

Probabilities, Distribution Monads, and Convex Categories

Bart Jacobs

*Institute for Computing and Information Sciences (iCIS),
Radboud University Nijmegen
P.O. Box 9010, 6500 GL Nijmegen, The Netherlands.
Email: B.Jacobs@cs.ru.nl
URL: <http://www.cs.ru.nl/B.Jacobs>*

Abstract

Probabilities are understood abstractly as forming a monoid in the category of effect algebras. They can be added, via a partial operation, and multiplied. This generalises key properties of the unit interval $[0, 1]$. Such effect monoids can be used to define a probability distribution monad, again generalising the situation for $[0, 1]$ -probabilities. It will be shown that there are translations back-and-forth, in the form of an adjunction, between effect monoids and “convex” monads. This convexity property is formalised, both for monads and for categories. In the end this leads to “triangles of adjunctions” (in the style of Coumans and Jacobs) relating all the three relevant structures: probabilities, monads, and categories.

1 Introduction

In the foundation of quantum mechanics so-called effect algebras [9,8] have emerged as mathematical structures that capture both probabilities and propositions in a single mathematical notion. The unit interval, with its partial addition operation, is a main example. Its multiplication operation is captured abstractly via the notion of effect monoid: a monoid in the category of effect algebras.

This paper establishes a tight connection between such effect monoids and distribution monads, taking probabilities not from $[0, 1]$ but from such an effect monoid. Distribution monads are frequently used in the abstract modeling of probabilistic state based systems (see [5] for an overview). This connection makes it possible to consider a wider range of systems, involving a more general notion of probability.

The kind of monads that arise in this way are characterised as “convex” monads. Actually, a notion of convex category is introduced first, in which certain homsets turn out to be effect monoids. This may be seen as the main technical result of the paper. In the end the various connections will be organised in Section 7 in terms of “triangles of adjunctions”, following the paradigm of [7].

This paper builds on [14], where an adjunction between effect algebras and convex functors was established. The present paper extends this adjunction to effect monoids and convex monads, and adds the notion of convex category. The resulting triangles in Section 7 describe a close connection between fundamental mathematical structures in the context of probabilistic systems.

2 Preliminaries

This paper assumes familiarity with basic category theory, see *e.g.* [19,4,18,6]. It uses coproducts/sums $(+, 0)$, with coprojections $\kappa_i: X_i \rightarrow X_1 + X_2$ and unique maps $!: 0 \rightarrow X$. It also uses monoidal structure (\otimes, I) , with the standard (associativity and unit) isomorphisms. The notion of monad will play a central role. We refer the reader to the literature for the general notion, and only explicitly describe the distribution monad \mathcal{D} on the category **Sets** because it plays such a fundamental role. On a set X one defines:

$$\mathcal{D}(X) = \{\varphi: X \rightarrow [0, 1] \mid \text{supp}(\varphi) \text{ is finite and } \sum_x \varphi(x) = 1\} \quad (1)$$

The subset $\text{supp}(\varphi) \subseteq X$ is the support of φ and contains the elements x with $\varphi(x) \neq 0$. An element φ may be understood as a formal convex sum $r_1x_1 + \dots + r_nx_n$ where $\text{supp}(\varphi) = \{x_1, \dots, x_n\}$ and $r_i = \varphi(x_i) \in [0, 1]$ is the probability associated with x_i , in such a way that $r_1 + \dots + r_n = 1$. An implicit convention is to identify $rx + sx$ in such formal convex sums with $(r + s)x$.

This mapping $X \mapsto \mathcal{D}(X)$ is functorial: for $f: X \rightarrow Y$ we may describe $\mathcal{D}(f): \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ as:

$$\left(r_1x_1 + \dots + r_nx_n \right) \mapsto \left(r_1f(x_1) + \dots + r_nf(x_n) \right).$$

The unit $\eta: X \rightarrow \mathcal{D}(X)$ and multiplication $\mu: \mathcal{D}^2(X) \rightarrow \mathcal{D}(X)$ of this monad can then be described as:

$$\begin{aligned} x &\xrightarrow{\eta} 1x \\ (r_1\varphi_1 + \dots + r_n\varphi_n) &\xrightarrow{\mu} \lambda x \in X. r_1 \cdot \varphi_1(x) + \dots + r_n \cdot \varphi_n(x) \end{aligned}$$

where the notation $\lambda x \in X. \dots$ is used for the function $x \mapsto \dots$.

