Coalgebraic Walks, in Quantum and Turing Computation⋆

Bart Jacobs

Institute for Computing and Information Sciences (iCIS),
Radboud University Nijmegen, The Netherlands.
Webaddress: `www.cs.ru.nl/B.Jacobs`

Abstract. The paper investigates non-deterministic, probabilistic and quantum walks, from the perspective of coalgebras and monads. Non-deterministic and probabilistic walks are coalgebras of a monad (powerset and distribution), in an obvious manner. It is shown that also quantum walks are coalgebras of a new monad, involving additional control structure. This new monad is also used to describe Turing machines coalgebraically, namely as controlled 'walks' on a tape.

1 Introduction

Coalgebras have emerged in theoretical computer science as a generic formalism for state-based computing, covering various flavours of computation, like deterministic, non-determinstic, probabilistic *etc.* In general, a coalgebra is a transition map of the form $X \longrightarrow \boxed{\cdots X \cdots X \cdots}$, where X is the state space and the box captures the form of computation involved. For instance, it is a powerset $\mathcal{P}(X)$ in case of non-determinism; many other coalgebraic classifications of systems are described in [11,1]. More formally, this box is a functor, or often even a monad (in this paper) giving composition as monoid structure on coalgebras. A question that is open for a long time is whether Turing machines can also be modeled coalgebraically. More recently, the same question has been asked for quantum computing.

This paper addresses both these questions and provides positive answers via illustrations, starting from the notion of a random walk. Such walks exist in non-deterministic, probabilistic and quantum form. A first goal is to describe all three variants in a common (coalgebraic) framework, using monads. This effort focuses on the quantum case, and leads to a new construction for monads (see Proposition 2) that yields an appropriate monad for quantum walks, involving a separate control structure.

Since quantum computation is inherently reversible, the framework of dagger categories is needed. Examples of such categories are described in Section 5, via suitable relations that capture 'bi-coalgebraic' computations. Among the different kinds of walks, only the quantum walks give rise a unitary map.

Finally, an analogy is observed between quantum walks and Turing machines: both involve a road/tape on which movement is steered by a separate control structure. This will be captured coalgebraically, via the newly defined monads.

⋆ In: FOSSACS 2011, LNCS proceedings.

The approach of the paper is rather phenomenological, focusing on examples. However, the material is supported by two general results (Propositions 2 and 3), one of which is moved to the appendix; it describes how coalgebras of a monad, with Kleisli composition, form a monoid in categories of algebras of the monad.

2 Three monads for computation types

Category theory, especially the theory of monads, plays an important role in the background of the current paper. The presentation however is intended to be accessible—to a large extent—without familiarity with monads. We do use three particular monads extensively, namely the powerset, multiset, and distribution monad, and so we describe them here explicitly—without making their monad structure explicit; *cognoscenti* will have no problem filling in this structure themselves.

The first monad is the finite powerset $\mathcal{P}_{fin}(X) = \{U \subseteq X \mid U \text{ is finite}\}$. Next, a multiset is like a subset except that elements may occur multiple times. Hence one needs a way of counting elements. Most generally this can be done in a semiring, but in the current setting we count in the complex numbers $\mathbb{C}$. Thus the collection of (complex-valued) multisets of a set X is defined in terms of formal linear combinations of elements of X, as in:

$$\mathcal{M}(X) = \Big\{ z_1|x_1\rangle + \cdots + z_n|x_n\rangle \;\Big|\; z_i \in \mathbb{C} \text{ and } x_i \in X \Big\}. \tag{1}$$

Such a multiset $\sum_i z_i|x_i\rangle \in \mathcal{M}(X)$ can equivalently be described as a function $X \to \mathbb{C}$ with finite support (*i.e.* with only finitely many non-zero values).

The "ket" notation $|x\rangle$, for $x \in X$, is just syntactic sugar, describing x as singleton multiset. It is Dirac's notation for vectors, that is standard in physics. The formal combinations in (1) can be added in an obvious way, and multiplied with a complex number. Hence $\mathcal{M}(X)$ is a vector space over $\mathbb{C}$, namely the free one on X.

The distribution monad $\mathcal{D}$ contains formal *convex* combinations:

$$\begin{aligned}&\mathcal{D}(X)\\ &= \Big\{ r_1|x_1\rangle + \cdots + r_n|x_n\rangle \;\Big|\; r_i \in [0,1] \text{ with } r_1 + \cdots + r_n = 1 \text{ and } x_i \in X \Big\}\end{aligned} \tag{2}$$

Such a convex combination is a discrete probability distribution on X.

Coalgebra provides a generic way of modeling state-based systems, namely as maps of the form $X \to T(X)$, where T is a functor (or often a monad). Basically, we only use the terminology of coalgebras, but not associated notions like bisimilarity, finality or coalgebraic modal logic. See [11] for more information.

3 Walk the walk

This section describes various ways of walking on a line—and not, for instance, on a graph—using non-deterministic, probabilistic or quantum decisions about next steps. Informally, one can think of a drunkard moving about. His steps are discrete, on a line represented by the integers $\mathbb{Z}$.

3.1 Non-deterministic walks

A system for non-deterministic walks is represented as a coalgebra $s\colon \mathbb{Z} \to \mathcal{P}_{\mathit{fin}}(\mathbb{Z})$ of the finite powerset monad $\mathcal{P}_{\mathit{fin}}$. For instance, the one-step-left-one-step-right walk is represented via the coalgebra:

$$s(k) = \{k-1, k+1\}$$

In such a non-deterministic system both possible successor states $k-1$ and $k+1$ are included, without any distinction between them. The coalgebra $s\colon \mathbb{Z} \to \mathcal{P}_{\mathit{fin}}(\mathbb{Z})$ forms an endomap $\mathbb{Z} \to \mathbb{Z}$ in the Kleisli category $\mathcal{K}\ell(\mathcal{P}_{\mathit{fin}})$ of the powerset monad. Repeated composition $s^n = s \bullet \cdots \bullet s\colon \mathbb{Z} \to \mathbb{Z}$ can be defined directly in $\mathcal{K}\ell(\mathcal{P}_{\mathit{fin}})$. Inductively, one can define s^n via Kleisli extension $s^{\#}$ as in:

$$\begin{array}{rcl} s^0 &=& \{-\} \\ s^{n+1} &=& s^{\#} \circ s^n \end{array} \qquad \text{where} \qquad \begin{array}{rl} s^{\#}\colon & \mathcal{P}_{\mathit{fin}}(\mathbb{Z}) \longrightarrow \mathcal{P}_{\mathit{fin}}(\mathbb{Z}) \\ \text{is} & U \longmapsto \bigcup\{s(m) \mid m \in U\}. \end{array}$$

