrix}$ with a truth functor $\top : \mathbb{B} \to \mathbb{E}$, that is, a functor $\top$ such that $p \dashv \top$ and $p\top = \mathsf{id}_{\mathbb{B}}$. Given arbitrary objects $E \in \mathbb{E}$ and $B \in \mathbb{B}$, we write $B \models E$ just in case, for every $u : B \to pE$ in $\mathbb{B}$, there is a morphism $\top_B \to u^*E$ in $\mathbb{E}_B$. Intuitively, $B \models E$ just in case for every "valuation" u, there is a "proof" of u^*E. Explicitly, interpreting E as a predicate $E(x)$ where x has type pE, then B satisfies E iff for every $u : B \to pE$, we have a derivation of the predicate $E(u(y))$ where y has type B.

As an illustration, we will show that this relation $\models$ is closely related to cocone projectivity. Consider a category $\mathbb{C}$ with pullbacks and coproducts and construct the discrete cocone fibration via the pullback below.

$$\begin{array}{ccc} \mathsf{DisCoc}(\mathbb{C}) & \longrightarrow & (\mathsf{Fam}(\mathbb{C}))^{\to} \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{cod}} \\ \mathbb{C} & \xrightarrow[\eta]{} & \mathsf{Fam}(\mathbb{C}) \end{array}$$

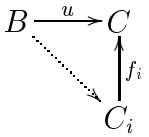
Here, η takes $C \in \mathbb{C}$ to the singleton family (C). The fibration $\mathsf{DisCoc}(\mathbb{C})$ consists of sets of arrows

$$c = \{f_i : C_i \longrightarrow C\}_{i \in I}$$

with common codomain, and the functor p takes such a cocone to its vertex.

This fibration p has a right adjoint $\top$ taking C to the cocone $\{\mathsf{id}_C : C \to C\}$. One may check that, given a cocone c and an object $B \in \mathbb{C}$, we have $B \models c$ just in case, for every $u : B \to C$, there is an $f_i \in C$ such that u factors through

f_i, i.e.,

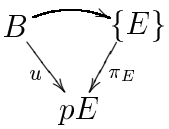


In other words, $B \models c$ iff B is *projective* with respect to the cocone c.

Cocone projectivity was recently discussed in categories of coalgebras in [Hug02], which provided a dualization of the work on cone injectivity and Birkhoff-type theorems found in [NS81].

In the case that our general fibration $\begin{smallmatrix}\mathbb{E}\\ p\downarrow\\ \mathbb{B}\end{smallmatrix}$ has subset types, the relation $\models$ is easily characterized. The following lemma shows that, in this situation, $B \models E$ iff B is projective with respect to the subset projection π_E associated with E.

Lemma 4.1 *Let $p{:}\mathbb{E}\to\mathbb{B}$ have subset types and let $u{:}B\to pE$ be given. Arrows $\top_B\to u^*E$ in $\mathbb{E}_B$ correspond to morphisms $B\to\{E\}$ in $\mathbb{B}$ such that the diagram below commutes.*



Hence, $B \models E$ if and only if every morphism $B\to pE$ in $\mathbb{B}$ factors through π_E, i.e., B is projective with respect to π_E.

Proof. The correspondence maps a vertical morphism $f{:}\top_B\to u^*E$ to

$$\{\overline{u}E \circ f\} \circ \eta_B$$

and an arrow $g{:}B\to\{E\}$ to the mediating morphism $\top_B\to u^*E$ induced by $\widetilde{g}{:}\top_B\to E$. □

The following theorem is the first Birkhoff-type result we have in a fibred setting. The proof is natural and simple. This suggests that fibred categories provide a suitable setting in which to understand Birkhoff-type results. More work is needed to justify this approach, and one would like to explore the deductive completeness results in addition to the model theoretic theorems. We hope to return to this topic in future work.

Theorem 4.2 *Let $\begin{smallmatrix}\mathbb{E}\\ p\downarrow\\ \mathbb{B}\end{smallmatrix}$ be a fibration with subset types and fibred diagonal fill-ins and let $\mathbb{B}$ have all (small) coproducts. Further suppose that for every $A \in \mathbb{B}$, the skeleton category $\mathsf{Skel}(\mathbb{E}_A)$ of the fibre over A is small, i.e., that there is a set $\mathsf{Skel}(\mathbb{E}_A) \subseteq \mathbb{E}_A$ of predicates over A such that for every $X \in \mathbb{E}_A$, there is a $Y \in \mathsf{Skel}(\mathbb{E}_A)$ with $X \cong Y$. Let $\mathbf{V}$ be a full subcategory of $\mathbb{B}$. Then $\mathbf{V} = \mathsf{Mod}\,\mathsf{Th}\,\mathbf{V}$ just in case $\mathbf{V}$ is closed under*