In general, for a monad $T = (T, \eta, \mu)$ on a category \mathbb{A} we write $\mathcal{Kl}(T)$ for the Kleisli category of \mathbb{A} . Its objects are $X \in \mathbb{A}$ and its morphisms $X \rightarrow Y$ are maps $X \rightarrow T(Y)$ in \mathbb{A} . There is a standard functor $\mathcal{J}: \mathbb{A} \rightarrow \mathcal{Kl}(\mathbb{A})$, given by $\mathcal{J}(X) = X$ and $\mathcal{J}(f) = \eta \circ f$. We write $\mathcal{Kl}_{\mathbb{N}}(T) \hookrightarrow \mathcal{Kl}(T)$ for the full subcategory with numbers $n \in \mathbb{N}$ as objects, considered as n -element set.

3 Probabilities as scalars

In a monoidal category, with tensor structure (\otimes, I) , the endomaps $I \rightarrow I$ on the tensor unit are usually called scalars. They obviously form a monoid, under composition. A remarkable result of Kelly and Laplaza [16] says that this monoid is actually commutative; it is called the ‘miracle’ of scalars in [2, §3.2]. More recent work in the area of quantum computation has led to renewed interest in such scalars, see for instance [1,2], where it is shown that the presence of biproducts makes this homset of scalars a semiring, and that daggers \dagger make it involutive. A systematic account is given in [7], relating semirings, additive monads and symmetric monoidal categories with biproducts, in a triangle of adjunctions. In a sense, the present work adapts this triangle, namely from scalars to probabilities, *i.e.* from semirings (formally: monoids in the category of commutative monoids) to effect monoids (monoids in the category of effect algebras).

This section presents an analogue of the Kelly-Laplaza result that will be used later on in a probabilistic setting. For instance, the following proposition applies to the Kleisli category of the distribution monad \mathcal{D} on **Sets**, with stochastic relations as morphisms. Maps $1 \rightarrow 2$ in $\mathcal{Kl}(\mathcal{D})$ correspond to probabilities, *i.e.* to maps in the unit interval $[0, 1]$. The monoid structure defined below corresponds to multiplication on $[0, 1]$. This will be elaborated after the proof.

Proposition 3.1 *Assume a category \mathbb{A} with a final object $1 \in \mathbb{A}$, for which the coproduct $2 = 1 + 1$ exists.*

(1) *The homset $\mathbb{A}(1, 2)$ is a monoid with zero object, where:*

$$1 \stackrel{\text{def}}{=} (1 \xrightarrow{\kappa_1} 2) \quad 0 \stackrel{\text{def}}{=} (1 \xrightarrow{\kappa_2} 2) \quad x \cdot y \stackrel{\text{def}}{=} (1 \xrightarrow{x} 2 \xrightarrow{[y, \kappa_2]} 2).$$

(2) *In case the category \mathbb{A} also carries a tensor \otimes such that:*

(a) *1 is tensor unit for \otimes , and*

(b) *\otimes distributes over $+$: the canonical maps $X \otimes Y + X \otimes Z \rightarrow X \otimes (Y + Z)$ are isomorphisms,*

then the monoid structure on $\mathbb{A}(1, 2)$ is commutative.

Proof. The first point follows from straightforward calculations:

$$\begin{aligned}
1 \cdot y &= [y, \kappa_2] \circ \kappa_1 = y \\
x \cdot 1 &= [\kappa_1, \kappa_2] \circ x = \text{id} \circ x = x \\
0 \cdot y &= [y, \kappa_2] \circ \kappa_2 = \kappa_2 = 0 \\
x \cdot 0 &= [\kappa_2, \kappa_2] \circ x = \kappa_2 \circ [\text{id}, \text{id}] \circ x = \kappa_2 \circ \text{id} = 0,
\end{aligned}$$

where the latter case uses that the composite $[\text{id}, \text{id}] \circ x: 1 \rightarrow 2 \rightarrow 1$ is the identity because $1 \in \mathbb{A}$ is final. Associativity holds since:

$$\begin{aligned}
(x \cdot y) \cdot z &= [z, \kappa_2] \circ (x \cdot y) = [z, \kappa_2] \circ [y, \kappa_2] \circ x \\
&= [[z, \kappa_2] \circ y, [z, \kappa_2] \circ \kappa_2] \circ x \\
&= [y \cdot z, \kappa_2] \circ x = x \cdot (y \cdot z).
\end{aligned}$$