Thus, $s^n(k) \subseteq \mathbb{Z}$ describes the points that can be reached from $k \in \mathbb{Z}$ in n steps:

$$\begin{array}{rcl} s^0(k) &=& \{k\} \\ s^1(k) &=& s(k) \;=\; \{k-1, k+1\} \\ s^2(k) &=& \bigcup\{s(m) \mid m \in s(k)\} \;=\; s(k-1) \cup s(k+1) \\ &=& \{k-2, k\} \cup \{k, k+2\} \;=\; \{k-2, k, k+2\} \\ s^3(k) &=& s(k-2) \cup s(k) \cup s(k+2) \;=\; \{k-3, k-1, k+1, k+3\} \qquad \textit{etc.} \end{array}$$

After n iterations we obtain a set with $n+1$ elements, each two units apart:

$$s^n(k) = \{k-n, k-n+2, k-n+4, \ldots, k+n-2, k+n\}.$$

Hence we can picture the non-deterministic walk, starting at $0 \in \mathbb{Z}$ by indicating the elements of $s^n(0)$ successively by $+$ signs:

$$\begin{array}{ccccccccc} \cdots & \text{-3} & \text{-2} & \text{-1} & 0 & 1 & 2 & 3 & \cdots \\ & & & & + & & & & \\ & & & + & & + & & & \\ & & + & & + & & + & & \\ & + & & + & & + & & + & \\ + & & + & & + & & + & & + \qquad \textit{etc.} \end{array} \tag{3}$$

What we have used is that coalgebras $X \to \mathcal{P}_{\mathit{fin}}(X)$ carry a monoid structure given by Kleisli composition. The set $\mathcal{P}_{\mathit{fin}}(X)$ is the free join semilattice on X. The set of coalgebras $X \to \mathcal{P}_{\mathit{fin}}(X)$ then also carries a semilattice structure, pointwise. These two monoid structures (join and composition) interact appropriately, making the set of coalgebras $X \to \mathcal{P}_{\mathit{fin}}(X)$ a semiring. This follows from a quite general result about monads, see Proposition 3 in the appendix. The semiring structure is used in Section 7 when we consider matrices of coalgebras.

3.2 Probabilistic walks

Probabilistic walks can be described by replacing the powerset monad $\mathcal{P}_{\mathit{fin}}$ by the (uniform) probability distribution monad $\mathcal{D}$, as in:

$$\mathbb{Z} \xrightarrow{\;d\;} \mathcal{D}(\mathbb{Z}) \qquad \text{given by} \qquad k \longmapsto \tfrac{1}{2}|\,k-1\,\rangle + \tfrac{1}{2}|\,k+1\,\rangle.$$

This coalgebra d is an endomap $\mathbb{Z} \to \mathbb{Z}$ in the Kleisli category $\mathcal{K}\ell(\mathcal{D})$ of the distribution monad. This yields a monoid structure, and iterations $d^n \colon \mathbb{Z} \to \mathbb{Z}$ in $\mathcal{K}\ell(\mathcal{D})$. The Kleisli extension function $d^{\#} \colon \mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$ can be described as:

$$\begin{aligned}
&d^{\#}\big(r_1|\,k_1\,\rangle + \cdots + r_n|\,k_n\,\rangle\big) \\
&= \tfrac{1}{2}r_1|\,k_1-1\,\rangle + \tfrac{1}{2}r_1|\,k_1+1\,\rangle + \cdots + \tfrac{1}{2}r_n|\,k_n-1\,\rangle + \tfrac{1}{2}r_n|\,k_n+1\,\rangle,
\end{aligned}$$

where on the right-hand-side we must, if needed, identify $r|\,k\,\rangle + s|\,k\,\rangle$ with $(r+s)|\,k\,\rangle$. One has $d^n = d \bullet \cdots \bullet d$, where $d \bullet d = d^{\#} \circ d$.

The iterations d^n, as functions $d^n \colon \mathbb{Z} \to \mathcal{D}(\mathbb{Z})$, yield successively:

$$\begin{aligned}
d^0(k) &= 1|\,k\,\rangle \\
d^1(k) &= d(k) \;=\; \tfrac{1}{2}|\,k-1\,\rangle + \tfrac{1}{2}|\,k+1\,\rangle \\
d^2(k) &= \tfrac{1}{4}|\,k-2\,\rangle + \tfrac{1}{4}|\,k\,\rangle + \tfrac{1}{4}|\,k\,\rangle + \tfrac{1}{4}|\,k+2\,\rangle \;=\; \tfrac{1}{4}|\,k-2\,\rangle + \tfrac{1}{2}|\,k\,\rangle + \tfrac{1}{4}|\,k+2\,\rangle \\
d^3(k) &= \tfrac{1}{8}|\,k-3\,\rangle + \tfrac{1}{8}|\,k-1\,\rangle + \tfrac{1}{4}|\,k-1\,\rangle + \tfrac{1}{4}|\,k+1\,\rangle + \tfrac{1}{8}|\,k+1\,\rangle + \tfrac{1}{8}|\,k+3\,\rangle \\
&= \tfrac{1}{8}|\,k-3\,\rangle + \tfrac{3}{8}|\,k-1\,\rangle + \tfrac{3}{8}|\,k+1\,\rangle + \tfrac{1}{8}|\,k+3\,\rangle \qquad \textit{etc.}
\end{aligned}$$

The general formula involves binomial coefficients describing probabilities:

$$\begin{aligned}
d^n(k) = \tfrac{\binom{n}{0}}{2^n}|\,k-n\,\rangle + \tfrac{\binom{n}{1}}{2^n}|\,k-n+2\,\rangle + \tfrac{\binom{n}{2}}{2^n}|\,k-n+4\,\rangle + \ldots + \\
\tfrac{\binom{n}{n-1}}{2^n}|\,k+n-2\,\rangle + \tfrac{\binom{n}{n}}{2^n}|\,k+n\,\rangle.
\end{aligned}$$

This provides a distribution since all probabilities involved add up to 1, because of the well-known sum formula for binomial coefficients:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

The resulting probabilistic walks starting in $0 \in \mathbb{Z}$ can be pictured like in (3), but this time with explicit probabilities:

$$\begin{tikzcd}[column sep=tiny, row sep=tiny]
\cdots \arrow[r, dash] & \text{-3} \arrow[r, dash] & \text{-2} \arrow[r, dash] & \text{-1} \arrow[r, dash] & 0 \arrow[r, dash] & 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots & \\
& & & & 1 \arrow[dl, dash] \arrow[dr, dash] & & & & & \\
& & & \frac{1}{2} \arrow[dl, dash] \arrow[dr, dash] & & \frac{1}{2} \arrow[dl, dash] \arrow[dr, dash] & & & & \\
& & \frac{1}{4} \arrow[dl, dash] \arrow[dr, dash] & & \frac{2}{4} \arrow[dl, dash] \arrow[dr, dash] & & \frac{1}{4} \arrow[dl, dash] \arrow[dr, dash] & & & \\
& \frac{1}{8} \arrow[dl, dash] \arrow[dr, dash] & & \frac{3}{8} \arrow[dl, dash] \arrow[dr, dash] & & \frac{3}{8} \arrow[dl, dash] \arrow[dr, dash] & & \frac{1}{8} \arrow[dl, dash] \arrow[dr, dash] & & \\
\frac{1}{16} & & \frac{4}{16} & & \frac{6}{16} & & \frac{4}{16} & & \frac{1}{16} & \textit{etc.}
\end{tikzcd} \tag{4}$$

The role of Pascal's triangle in the description of the probability distributions for such random walks is of course well-known.