(i) *codomains of arrows* $B \to \{X\}$ *whose adjoint transpose* $\top_B \to X$ *is op-Cartesian, where* $B \in \mathbf{V}$,

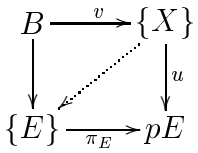
(ii) *codomains of split epis and*

(iii) *coproducts.*

Furthermore, we may replace (ii) above with closure under isomorphisms if p *is a pre-order fibration—which is the standard classical situation with predicates as subsets.*

Proof. Assume that $\mathbf{V} = \mathsf{Mod}\,\mathsf{Th}\,\mathbf{V}$. We start by proving point (i). Accordingly, suppose that $B \in \mathbf{V}$ and let $v : B \to \{X\}$ be given such that $\widetilde{v} : \top_B \to X$ is op-Cartesian. We will prove that if $B \models E$, then also $\{X\} \models E$, to conclude that $\{X\} \in \mathbf{V}$ as well.

Let $u : \{X\} \to pE$ be given, where $B \models E$. By Lemma 4.1, there is a morphism $B \to \{E\}$ making the square below commute.



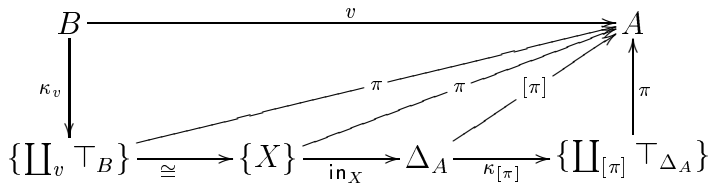
Apply Proposition 3.11 to obtain the diagonal fill-in shown and Lemma 4.1 again to conclude $\{X\} \models E$.

The proofs that $\mathbf{V}$ is closed under codomains of split epis and coproducts are straightforward.

For the converse, assume that $\mathbf{V}$ is closed under (i) – (iii) and we will show that $\mathbf{V} = \mathsf{Mod}\,\mathsf{Th}\,\mathbf{V}$. The inclusion $\mathbf{V} \subseteq \mathsf{Mod}\,\mathsf{Th}\,\mathbf{V}$ is obvious, so let $A \in \mathsf{Mod}\,\mathsf{Th}\,\mathbf{V}$ and we will show that $A \in \mathbf{V}$.

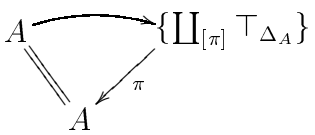
Let Δ_A denote the coproduct $\displaystyle\sum_{\substack{X \in \mathsf{Skel}(\mathbb{E}_A) \\ \{X\} \in \mathbf{V}}} \{X\}$ and let $[\pi] : \Delta_A \to A$ be the evident morphism. We will first show that $\mathbf{V} \models \coprod_{[\pi]} \top_{\Delta_A}$.

Let $B \in \mathbf{V}$ and $v : B \to A$ be given, and we must show that v factors through $\pi : \{\coprod_{[\pi]} \top_{\Delta_A}\} \to A$. By assumption, there is an $X \in \mathsf{Skel}(\mathbb{E}_A)$ such that $\coprod_v \top_B \cong X$. Since $B \in \mathbf{V}$, we have $\{\coprod_v \top_B\} \in \mathbf{V}$ (by (i)) and so $X \in \mathbf{V}$, since $\mathbf{V}$ is closed under isomorphisms. Let $\mathsf{in}_X : \{X\} \to \Delta_A$ be the injection of $\{X\}$ into the coproduct Δ_A. Then we have the following factorization.



Next, we note that $\{\coprod_{[\pi]} \top_{\Delta_A}\} \in \mathbf{V}$. Indeed, $\Delta_A \in \mathbf{V}$ by (iii), and so $\{\coprod_{[\pi]} \top_{\Delta_A}\} \in \mathbf{V}$ by (i).

Now, since $A \in \mathsf{Mod}\,\mathsf{Th}\,\mathbf{V}$, we have that $A \models \coprod_{[\pi]} \top_{\Delta_A}$. Thus, there is a map $A \to \{\coprod_{[\pi]} \top_{\Delta_A}\}$ making the diagram below commute.