The second point follows because in presence of tensors \otimes both $x \cdot y = [y, \kappa_2] \circ x$ and $y \cdot x = [x, \kappa_2] \circ y$ are equal to the composite:

$$1 \xrightarrow{\cong} 1 \otimes 1 \xrightarrow{x \otimes y} 2 \otimes 2 \xrightarrow{\cong} 2 + 2 \xrightarrow{[\text{id}, \kappa_2 \circ !]} 2. \quad (2)$$

This equality will be shown for $x \cdot y$ via a diagram chase. All the maps labeled with ‘ \cong ’ are canonical isomorphisms, in:

$$\begin{array}{ccccc}
1 & \xrightarrow{x} & 2 & & \\
\cong \downarrow & & \cong \downarrow & & \\
1 \otimes 1 & \xrightarrow{x \otimes \text{id}} & 2 \otimes 1 & & \\
x \otimes y \downarrow & \swarrow \text{id} \otimes y & \cong \downarrow & & \\
2 \otimes 2 & & 1 \otimes 1 + 1 \otimes 1 & \xrightarrow{\cong} & 1 + 1 \\
\cong \downarrow & \swarrow \text{id} \otimes y + \text{id} \otimes y & \text{id} \otimes y + ! \downarrow & & \\
1 \otimes 2 + 1 \otimes 2 & \xrightarrow{\text{id} + !} & 1 \otimes 2 + 1 & & \\
\cong \downarrow & & \cong \downarrow & & \\
(1 \otimes 1 + 1 \otimes 1) + (1 \otimes 1 + 1 \otimes 1) & & & & \\
\cong \downarrow & & & & \\
(1 + 1) + (1 + 1) & \xrightarrow{\text{id} + !} & 2 + 1 & \xrightarrow{[\text{id}, \kappa_2]} & 2 \\
& & \swarrow y + \text{id} & & \downarrow [y, \kappa_2]
\end{array}$$

The same argument can be used for $y \cdot x$, by exchanging x, y everywhere in the previous diagram, except in the label $x \otimes y$. This shows $x \cdot y = y \cdot x$. \square

Notice that the tensors \otimes do not have to be symmetric for this commutativity result—just like in the Kelly-Laplaza case.

Remark 3.2 (1) We briefly illustrate that the monoid structure in this proposition yields multiplication on $[0, 1]$ when applied to the Kleisli category $\mathcal{Kl}(\mathcal{D})$ of the distribution monad \mathcal{D} from (1). The latter satisfies $\mathcal{D}(1) \cong$

1, so that 1 is final in $\mathcal{Kl}(\mathcal{D})$. A map $1 \rightarrow \mathcal{D}(2)$ corresponds to—and is often identified with—an element in $\mathcal{D}(2)$, which can be written as formal convex sum $x\mathbf{1} + (1-x)\mathbf{2}$, where $2 = \{\mathbf{1}, \mathbf{2}\}$ and $x \in [0, 1]$. The multiplication \cdot from Proposition 3.1 involves a Kleisli composition:

$$x \cdot y = \left(1 \xrightarrow{x} \mathcal{D}(2) \xrightarrow{\mathcal{D}([y, \kappa_2])} \mathcal{D}^2(2) \xrightarrow{\mu} \mathcal{D}(2) \right)$$

that can be described more explicitly as:

$$\begin{aligned} (\mu \circ \mathcal{D}([y, \kappa_2]) \circ x) &= (\mu \circ \mathcal{D}([y, \kappa_2]))(x\mathbf{1} + (1-x)\mathbf{2}) \\ &= \mu(x(y\mathbf{1} + (1-y)\mathbf{2}) + (1-x)(\mathbf{01} + \mathbf{12})) \\ &= xy\mathbf{1} + (x(1-y) + (1-x))\mathbf{2} \\ &= xy\mathbf{1} + (1-xy)\mathbf{2}, \end{aligned}$$

since $x(1-y) + (1-x) = x - xy + 1 - x = 1 - xy$. Hence the result is an appropriate convex sum, capturing the multiplication of $x, y \in [0, 1]$. Of course we know that this multiplication is commutative—but also that the distribution monad is commutative so that its Kleisli category has tensors.