3.3 Quantum walks

In the quantum case the states $k \in \mathbb{Z}$ appear as base vectors, written as $|k\rangle \in \mathcal{M}(\mathbb{Z})$, in the free vector space $\mathcal{M}(\mathbb{Z})$ on $\mathbb{Z}$, see Section 2. Besides these vectors, one qubit, with base vectors $|\downarrow\rangle$ and $|\uparrow\rangle$, is used for the direction of the walk. Thus, the space that is typically used in physics (see [5,12]) for quantum walks is:

$$\mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \qquad \text{with basis elements} \qquad |\downarrow\rangle \otimes |k\rangle,\ |\uparrow\rangle \otimes |k\rangle,$$

where we may understand $|\uparrow\rangle = \binom{1}{0} \in \mathbb{C}^2$ and $|\downarrow\rangle = \binom{0}{1} \in \mathbb{C}^2$.

A single step of a quantum walk is then written as an endomap:

$$\begin{array}{rcl} \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) & \xrightarrow{\qquad\qquad q \qquad\qquad} & \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \\ |\uparrow\rangle \otimes |k\rangle & \longmapsto & \frac{1}{\sqrt{2}}|\uparrow\rangle \otimes |k-1\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes |k+1\rangle \\ |\downarrow\rangle \otimes |k\rangle & \longmapsto & \frac{1}{\sqrt{2}}|\uparrow\rangle \otimes |k-1\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes |k+1\rangle \end{array} \tag{5}$$

Implictly the Hadamard transform $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is applied to the qubits in $\mathbb{C}^2$. A tree of probabilities is now obtained by repeatedly applying q, say to a start state $|\uparrow\rangle \otimes |0\rangle$, and subsequently measuring $|k\rangle$. We write $Prob_k$ for the probability of seeing $|k\rangle$ as outcome.

Thus, after one step we have:

$$q(|\uparrow\rangle \otimes |0\rangle) = \tfrac{1}{\sqrt{2}}|\uparrow\rangle \otimes |-1\rangle + \tfrac{1}{\sqrt{2}}|\downarrow\rangle \otimes |1\rangle,$$

giving probabilities $Prob_{-1} = Prob_1 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$. After two steps we get:

$$\begin{aligned} q^2(|\uparrow\rangle \otimes |0\rangle) &= \tfrac{1}{\sqrt{2}}q(|\uparrow\rangle \otimes |-1\rangle) + \tfrac{1}{\sqrt{2}}q(|\downarrow\rangle \otimes |1\rangle) \\ &= \tfrac{1}{2}|\uparrow\rangle \otimes |-2\rangle + \tfrac{1}{2}|\downarrow\rangle \otimes |0\rangle + \tfrac{1}{2}|\uparrow\rangle \otimes |0\rangle - \tfrac{1}{2}|\downarrow\rangle \otimes |2\rangle \\ &= \tfrac{1}{2}|\uparrow\rangle \otimes |-2\rangle + \tfrac{1}{2}(|\uparrow\rangle + |\downarrow\rangle) \otimes |0\rangle - \tfrac{1}{2}|\downarrow\rangle \otimes |2\rangle, \end{aligned}$$

with probabilities:

$$Prob_{-2} = \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{4} \qquad Prob_0 = \left|\tfrac{1}{2}\right|^2 + \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{2} \qquad Prob_2 = \left|-\tfrac{1}{2}\right|^2 = \tfrac{1}{4}.$$

After 3 steps the outcomes begin to differ from the probabilistic outcomes, see (4), due to interference between the different summands:

$$\begin{aligned} & q^3(|\uparrow\rangle \otimes |0\rangle) \\ &= \tfrac{1}{2}q(|\uparrow\rangle \otimes |-2\rangle) + \tfrac{1}{2}q(|\downarrow\rangle \otimes |0\rangle) \\ &\qquad + \tfrac{1}{2}q(|\uparrow\rangle \otimes |0\rangle) - \tfrac{1}{2}q(|\downarrow\rangle \otimes |2\rangle) \\ &= \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |-3\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |-1\rangle + \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |-1\rangle - \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |1\rangle \\ &\qquad + \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |-1\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |1\rangle - \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |1\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |3\rangle \\ &= \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |-3\rangle + \tfrac{1}{\sqrt{2}}|\uparrow\rangle \otimes |-1\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |-1\rangle \\ &\qquad - \tfrac{1}{2\sqrt{2}}|\uparrow\rangle \otimes |1\rangle + \tfrac{1}{2\sqrt{2}}|\downarrow\rangle \otimes |3\rangle, \end{aligned}$$

leading to probabilities:

$$Prob_{-3} = Prob_1 = Prob_3 = \left|\tfrac{1}{2\sqrt{2}}\right|^2 = \tfrac{1}{8} \qquad Prob_{-1} = \left|\tfrac{1}{\sqrt{2}}\right|^2 + \left|\tfrac{1}{2\sqrt{2}}\right|^2 = \tfrac{5}{8}.$$

Thus there is a 'drift' to the left, see the following table of probabilities starting from the initial state $|\uparrow\rangle \otimes |0\rangle \in \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z})$.

$$\begin{array}{ccccccccccc}
\cdots & \text{—} \;\text{-3} & \text{—}\;\text{-2} & \text{—}\;\text{-1} & \text{—}\;0 & \text{—}\;1 & \text{—}\;2 & \text{—}\;3 & \text{—}\;\cdots & & \\
 & & & & 1 & & & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & & & \\
 & & \frac{1}{4} & & \frac{2}{4} & & \frac{1}{4} & & & & \\
 & \frac{1}{8} & & \frac{5}{8} & & \frac{1}{8} & & \frac{1}{8} & & & \\
\frac{1}{16} & & \frac{5}{8} & & \frac{1}{8} & & \frac{1}{8} & & \frac{1}{16} & & \textit{etc.}
\end{array} \tag{6}$$

The matrix involved—Hadamard's H in this case—determines the drifting, and thus how the tree is traversed.

4 A coalgebraic/monadic description of quantum walks

In the previous section we have seen the standard way of describing quantum walks, namely via endomaps $\mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \to \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z})$. The question arises if such walks can also be described coalgebraically, of the form $\mathbb{Z} \to T(\mathbb{Z})$, for a suitable monad T, just like for non-deterministic and probabilitistic walks in Subsections 3.1 and 3.2. This section will show how to do so. The following observation forms the basis.

Proposition 1. *1. For each $n \in \mathbb{N}$, there is an isomorphism of vector spaces:*

$$\mathbb{C}^n \otimes \mathcal{M}(X) \;\cong\; \mathcal{M}(n \cdot X),$$

natural in X—where $n \cdot X$ is the n-fold coproduct $X + \cdots + X$, also known as copower of the set X.