Hence, π is a split epi and thus, by (ii), $A \in \mathbf{V}$.

In case the fibration p is pre-ordered, the projection $\pi : \{\coprod_{[\pi]} \top_{\Delta_A}\} \to A$ is monic by [Jac99, Lemma4.6.2(i)]. Thus, the above splitting yields an isomorphism $\{\coprod_{[\pi]} \top_{\Delta_A}\} \cong A$, and so, in this case, (i), (iii) and closure under isomorphisms yields $\mathbf{V} = \mathsf{Mod}\,\mathsf{Th}\,\mathbf{V}$. □

Remark 4.3 The proof of Theorem 4.2 implicitly constructs a weak right adjoint to the inclusion $\mathbf{V} \hookrightarrow \mathbb{B}$. The object part of the functor $\mathbb{B} \to \mathbf{V}$ is given by

$$A \longmapsto \{\coprod_{[\pi]} \top_{\Delta_A}\}.$$

References

[AH00] Steve Awodey and Jesse Hughes. The coalgebraic dual of Birkhoff's variety theorem. Technical Report CMU-PHIL-109, Carnegie Mellon University, Pittsburgh, PA, 15213, November 2000.

[AHS90] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and Concrete Categories.* Wiley Interscience, 1990.

[AP01] Jiří Adámek and Hans-E. Porst. From varieties of algebras to covarieties of coalgebras. In Corradini et al. [CLM01]. Available at `http://www.elsevier.nl/locate/entcs/volume44.html`.

[AR94] Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*, volume 189 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 1994.

[BH76] B. Banaschewski and H. Herrlich. Subcategories defined by implications. *Houston Journal of Mathematics*, 2(2), 1976.

[Bir35] G. Birkhoff. On the structure of abstract algebras. *Proceedings of the Cambridge Philosophical Society*, 31:433–454, 1935.

[Bor94] F. Borceux. *Handbook of Categorical Algebra I-III.* Cambridge University Press, 1994.

[CLM01] A. Corradini, M. Lenisa, and U. Montanari, editors. *Coalgebraic Methods in Computer Science (CMCS'2001)*, number 44 in Elect. Notes in Theor. Comp. Sci. Elsevier, Amsterdam, 2001. Available at `http://www.elsevier.nl/locate/entcs/volume44.html`.

[GS01] H. Peter Gumm and Tobias Schröder. Covarieties and complete covarieties. *Theoret. Comput. Sci.*, 260(1-2):71–86, 2001. Coalgebraic methods in computer science (Lisbon, 1998).

[Gum01] H. Peter Gumm. Equational and implicational classes of coalgebras. *Theoret. Comput. Sci.*, 260(1-2):57–69, 2001. Coalgebraic methods in computer science (Lisbon, 1998).

[Hug01] Jesse Hughes. Modal operators for coequations. In Corradini et al. [CLM01]. Available at `http://www.elsevier.nl/locate/entcs/volume44.html`.

[Hug02] Jesse Hughes. Some co-birkhoff type theorems. *Electronic Notes in Theoretical Computer Science*, 2002. To appear.

[Jac99] Bart Jacobs. *Categorical Logic and Type Theory*. Elsevier, 1999.

[KR02] Alexander Kurz and Jiří Rosicky. Modal predicates and co-equations. *Electronic Notes in Theoretical Computer Science*, 2002. To appear.

[Kur00] Alexander Kurz. *Logics for Coalgebras and Applications for Computer Science*. PhD thesis, Ludwig-Maximilians-Universität München, 2000.

[Kur01] Alexander Kurz. A co-variety-theorem for modal logic. In *Advances in Modal Logic 2*. Center for the Study of Language and Information, Stanford University, 2001. Selected Papers from AiML 2, Uppsala, 1998.

[Man76] Ernie Manes. *Algebraic Theories*. Number 26 in Graduate Texts in Mathematics. Springer-Verlag, 1976.

[ML71] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer-Verlag, 1971.

[NS81] I. Németi and I. Sain. Cone-implicational subcategories and some Birkhoff-type theorems. In *Universal Algebra (Proc. Coll. Esztergom 1977)*, volume 29 of *Colloq. Math. Soc. J. Bolyai*, pages 535–578, 1981.

[Roş00] Grigore Roşu. Equational axiomatizability for coalgebra. *Theoretical Computer Science*, 260, 2000.

[Rut00] J.J.M.M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249:3–80, 2000.

[Wec92] Wolfgang Wechler. *Universal algebra for computer scientists*. Springer-Verlag, 1992.