- (2) The composition defined in Proposition 3.1 may be understood as composition of endomaps $1 \rightarrow 1$ in the Kleisli category of the lift monad $1 + (-)$ on the category \mathbb{A} . It may also be extended to an action, namely of the monoid $\mathbb{A}(1, 2)$ on homsets $\mathbb{A}(X, 2)$, via:

$$x * f \stackrel{\text{def}}{=} \left(X \xrightarrow{f} 2 \xrightarrow{[x, \kappa_2]} 2 \right).$$

Clearly, $1 * f = f$ and $(x \cdot y) * f = y * (x * f)$. We do not need this action $*$ in the sequel—but we do show that it yields a module structure in Remark 5.6 (2).

4 Effect algebras and effect monoids

This section recalls the basics of effect algebras (from [9], see also [8] for an overview, references and background information), and introduces effect monoids as monoids in the category of effect algebras. To start, we need the notion of partial commutative monoid (PCM). It consists of a set M with a zero element $0 \in M$ and a partial binary operation $\odot: M \times M \rightarrow M$ satisfying the three requirements below—involving the notation $x \perp y$ for: $x \odot y$ is defined.

- (1) Commutativity: $x \perp y$ implies $y \perp x$ and $x \odot y = y \odot x$;

- (2) Associativity: $y \perp z$ and $x \perp (y \otimes z)$ implies $x \perp y$ and $(x \otimes y) \perp z$ and also $x \otimes (y \otimes z) = (x \otimes y) \otimes z$;
- (3) Zero: $0 \perp x$ and $0 \otimes x = x$;

When $x \perp y$ we say that elements x, y are orthogonal. More generally, a (finite) subset of a PCM is called orthogonal if all its elements are pairwise orthogonal. In writing $x \otimes y$ it is usually implicitly assumed that $x \otimes y$ is defined, *i.e.* that x, y are orthogonal.

An example of a PCM is the unit interval $[0, 1]$ of real numbers, where \otimes is the partially defined sum $+$.

Definition 4.1 *An effect algebra is a partial commutative monoid $(E, 0, \otimes)$ with an “orthosupplement”: a unary operation $(-)^{\perp}: E \rightarrow E$ satisfying:*

- (1) $x^{\perp} \in E$ is the unique element in E with $x \otimes x^{\perp} = 1$, where $1 = 0^{\perp}$;
- (2) $x \perp 1 \Rightarrow x = 0$.

Effect algebras generalise both probabilities and propositions.

Example 4.2 *We briefly discuss several classes of examples.*

- (1) *A singleton set forms an example of a degenerate effect algebra, with $0 = 1$. A two element set $2 = \{0, 1\}$ is also an example.*
- (2) *A more interesting example is the unit interval $[0, 1] \subseteq \mathbb{R}$ of real numbers, with $r^{\perp} = 1 - r$ and $r \otimes s$ is defined as $r + s$ in case this sum is in $[0, 1]$. In fact, for each positive number $M \in \mathbb{R}$ the interval $[0, M]_{\mathbb{R}} = \{r \in \mathbb{R} \mid 0 \leq r \leq M\}$ is an example of an effect algebra, with $r^{\perp} = M - r$. More generally, so-called “interval effect algebras”, see *e.g.* [10] or [8, 1.4] can be obtained from ordered Abelian groups. This includes the “effects” on a Hilbert space H , consisting of the positive operators $H \rightarrow H$ below the identity.*
- (3) *A separate class of examples has a join as sum \vee . Let $(L, \vee, 0, (-)^{\perp})$ be an ortholattice: $\vee, 0$ are finite joins and complementation $(-)^{\perp}$ satisfies $x \leq y \Rightarrow y^{\perp} \leq x^{\perp}$, $x^{\perp\perp} = x$ and $x \vee x^{\perp} = 1 = 0^{\perp}$. This L is called an orthomodular lattice if $x \leq y$ implies $y = x \vee (x^{\perp} \wedge y)$. Such an orthomodular lattice forms an effect algebra in which $x \otimes y$ is defined if and only if $x \perp y$ (*i.e.* $x \leq y^{\perp}$, or equivalently, $y \leq x^{\perp}$); and in that case $x \otimes y = x \vee y$. This restriction of \vee is needed for the validity of requirements (1) and (2) in Definition 4.1.*