2. As a consequence, there is a bijective correspondence between:

$$\frac{\textit{linear maps} \quad \mathbb{C}^n \otimes \mathcal{M}(X) \longrightarrow \mathbb{C}^n \otimes \mathcal{M}(Y)}{\textit{functions} \quad X \longrightarrow \mathcal{M}(n \cdot Y)^n}$$

Proof. 1. For convenience we restrict to $n = 2$. We shall write $\oplus$ for the product of vector spaces, which is at the same time a coproduct of spaces (and hence a 'biproduct'). There is the following chain of (natural) isomorphisms

$$\begin{aligned}
\mathbb{C}^2 \otimes \mathcal{M}(X) &= (\mathbb{C} \oplus \mathbb{C}) \otimes \mathcal{M}(X) \\
&\cong \big(\mathbb{C} \otimes \mathcal{M}(Z)\big) \oplus \big(\mathbb{C} \otimes \mathcal{M}(X)\big) && \text{since } \otimes \text{ distributes over } \oplus \\
&\cong \mathcal{M}(X) \oplus \mathcal{M}(X) && \text{since } \mathbb{C} \text{ is tensor unit} \\
&\cong \mathcal{M}(X + X),
\end{aligned}$$

where the last isomorphism exists because $\mathcal{M}$ is a free functor **Sets** $\to$ **Vect**, and thus preserves coproducts.

2. Directly from the previous point, since:

$$\begin{array}{ll}
\mathbb{C}^n \otimes \mathcal{M}(X) \longrightarrow \mathbb{C}^n \otimes \mathcal{M}(Y) & \text{in } \mathbf{Vect} \\
\hline\hline
\mathcal{M}(n \cdot X) \longrightarrow \mathcal{M}(n \cdot Y) & \text{in } \mathbf{Vect}\text{, by point 1} \\
\hline\hline
n \cdot X \longrightarrow \mathcal{M}(n \cdot Y) & \text{in } \mathbf{Sets}\text{, since } \mathcal{M} \text{ is free} \\
\hline\hline
X \longrightarrow \mathcal{M}(n \cdot Y)^n & \text{in } \mathbf{Sets}
\end{array}$$

$\square$

Corollary 1. *There is a bijective correspondence between linear endomaps*

$$\mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \longrightarrow \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z})$$

as used for quantum walks in Subsection 3.3, and coalgebras

$$\mathbb{Z} \longrightarrow \mathcal{M}(\mathbb{Z} + \mathbb{Z})^2$$

of the functor $\mathcal{M}(2 \cdot -)^2$.

The coalgebra $\mathbb{Z} \to \mathcal{M}(\mathbb{Z}+\mathbb{Z})^2$ corresponding to the linear endomap $q\colon \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \to \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z})$ from Subsection 3.3 can be described explicitly as follows.

$$\begin{array}{l}
\mathbb{Z} \xrightarrow{\qquad\qquad\qquad\qquad\qquad} \mathcal{M}(\mathbb{Z}+\mathbb{Z})^2 \\
m \longmapsto \Big\langle \frac{1}{\sqrt{2}}\kappa_1|m-1\rangle + \frac{1}{\sqrt{2}}\kappa_2|m+1\rangle,\; \frac{1}{\sqrt{2}}\kappa_1|m-1\rangle - \frac{1}{\sqrt{2}}\kappa_2|m+1\rangle \Big\rangle
\end{array} \tag{7}$$

The κ_i, for $i = 1, 2$, are coprojections that serve as tags for 'left' and 'right' in a coproduct (disjoint union) $\mathbb{Z}+\mathbb{Z}$. Notice that in this re-description tensor spaces and their bases have disappeared completely.

Of course, at this stage one wonders if the the functor $\mathcal{M}(2\cdot-)^2$ in Corollary 1 is also a monad—like powerset and distribution. This turns out to be the case, as an instance of the following general "monad transformer" result.

Proposition 2. *Let $\mathbf{A}$ be a category with finite powers $X^n = X \times \cdots \times X$ and copowers $n \cdot X = X + \cdots + X$. For a monad $T\colon \mathbf{A} \to \mathbf{A}$ there is for each $n \in \mathbb{N}$ a new monad $T[n]\colon \mathbf{A} \to \mathbf{A}$ by:*

$$T[n](X) = \Big(T(n \cdot X)\Big)^n$$

with unit and Kleisli extension:

$$\eta[n]_X = \langle T(\kappa_i) \circ \eta_X \rangle_{i \le n} \qquad\qquad f^{\#} = \big(\mu_{T(n \cdot Y)} \circ T([f_i]_{i \le n})\big)^n.$$

where in the latter case f is a map $f = \langle f_i \rangle_{i \le n}\colon X \to T[n](Y)$.

Proof. For convenience, and in order to be more concrete, we restrict to $n = 2$. We leave it to the reader to verify that $\eta[2]$ is natural and that its extension is the identity: $\eta[2]^{\#} = \mathrm{id}$. Of the two remaining properties of Kleisli extension, $f^{\#} \circ \eta[2] = f$ and $(g^{\#} \circ f)^{\#} = g^{\#} \circ f^{\#}$, we prove the first one:

$$\begin{aligned}
f^{\#} \circ \eta[2] &= (\mu \circ T([f_1, f_2])) \times (\mu \circ T([f_1, f_2])) \circ \langle T(\kappa_1) \circ \eta, T(\kappa_2) \circ \eta \rangle \\
&= \langle \mu \circ T([f_1, f_2]) \circ T(\kappa_1) \circ \eta, \mu \circ T([f_1, f_2]) \circ T(\kappa_2) \circ \eta \rangle \\
&= \langle \mu \circ T(f_1) \circ \eta, \mu \circ T(f_2) \circ \eta \rangle \\
&= \langle \mu \circ \eta \circ f_1, \mu \circ \eta \circ f_2 \rangle \\
&= \langle f_1, f_2 \rangle \\
&= f. \qquad \square
\end{aligned}$$

Kleisli extension yields the multiplication map $T[n]^2(X) \to T[n](X)$ as extension $\mathrm{id}^{\#}$ of the identity on $T[n](X)$. Concretely, it can be described as:

$$\Big[T\Big(n \cdot \big(T(n \cdot X)\big)^n\Big)\Big]^n \xrightarrow{\ \big[\mu \circ T([\pi_i]_{i \le n})\big]^n\ } \Big[T(n \cdot X)\Big]^n$$

The number $n \in \mathbb{N}$ in $T[n]$ yields a form of control via n states, like in the quantum walks in Subsection 3.3 where $n = 2$ and $T = \mathcal{M}$. Indeed, there is a similarity with the state monad transformer $X \mapsto T(S \times X)^S$, for T a monad and S a fixed set of states (see *e.g.* [8]). If S is a finite set, say with size $n = |S|$, $T(S \times -)^S$ is the same as the monad $T[n] = T(n \cdot -)^n$ in Proposition 2 since the product $S \times X$ in **Sets** is the same as the copower $n \cdot X$.