In particular, the lattice of closed subsets of a Hilbert space is an orthomodular lattice and thus an effect algebra. This applies more generally to the kernel subobjects of an object in a dagger kernel category [11].

- (4) *Since Boolean algebras are (distributive) orthomodular lattices, they are also effect algebras. By distributivity, elements in a Boolean algebra are orthogonal if and only if they are disjoint, *i.e.* $x \perp y$ iff $x \wedge y = 0$. In*

particular, the Boolean algebra of measurable subsets of a measure space forms an effect algebra, where $U \otimes V$ is defined if $U \cap V = \emptyset$, and is then equal to $U \cup V$.

Definition 4.3 A homomorphism $E \rightarrow D$ of effect algebras is given by a function $f: E \rightarrow D$ between the underlying sets satisfying $f(1) = 1$, and if $x \perp x'$ in E then both $f(x) \perp f(x')$ in D and $f(x \otimes x') = f(x) \otimes f(x')$.

Effect algebras and their homomorphisms form a category, called **EA**.

Homomorphisms are like measurable maps. Indeed, for the effect algebra Σ associated in Example 4.2 (4) with a measure space (X, Σ) , effect algebra homomorphisms $f: \Sigma \rightarrow [0, 1]$ satisfy $f(U \cup V) = f(U) + f(V)$ in case U, V are disjoint—because then $U \otimes V$ is defined and equals $U \cup V$. In general, effect algebra homomorphisms $E \rightarrow [0, 1]$ to the unit interval are often called states. They give rise to a (dual) adjunction with convex sets, see [14], via $[0, 1]$ as dualising object.

Homomorphisms of effect algebras preserve all the relevant structure.

Lemma 4.4 Let $f: E \rightarrow D$ be a homomorphism of effect algebras. Then:

$$f(x^\perp) = f(x)^\perp \quad \text{and thus} \quad f(0) = 0.$$

Proof. From $1 = f(1) = f(x \otimes x^\perp) = f(x) \otimes f(x^\perp)$ we get $f(x^\perp) = f(x)^\perp$ by uniqueness of orthosupplements. Hence: $f(0) = f(1^\perp) = f(1)^\perp = 1^\perp = 0$. \square

It is not hard to see that the one-element effect algebra 1 is final, and the two-element effect algebra 2 is initial.

4.1 Effect monoids

In [15] it is shown that the category **EA** is complete and cocomplete, and has a symmetric monoidal structure. The tensor \otimes can be characterised via the following bijective correspondence.

$$\frac{E_1 \otimes E_2 \xrightarrow{f} D \quad \text{in } \mathbf{EA}}{E_1 \times E_2 \xrightarrow{g} D \quad \text{bihomomorphism}}$$

Such a bihomomorphism is a function $g: E_1 \times E_2 \rightarrow D$ between the underlying sets for which $g(1, 1) = 1$ and both $g(x_1, -): E_2 \rightarrow D$ and $g(-, x_2): E_1 \rightarrow D$ are homomorphisms of PCMs.

The tensor unit is the two-element effect algebra $2 \in \mathbf{EA}$. Since it is also initial in **EA** we have “tensors with coprojections”. With such a tensor product one

can combine for instance probabilities and propositions, in an effect algebra $[0, 1] \otimes \mathcal{P}(X)$ of weighted predicates on X .

Definition 4.5 *An effect monoid is a monoid object $2 \xrightarrow{1} M \xleftarrow{\cdot} M \otimes M$ in the category \mathbf{EA} . Explicitly, it is given by an effect algebra $M \in \mathbf{EA}$ with a bihomomorphism $\cdot: M \times M \rightarrow M$ that satisfies the familiar monoid equations: $1 \cdot x = x = x \cdot 1$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, where $1 \in M$ is the top element.*

It is obvious what it means for an effect monoid to be commutative. Also the notion of homomorphism of effect monoids is straightforward: it is a map of effect algebras that commutes with multiplication. Thus we have categories **EMon** and **CEMon** of (commutative) effect monoids and their homomorphisms.

The unit interval is an example of a commutative effect monoid: multiplication $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a bihomomorphism of effect algebras, since if $x_1 \perp x_2$, i.e. $x_1 + x_2 \leq 1$, then for each $y \in [0, 1]$ one has $x_1 \cdot y + x_2 \cdot y \leq 1$, so that $x_1 \cdot y \perp x_2 \cdot y$, and $(x_1 \otimes x_2) \cdot y = (x_1 + x_2) \cdot y = x_1 \cdot y + x_2 \cdot y = x_1 \cdot y \otimes x_2 \cdot y$.

The effects on a Hilbert space (positive operators below the identity) provide more examples, of non-commutative effect monoids.

5 Convex categories

In Section 3 we have seen how certain categorical structure makes a homset of maps $1 \rightarrow 2 = 1 + 1$ in a category into a monoid. In this section we investigate what is required to make this monoid into an effect monoid. We thus assume that we have a category \mathbb{A} with finite coproducts $(+, 0)$ and a final object $1 \in \mathbb{A}$. This object $1 \in \mathbb{A}$ can be used to obtain (representations of) natural numbers $\underline{n} \in \mathbb{A}$, for $n \in \mathbb{N}$. One simply puts:

$$\underline{0} = 0 \quad \text{and} \quad \underline{n+1} = \underline{n} + 1.$$

We shall use these “numbers” $\underline{n} \in \mathbb{A}$ with coprojections $\kappa_i: 1 \rightarrow \underline{n}$ for $1 \leq i \leq n$. The following maps will be very important in the sequel.

$$\underline{n+1} \xrightarrow{\nabla_i} \underline{2} \quad \text{where} \quad \nabla_i \circ \kappa_j = \begin{cases} \kappa_1 & \text{if } i = j \\ \kappa_2 & \text{otherwise} \end{cases} \quad \text{i.e.} \quad \begin{array}{c} n+1 \\ n \\ \vdots \\ i \\ \vdots \\ 1 \end{array} \begin{array}{c} \diagdown \\ \diagdown \\ \text{---} \\ \diagup \\ \diagup \\ \diagup \end{array} \begin{array}{c} \\ \\ 2 \\ \\ 1 \end{array} \quad (3)$$

(where $1 \leq i \leq n$ and $1 \leq j \leq n+1$). We often use the two maps $\nabla_1, \nabla_2: \underline{3} \rightarrow \underline{2}$, for the special case $n = 2$; they can be described explicitly as $\nabla_1 = [\text{id}, \kappa_2]$ and $\nabla_2 = [[\kappa_2, \kappa_1], \kappa_2]$.

Writing the underlining \underline{n} gets tedious, so we often drop it when no confusion arises, and write $n \in \mathbb{A}$ for the n -fold coproduct of $1 \in \mathbb{A}$.

In **Sets** we identify n (to be more precise: \underline{n}) with the set $\{1, 2, \dots, n\}$. The coprojection $\kappa_i: 1 \rightarrow n$ is then simply i . The maps $\nabla_i: n+1 \rightarrow 2$ from (3), for $1 \leq i \leq n$, satisfy $\nabla_i(j) = 1$ if $i = j$ and $\nabla_i(j) = 2$ if $i \neq j$, as pictured.