Next we recall that $\mathcal{M}$ is an additive monad. This means that it maps finite coproducts to products: $\mathcal{M}(0) \cong 1$ and $\mathcal{M}(X + Y) \cong \mathcal{M}(X) \times \mathcal{M}(Y)$, in a canonical manner, see [2] for the details. This is relevant in the current setting, because the endomap for quantum walks from Subsection 3.3 can now be described also as a 4-tuple of coalgebras $\mathbb{Z} \to \mathcal{M}(\mathbb{Z})$, since:

$$\begin{array}{c}
\mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \longrightarrow \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \\
\hline\hline
\mathbb{Z} \longrightarrow \mathcal{M}(\mathbb{Z} + \mathbb{Z})^2 \\
\hline\hline
\mathbb{Z} \longrightarrow \big(\mathcal{M}(\mathbb{Z}) \times \mathcal{M}(\mathbb{Z})\big)^2 \\
\cong \\
\mathcal{M}(\mathbb{Z})^4
\end{array}
\begin{array}{l}
\text{(by Corollary 1)} \\
\text{(by additivity of } \mathcal{M}\text{)}
\end{array}$$

We shall write these four coalgebras corresponding to the endomap q in (5) as $c_{ij} \colon \mathbb{Z} \to \mathcal{M}(\mathbb{Z})$, for $i, j \in \{1, 2\}$. Explicitly, they are given as follows.

$$\begin{array}{ll}
c_{11}(k) = \frac{1}{\sqrt{2}}|\,k-1\,\rangle & \qquad c_{12}(k) = \frac{1}{\sqrt{2}}|\,k-1\,\rangle \\
c_{21}(k) = \frac{1}{\sqrt{2}}|\,k+1\,\rangle & \qquad c_{22}(k) = -\frac{1}{\sqrt{2}}|\,k+1\,\rangle.
\end{array}$$

As the notation already suggests, we can consider these four coalgebras as entries in a 2×2 matrix of coalgebras, in the following manner:

$$c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \lambda k.\, \frac{1}{\sqrt{2}}|\,k-1\,\rangle & \lambda k.\, \frac{1}{\sqrt{2}}|\,k-1\,\rangle \\ \lambda k.\, \frac{1}{\sqrt{2}}|\,k+1\,\rangle & \lambda k.\, -\frac{1}{\sqrt{2}}|\,k+1\,\rangle \end{pmatrix}. \qquad (8)$$

Thus, the first column describes the output for input of the form $|\uparrow\rangle \otimes |k\rangle = \binom{|k\rangle}{0}$, and the second column describes the result for $|\downarrow\rangle \otimes |k\rangle = \binom{0}{|k\rangle}$. By multiplying this matrix with itself one achieves iteration as used in Subsection 3.3. This matrix notation is justified by the following observation.

Lemma 1. *The set $\mathcal{M}(X)^X$ of $\mathcal{M}$-coalgebras on a set X forms a semiring. Addition is done pointwise, using addition on $\mathcal{M}(X)$, and multiplication is Kleisli composition $\bullet$ for $\mathcal{M}$, given by $(d \bullet c)(x)(z) = \sum_y c(x)(y) \cdot d(y)(z)$.* $\square$

The proof is skipped because this lemma is a special instance of a more general result, namely Proposition 3 in the appendix.

5 Reversibility of computations

So far we have described different kinds of walks as different coalgebras $\mathbb{Z} \to \mathcal{P}_{\mathit{fin}}(\mathbb{Z})$, $\mathbb{Z} \to \mathcal{D}(\mathbb{Z})$, and $\mathbb{Z}+\mathbb{Z} \to \mathcal{M}(\mathbb{Z}+\mathbb{Z})$. Next we investigate reversibility of these coalgebras. It turns out that all these coalgebras are reversible, via a dagger operation, but only the quantum case involves a 'unitary' operation, where the dagger yields the inverse. The three dagger categories that we describe below are captured in [4] as instance of a general construction of a category of 'tame' relations. Here we only look at the concrete descriptions.

We start with the non-deterministic case. Let **BifRel** be the category of sets and bifinite relations. Object are sets X, and morphisms $X \to Y$ are relations $r\colon X \times Y \to 2 = \{0, 1\}$ such that:

- for each $x \in X$ the set $\{y \in Y \mid r(x, y) \neq 0\}$ is finite;
- also, for each $y \in Y$ the set $\{x \in X \mid r(x, y) \neq 0\}$ is finite.

This means that r factors both as function $X \to \mathcal{P}_{\mathit{fin}}(Y)$ and as $Y \to \mathcal{P}_{\mathit{fin}}(X)$. Relational composition and equality relations make **BifRel** a category. For a map $r\colon X \to Y$ there is an associated map $r^\dagger\colon Y \to X$ in the reverse direction, obtained by swapping arguments: $r^\dagger(y, x) = r(x, y)$. This makes **BifRel** a dagger category.

The non-deterministic walks coalgebra $s\colon \mathbb{Z} \to \mathcal{P}_{\mathit{fin}}(\mathbb{Z})$ from Subsection 3.1 is in fact such bifinite relation $\mathbb{Z} \to \mathbb{Z}$ in **BifRel**. Explicitly, as a map $s\colon \mathbb{Z}\times\mathbb{Z} \to 2$, also called s, it is given by $s(n, m) = 1$ iff $m = n-1$ or $m = n+1$. The associated dagger map $s^\dagger$, in the reverse direction, is $s^\dagger(n, m) = 1$ iff $s(m, n) = 1$ iff $n = m-1$ or $n = m+1$; it is the same relation. In general, a map f in a dagger category is unitary if $f^\dagger$ is the inverse of f. The non-deterministic walks map s is not unitary, since, for instance:

$$\begin{aligned}(s \circ s^\dagger)(n, n') = 1 &\Leftrightarrow \exists_m.\, s^\dagger(n, m) \wedge s(m, n') \\ &\Leftrightarrow s(n-1, n') \vee s(n+1, n') \\ &\Leftrightarrow n' = n-2 \vee n' = n \vee n' = n+2.\end{aligned}$$

This is not the identity map $\mathbb{Z} \to \mathbb{Z}$ given by $\mathrm{id}_{\mathbb{Z}}(n, n') = 1$ iff $n = n'$.

We turn to the probabilistic case, using a dagger category **dBisRel** of discrete bistochastic relations. Objects are sets X and morphisms $X \to Y$ are maps $r\colon X \times Y \to [0,1]$ that factor both as $X \to \mathcal{D}(Y)$ and as $Y \to \mathcal{D}(X)$. Concretely, this means that for each $x \in X$ there are only finitely many $y \in Y$ with $r(x,y) \neq 0$ and $\sum_y r(x,y) = 1$, and similarly in the other direction. These maps form a category, with composition given by matrix multiplication and identity maps by equality relations. The resulting category **dBisRel** has a dagger by reversal of arguments, like in **BifRel**.