Definition 5.1 *Let \mathbb{A} be a category with finite coproducts and final object $1 \in \mathbb{A}$. It will be called a convex category if it satisfies the following two requirements.*

- (1) *For each $n \in \mathbb{N}$, the above maps $\nabla_i: n+1 \rightarrow 2$ for $i \leq n$ are jointly monic: if $\nabla_i \circ f = \nabla_i \circ g$ for each $i \leq n$, then $f = g$.*
- (2) *The following three diagrams in \mathbb{A} are pullbacks.*

$$\begin{array}{ccc}
 n \xrightarrow{\kappa_1} n+1 & 1 \xrightarrow{\kappa_1} 1+n & n+m \xrightarrow{!+\text{id}} 1+m \\
 \downarrow \lrcorner & \parallel \lrcorner & \text{id}+! \downarrow \lrcorner \\
 1 \xrightarrow{\kappa_1} 1+1=2 & 1 \xrightarrow{\kappa_1} 1+1=2 & n+1 \xrightarrow{!+\text{id}} 1+1=2
 \end{array} \quad (4)$$

The intuition behind the first requirement is that two entities of $n+1$ elements are equal, when n of its elements are equal. This is typically the case for convex sums. As we shall see, the Kleisli category $\mathcal{Kl}(\mathcal{D})$ of the distribution monad—given by formal convex sums—forms an important example. The three pullbacks in the second requirement also make sense in such a context of convex sums. For instance the first one can be read as: if an entity of $n+1$ element is determined by the first n of them, then the last one is irrelevant.

Our claim is that each homset $\mathbb{A}(X, 2)$ in a convex category \mathbb{A} is an effect algebra. This requires some work (the essence of which is already in [14], but in different form). First, two maps $f_1, f_2: X \rightarrow 2$ in \mathbb{A} will be called orthogonal, written as $f_1 \perp f_2$, if there is a “bound” $b: X \rightarrow 3$ in \mathbb{A} with $\nabla_1 \circ b = f_1$ and $\nabla_2 \circ b = f_2$. We notice that such a bound is necessarily unique by the jointly-monic requirement for the ∇ ’s in Definition 5.1 (1). Further, this definition of orthogonality obviously generalises to n maps $f_i: X \rightarrow 2$, requiring a bound $X \rightarrow n+1$.

If maps $f_1, f_2: X \rightarrow 2$ have bound b , then we define:

$$f_1 \odot f_2 \stackrel{\text{def}}{=} \left(X \xrightarrow{b} 3 \xrightarrow{!+\text{id}} 2 \right) \quad (5)$$

where $!+\text{id} = [[\kappa_1, \kappa_1], \kappa_2]: 2+1=3 \rightarrow 2=1+1$ sends $1, 2 \mapsto 1$ and $3 \mapsto 2$. This style of defining a partial operation via a bound is reminiscent of partially

additive categories [3]. In this homset $\mathbb{A}(X, 2)$ we further define:

$$1 \stackrel{\text{def}}{=} \left(X \xrightarrow{!} 1 \xrightarrow{\kappa_1} 2 \right) \quad 0 \stackrel{\text{def}}{=} \left(X \xrightarrow{!} 1 \xrightarrow{\kappa_2} 2 \right) \quad f^\perp \stackrel{\text{def}}{=} \left(X \xrightarrow{f} 2 \xrightarrow{\cong} [\kappa_2, \kappa_1] \rightarrow 2 \right).$$

Theorem 5.2 *Let \mathbb{A} be a convex category.*

- (1) *With the above definitions of $\otimes, 0, 1, (-)^\perp$, each homset $\mathbb{A}(X, 2)$ is an effect algebra.*
- (2) *The special case $\mathbb{A}(1, 2)$, for $X = 1$, is an effect monoid, using the monoid structure from Proposition 3.1.*

Proof. We check some of the requirements that must hold for effect algebras.

The partial sum \otimes is commutative, since if $b: X \rightarrow 3$ is a bound for $f, g: X \rightarrow 2$, then $b' = ([\kappa_2, \kappa_1] + \text{id}) \circ b: X \rightarrow 3$ is a bound for g, f , with the same sum:

$$g \otimes f = (! + \text{id}) \circ b' = (! + \text{id}) \circ ([\kappa_2, \kappa_1] + \text{id}) \circ b = (! + \text{id}) \circ b = f \otimes g.$$

The zero $0 = \kappa_2 \circ !: X \rightarrow 2$ is a zero element for \otimes , since for an arbitrary element $f: X \rightarrow 2$ there is a bound $b = (\kappa_2 + \text{id}) \circ f: X \rightarrow 2 + 1 = 3$ for 0 and f , with sum f .