The probabilistic walks map $d\colon \mathbb{Z} \to \mathcal{D}(\mathbb{Z})$ from Subsection 3.2 is an endomap $d\colon \mathbb{Z} \to \mathbb{Z}$ in **dBisRel**, given as:

$$\mathbb{Z} \times \mathbb{Z} \xrightarrow{d} [0,1] \qquad \text{by} \qquad d(n,m) = \begin{cases} \frac{1}{2} & \text{if } m = n-1 \text{ or } m = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Also in this case d is not unitary; for instance we do not get equality in:

$$\begin{aligned} (d \circ d^\dagger)(n,n') &= \textstyle\sum_m d^\dagger(n,m) \cdot d(m,n') \\ &= \tfrac{1}{2} \cdot d(n-1,n') + \tfrac{1}{2} \cdot d(n+1,n') \\ &= \begin{cases} \frac{1}{4} & \text{if } n' = n-2 \text{ or } n' = n+2 \\ \frac{1}{2} & \text{if } n' = n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Finally we turn to the quantum case, for which we use the dagger category **BifMRel** of sets and $\mathbb{C}$-valued multirelations. Objects are sets, and morphisms $r\colon X \to Y$ are maps $r\colon X \times Y \to \mathbb{C}$ which factor both as $X \to \mathcal{M}(Y)$ and as $Y \to \mathcal{M}(X)$. This means that for each x there are finitely many y with $r(x,y) \neq 0$, and similarly, for each y there are finitely many x with $r(x,y) \neq 0$. Composition and identities are as before. The dagger now not only involves argument swapping, but also conjugation in $\mathbb{C}$, as in $r^\dagger(y,x) = \overline{r(x,y)}$.

We have already seen that the quantum walks endomap $\mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \to \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z})$ corresponds to a coalgebra $q\colon \mathbb{Z} + \mathbb{Z} \to \mathcal{M}(\mathbb{Z} + \mathbb{Z})$. We now represent it as endo map $q\colon \mathbb{Z} + \mathbb{Z} \to \mathbb{Z} + \mathbb{Z}$ in **BifMRel** given by:

$$\big(\mathbb{Z} + \mathbb{Z}\big) \times \big(\mathbb{Z} + \mathbb{Z}\big) \xrightarrow{q} \mathbb{C} \quad \text{where} \quad \begin{cases} q(\kappa_1 n, \kappa_1(n-1)) = \frac{1}{\sqrt{2}} \\ q(\kappa_1 n, \kappa_2(n+1)) = \frac{1}{\sqrt{2}} \\ q(\kappa_2 n, \kappa_1(n-1)) = \frac{1}{\sqrt{2}} \\ q(\kappa_2 n, \kappa_2(n+1)) = -\frac{1}{\sqrt{2}} \end{cases}$$

(Only the non-zero values are described.) The dagger $q^\dagger$ is:

$$\begin{array}{ll} q^\dagger(\kappa_1(n-1), \kappa_1 n) = \frac{1}{\sqrt{2}} & \qquad q^\dagger(\kappa_2(n+1), \kappa_1 n) = \frac{1}{\sqrt{2}} \\ q^\dagger(\kappa_1(n-1), \kappa_2 n) = \frac{1}{\sqrt{2}} & \qquad q^\dagger(\kappa_2(n+1), \kappa_2 n) = -\frac{1}{\sqrt{2}} \end{array}$$

In the quantum case we do get a unitary map. This involves several elementary verifications, of which we present an illustration:

$$\begin{aligned}
\big(q \circ q^{\dagger}\big)(\kappa_1 m, \kappa_1 n) &= \textstyle\sum_x q^{\dagger}(\kappa_1 m, x) \cdot q(x, \kappa_1 n) \\
&= q^{\dagger}(\kappa_1 m, \kappa_1(m+1)) \cdot q(\kappa_1(m+1), \kappa_1 n) \\
&\qquad + q^{\dagger}(\kappa_1 m, \kappa_2(m+1)) \cdot q(\kappa_2(m+1), \kappa_1 n) \\
&= \begin{cases} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & \text{if } n = m \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \kappa_1 n = \kappa_1 m \\ 0 & \text{otherwise} \end{cases} \\
&= \mathrm{id}(\kappa_1 m, \kappa_1 n).
\end{aligned}$$

In a similar way one obtains $\big(q \circ q^{\dagger}\big)(\kappa_1 m, \kappa_2 n) = 0 = \mathrm{id}(\kappa_1 m, \kappa_2 n)$, *etc.*

6 Summary, so far

At this stage, before proceeding, we sum up what we have seen so far. Nondeterministic and probabilistic walks are described quite naturally as coalgebras of a monad, namely of the (finite) powerset $\mathcal{P}_{\mathit{fin}}$ and distribution monad $\mathcal{D}$, respectively. Quantum walks are usually described (by physicists) as endomaps $\mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z}) \to \mathbb{C}^2 \otimes \mathcal{M}(\mathbb{Z})$. But we have illustrated that they can equivalently be described as coalgebras $\mathbb{Z} \to \mathcal{M}(2 \cdot \mathbb{Z})^2$ of a monad. Thus there is a common, generic framework in which to describe various walks, namely as coalgebras of monads. The monad yields a monoid structure on these coalgebras—via Kleisli composition—which enables iteration. This monoid structure can be described quite generally, for arbitrary monads, in the category of algebras of the monad, see Proposition 3.

All these walks coalgebras are in fact endo maps in a suitable dagger category. Only in the quantum case the walks form a unitary morphism.

Coalgebras of the form $\mathbb{Z} \to \mathcal{M}(2 \cdot \mathbb{Z})^2$ can equivalently be described as maps $\mathbb{Z} + \mathbb{Z} \to \mathcal{M}(\mathbb{Z} + \mathbb{Z})$ or as a quadruple of coalgebras $\mathbb{Z} \to \mathcal{M}(\mathbb{Z})$. Four such coalgebras are obtained because there is a qubit (in $\mathbb{C}^2$) involved that controls the walking, see Subsection 3.3. More generally, if the control happens via $\mathbb{C}^n$, one obtains n^2 coalgebras in a $n \times n$ matrix. A next step is to observe a similarity to what happens in Turing machines: there one has a finite-state automaton that controls a head which reads/writes/moves on a tape. This similarity will be explored further in the next section, where we use the understanding of walks, using the monad construction $T[n]$ from Proposition 2, to capture Turing machines coalgebraically, as a "head walking on a tape".

7 Turing machines as coalgebras

The idea we wish to explore further is that coalgebras of the form $X \to T[n](X) = T(n \cdot X)^n$ of the monad $T[n]$ from Propostion 2 can be understood as computations of type T on state space X with n auxiliary states that control the computation on X. This idea will be illustrated below for Turing machines.

We shall give a simple example of a non-deterministic Turing machine, for the finite powerset monad $T = \mathcal{P}_{\mathit{fin}}$. We use a tape with binary entries that stretches in 2 dimension, and use the integers $\mathbb{Z}$ (like in walks) as index. Thus the type $\mathbb{T}$ of tapes is given by $\mathbb{T} = 2^{\mathbb{Z}} \times \mathbb{Z}$, consisting of pairs (t, p) where $t \colon \mathbb{Z} \to 2 = \{0, 1\}$ is the tape itself and $p \in \mathbb{Z}$ the current position of the head. One could use a more general set Σ of tape symbols, and use maps $\mathbb{Z} \to \Sigma$ as tapes. Commonly one only uses a limited number of operations on a tape, given as abL or abR, with meaning: if a is read at the current position, then write b, and subsequently move one position left (or right) on the tape. Such operations can be used as labels of transitions between control states. An example non-deterministic Turing machine that can stop if it encounters two successive 0s to the right of the head can be described by the following graph with three state $1, 2, 3$.

$$\begin{tikzcd}
1 \arrow[loop above, "00R"] \arrow[loop below, "11R"'] \arrow[r, "00R"] & 2 \arrow[r, "00R"] & 3
\end{tikzcd}$$

We do not include final states explicitly, but clearly the right-most state 3 does not have any transitions and can thus be seen as final.

In line with the description of quantum walks, we shall use four equivalent ways of describing this Turing machine.

1. As an endomap, in the category of join semilattices (which is the category of algebras of the monad $\mathcal{P}_{\mathit{fin}}$ involved), described on base elements as:

$$\begin{array}{rcl}
2^3 \otimes \mathcal{P}_{\mathit{fin}}(\mathbb{T}) & \longrightarrow & 2^3 \otimes \mathcal{P}_{\mathit{fin}}(\mathbb{T}) \\
1 \otimes (t,p) & \longmapsto & \begin{cases} \big(1 \otimes (t, p+1)\big) \vee \big(2 \otimes (t, p+1)\big) & \text{if } t(p) = 0 \\ 1 \otimes (t, p+1) & \text{otherwise} \end{cases} \\
2 \otimes (t,p) & \longmapsto & \begin{cases} 3 \otimes (t, p+1) & \text{if } t(p) = 0 \\ \bot & \text{otherwise} \end{cases} \\
3 \otimes (t,p) & \longmapsto & \bot.
\end{array}$$

2. As a coalgebra of the monad $\mathcal{P}_{\mathit{fin}}(3 \cdot -)^3$, namely:

$$\begin{array}{rcl}
\mathbb{T} & \longrightarrow & \mathcal{P}_{\mathit{fin}}(\mathbb{T} + \mathbb{T} + \mathbb{T})^3 \\
(t,p) & \longmapsto & \Big\langle \{\kappa_1(t, p+1)\} \cup \{\kappa_2(t, p+1) \mid t(p) = 0\}, \\
& & \qquad \{\kappa_3(t, p+1) \mid t(p) = 0\},\ \emptyset \Big\rangle
\end{array}$$

3. As a 3×3 matrix of coalgebras $\mathbb{T} \to \mathcal{P}_{\mathit{fin}}(\mathbb{T})$, using that the monad $\mathcal{P}_{\mathit{fin}}$ is additive (see [2]), so that $\mathcal{P}_{\mathit{fin}}(\mathbb{T}+\mathbb{T}+\mathbb{T})^3 \cong \big(\mathcal{P}_{\mathit{fin}}(\mathbb{T}) \times \mathcal{P}_{\mathit{fin}}(\mathbb{T}) \times \mathcal{P}_{\mathit{fin}}(\mathbb{T})\big)^3 \cong$

$\mathcal{P}_{fin}(\mathbb{T})^9$.

$$\begin{pmatrix} \lambda(t,p).\,\{(t,p+1)\} & \lambda(t,p).\,\emptyset & \lambda(t,p).\,\emptyset \\ \lambda(t,p).\,\begin{cases} \{(t,p+1)\} & \text{if } t(p)=0 \\ \emptyset & \text{otherwise} \end{cases} & \lambda(t,p).\,\emptyset & \lambda(t,p).\,\emptyset \\ \lambda(t,p).\,\emptyset & \lambda(t,p).\,\begin{cases} \{(t,p+1)\} & \text{if } t(p)=0 \\ \emptyset & \text{otherwise} \end{cases} & \lambda(t,p).\,\emptyset \end{pmatrix}$$

The entry at column i and row j describes the coalgebra for the transition from control state i to j. This matrix representation of coalgebras makes sense because the set of coalgebras $X \to \mathcal{P}_{fin}(X)$ forms a semiring, as remarked at the end of Subsection 3.1.

4. As endo map $3 \cdot \mathbb{T} \to 3 \cdot \mathbb{T}$ in the category **BifRel**, that is as bifinite relation $r\colon \big(\mathbb{T}+\mathbb{T}+\mathbb{T}\big) \times \big(\mathbb{T}+\mathbb{T}+\mathbb{T}\big) \to \mathbb{C}$, given by the following non-zero cases.

$$r\big(\kappa_1(t,p), \kappa_1(t,p+1)\big), \qquad r\big(\kappa_1(t,p), \kappa_2(t,p+1)\big) \text{ if } t(p)=0,$$
$$r\big(\kappa_2(t,p), \kappa_3(t,p+1)\big) \text{ if } t(p)=0.$$

Via such modelling one can iterate the mappings involved and thus calculate successor states. We give an example calculation, using the second representation $\mathbb{T} \to \mathcal{P}_{fin}(\mathbb{T}+\mathbb{T}+\mathbb{T})^3$. An element of $\mathbb{T}$ will be described (partially) via expressions like $\cdots 0\underline{1}011 \cdots$, where the underlining indicates the current position of the head. Starting in the first state, represented by the label κ_1, we get:

$$\begin{aligned} \kappa_1(\cdots \underline{1}01001 \cdots) &\longmapsto \{\kappa_1(\cdots 1\underline{0}1001 \cdots)\} \\ &\longmapsto \{\kappa_1(\cdots 10\underline{1}001 \cdots), \kappa_2(\cdots 10\underline{1}001 \cdots)\} \\ &\longmapsto \{\kappa_1(\cdots 101\underline{0}01 \cdots)\} \\ &\longmapsto \{\kappa_1(\cdots 1010\underline{0}1 \cdots), \kappa_2(\cdots 1010\underline{0}1 \cdots)\} \\ &\longmapsto \{\kappa_1(\cdots 10100\underline{1} \cdots), \kappa_2(\cdots 10100\underline{1} \cdots), \kappa_3(\cdots 10100\underline{1} \cdots)\} \end{aligned}$$

Etcetera. Hopefully it is clear that this coalgebraic/monadic/relational modelling of Turing machines is quite flexible. For instance, by changing the monad one gets other types of computation on a tape: by taking the multiset monad $\mathcal{M}$, and requiring unitarity, one obtains quantum Turing machines (as in [10]). For instance, coalgebraic walks like in Subsection 3.3 can be seen as a 2-state quantum Turing machine with a singleton set of symbols (and thus only the head's position forming the tape-type $\mathbb{T} = \mathbb{Z}$).

The above (equivalent) representations of a Turing machine via the monad construction $T[n]$ distinguishes between the tape and the finitely many states of a machine. In contrast, for instance in [9], a Turing machine is represented as a coalgebra of the form $X \longrightarrow \mathcal{P}_{fin}\big(X \times \mathit{\Gamma} \times \{\triangleleft, \triangleright\}\big)^{\mathit{\Gamma}}$, where $\mathit{\Gamma}$ is a set of input symbols, and $\triangleleft, \triangleright$ represent left and right moves. There is only one state space X, which implicitly combines both the tape and the states that steer the computation.

8 Conclusions

The investigation of non-deterministic, probabilistic and quantum walks has led to a coalgebraic description of quantum computation, in the form of qubits acting on a set, via a new monad construction $T[n]$. It adds n-ary steering to T-computations, not only for quantum walks but also in n-state Turing machines (as controled 'walks' on a tape). The coalgebraic approach emphasises only the one-directional aspect of computation. Via suitable categories of 'bi-coalgebraic' relations this bidirectional aspect can be made explicit, and the distinctive unitary character of quantum computation becomes explicit. For the future, the role of final coalgebras requires clarity, especially for the new monad $T[n]$, for instance for computing stationary (limiting) distributions. How to describe (unidirectional) measurements coalgebraically will be described elsewhere.

References

1. F. Bartels, A. Sokolova, and E. de Vink. A hierarchy of probabilistic system types. *Theor. Comp. Sci.*, 327(1-2):3–22, 2004.
2. D. Coumans and B. Jacobs. Scalars, monads and categories, 2010. `arxiv.org/abs/1003.0585`.
3. B. Jacobs. Semantics of weakening and contraction. *Ann. Pure & Appl. Logic*, 69(1):73–106, 1994.
4. B. Jacobs. Dagger categories of tame relations. `arxiv.org/abs/1101.1077`, 2011.
5. J. Kempe. Quantum random walks – an introductory overview. *Contemporary Physics*, 44:307–327, 2003.
6. A. Kock. Bilinearity and cartesian closed monads. *Math. Scand.*, 29:161–174, 1971.
7. A. Kock. Closed categories generated by commutative monads. *Journ. Austr. Math. Soc.*, XII:405–424, 1971.
8. S. Liang, P. Hudak, and M. Jones. Monad transformers and modular interpreters. In *Principles of Programming Languages*, pages 333–343. ACM Press, 1995.
9. D. Pavlović, M. Mislove, and J. Worrell. Testing semantics: Connecting processes and process logics. In M. Johnson and V. Vene, editors, *Algebraic Methods and Software Technology*, number 4019 in Lect. Notes Comp. Sci., pages 308–322. Springer, Berlin, 2006.
10. S. Perdrix. Partial observation of quantum Turing machine and weaker wellformedness condition. Proceedings of: Quantum Physics and Logic and Development of Computational Models, see `http://web.comlab.ox.ac.uk/people/simon.perdrix/publi/weakerQTM.pdf`, 2008.
11. J. Rutten. Universal coalgebra: a theory of systems. *Theor. Comp. Sci.*, 249:3–80, 2000.
12. S. Venegas-Andraca. *Quantum Walks for Computer Scientists*. Morgan & Claypool, 2008.

A Coalgebras of a monad form a monoid in algebras

Let $\mathbf{A} = (\mathbf{A}, I, \otimes, \multimap)$ be a symmetric monoidal closed category. A monad $T = (T, \mu, \eta)$ is called monoidal (or commutative) if it comes with a 'double strength' natural transformation $\mathsf{dst}\colon T(X)\otimes T(Y) \to T(X\otimes Y)$ commuting appropriately with the monoidal isomorphisms and with the unit η and multiplication μ. We abbreviate $\mathsf{st} = \mathsf{dst} \circ (\mathrm{id} \otimes \eta)\colon T(X) \otimes Y \to T(X \otimes Y)$ and $\mathsf{st}' = \mathsf{dst} \circ (\eta \otimes \mathrm{id})\colon X \otimes T(Y) \to T(X \otimes Y)$. One can also express this double strength as $\mathsf{dst} = \mu \circ T(\mathsf{st}) \circ \mathsf{st}' = \mu \circ T(\mathsf{st}') \circ \mathsf{st}$, see [3] for details.

We assume that the categories $\mathbf{A}$ and $\mathit{Alg}(T)$ has enough coequalisers so that $\mathit{Alg}(T)$ is also symmetric monoidal via the canonical constructions from [7,6], with tensor $\otimes^T$ and tensor unit $I^T = T(I)$. The key property of this tensor of algebras $\otimes^T$ is that there is a bijective correspondence:

$$\frac{\big(TX \overset{a}{\to} X\big) \otimes^T \big(TY \overset{b}{\to} Y\big) \xrightarrow{\;f\;} \big(TZ \overset{c}{\to} Z\big) \quad \text{in } \mathit{Alg}(T)}{X \otimes Y \xrightarrow[g]{} Z \quad \text{bihomomorphism}} \tag{9}$$

Such a map $g\colon X \otimes Y \to Z$ is a bihomomorphism if the following diagram commutes.

$$\begin{tikzcd}
T(X)\otimes T(Y) \arrow[r, "\mathsf{dst}"] \arrow[d, "a\otimes b"'] & T(X\otimes Y) \arrow[r, "T(g)"] & T(Z) \arrow[d, "c"] \\
X\otimes Y \arrow[rr, "g"] & & Z
\end{tikzcd}$$

The next result may be read as: internal T-coalgebras form a monoid in $\mathit{Alg}(T)$.

Proposition 3. *In the situation described above,*

1. *for each $X \in \mathbf{A}$, the object $T(X)^X = X \multimap T(X)$ in $\mathbf{A}$ "of T-coalgebras" carries an algebra structure $a_X\colon T\big(T(X)^X\big) \to T(X)^X$, obtained by abstraction $\Lambda(-)$ as:*

$$a_X = \Lambda\Big(T\big(T(X)^X\big) \otimes X \xrightarrow{\mathsf{st}} T\big(T(X)^X \otimes X\big) \xrightarrow{T(ev)} T^2(X) \xrightarrow{\mu} T(X)\Big).$$

2. *This algebra $a_X \in \mathit{Alg}(T)$ carries a monoid structure in $\mathrm{Alg}(T)$ given by Kleisli composition, with monoid unit $u\colon I^T \to T(X)^X$ defined as:*

$$u = \Lambda\Big(T(I) \otimes X \xrightarrow{\mathsf{st}} T(I \otimes X) \xrightarrow[\cong]{T(\lambda)} T(X)\Big)$$

The monoid multiplication $m\colon T(X)^X \otimes^T T(X)^X \to T(X)^X$ is obtained via the correspondence (9) from the bihomomorphism $T(X)^X \otimes T(X)^X \to T(X)^X$ that one gets by abstraction from:

$$\begin{tikzcd}
\big(T(X)^X \otimes T(X)^X\big) \otimes X \arrow[d, "\alpha^{-1}"', "\cong"] & & & T(X) \\
T(X)^X \otimes \big(T(X)^X \otimes X\big) \arrow[r, "\mathrm{id}\otimes ev"] & T(X)^X \otimes T(X) \arrow[r, "\mathsf{st}'"] & T\big(T(X)^X \otimes X\big) \arrow[r, "T(ev)"] & T^2(X) \arrow[u, "\mu"']
\end{tikzcd}$$