Associativity of \otimes requires more work. Assume $f, g, h: X \rightarrow 2$ are given with $f \perp g$, say with bound $a: X \rightarrow 3$, and $(f \otimes g) \perp h$, with bound $b: X \rightarrow 3$. The latter implies $\nabla_1 \circ b = f \otimes g = (! + \text{id}) \circ a$. Thus we have a situation:

$$\begin{array}{ccccc}
 X & & & & \\
 \swarrow & \xrightarrow{a} & & & \searrow \\
 & & 2 + 2 & \xrightarrow{\text{id}+!} & 2 + 1 \\
 \downarrow b & \xrightarrow{c} & \downarrow \text{id}+! & \lrcorner & \downarrow \text{id}+! \\
 & & 2 + 1 & \xrightarrow{\cong} & 1 + 2 & \xrightarrow{\text{id}+!} & 2 \\
 & & & \searrow & & & \\
 & & & \nabla_1 & & &
 \end{array}$$

Because the square is a pullback in \mathbb{A} there is a unique map c as indicated. We now take $c' = [[\kappa_2, \kappa_1 \circ \kappa_1], \kappa_2 + \text{id}] \circ c: X \rightarrow 2 + 2 \rightarrow 2 + 1 = 3$. It is not hard to see that it is a bound for g and h . We next take $c'' = [\kappa_1, \kappa_2 + \text{id}] \circ$

$c: X \rightarrow 2 + 2 \rightarrow 2 + 1 = 3$. This c'' is a bound for f and $g \otimes h$. Finally,

$$\begin{aligned}
f \otimes (g \otimes h) &= (! + \text{id}) \circ c'' \\
&= (! + \text{id}) \circ [\kappa_1, \kappa_2 + \text{id}] \circ c \\
&= [\kappa_1 \circ !, \text{id} + \text{id}] \circ c \\
&= [\kappa_1, \text{id}] \circ (! + \text{id}) \circ c \\
&= [\kappa_1, \text{id}] \circ [\text{id} + \kappa_1, \kappa_2 \circ \kappa_2] \circ b \\
&= [[\kappa_1, \kappa_1], \kappa_2] \circ b \\
&= [\kappa_1 \circ !, \kappa_2] \circ b \\
&= (! + \text{id}) \circ b \\
&= (f \otimes g) \otimes h.
\end{aligned}$$

The equation $f \otimes f^\perp = 1$ can be proven via the bound $b = \kappa_1 \circ f: X \rightarrow 3$.

We also have to check that f^\perp is the only element $g: X \rightarrow 2$ with $f \otimes g = 1$. Suppose $f \otimes g = 1$, say via a bound $b: X \rightarrow 3$ satisfying $\nabla_1 \circ b = f$ and $\nabla_2 \circ b = g$, and $(! + \text{id}) \circ b = 1 = \kappa_1 \circ !$. Since the sum is 1 we can do a ‘‘bound restriction’’ using the first pullback in Definition 5.1, as in:

$$\begin{array}{ccc}
X & \xrightarrow{b} & 2 + 1 \\
\text{---} c \text{---} & \searrow & \downarrow \kappa_1 \\
2 & \xrightarrow{\kappa_1} & 2 + 1 \\
\text{---} ! \text{---} & \searrow & \downarrow ! + \text{id} \\
1 & \xrightarrow{\kappa_1} & 1 + 1
\end{array} \quad (6)$$

Thus we can write the bound b as $b = \kappa_1 \circ c$, and so we get $f^\perp = g$, as required:

$$\begin{aligned}
f^\perp &= [\kappa_2, \kappa_1] \circ f \\
&= [\kappa_2, \kappa_1] \circ \nabla_1 \circ b \\
&= [\kappa_2, \kappa_1] \circ [\text{id}, \kappa_2] \circ \kappa_1 \circ c \\
&= [\kappa_2, \kappa_1] \circ c \\
&= [[\kappa_2, \kappa_1], \kappa_2] \circ \kappa_1 \circ c \\
&= \nabla_2 \circ b \\
&= g
\end{aligned}$$

We proceed by showing $1 \perp f \Rightarrow f = 0$. Assume thus $1 \perp f$, say via a bound $b: X \rightarrow 3$ satisfying $\nabla_1 \circ b = 1 = \kappa_1 \circ !$ and $\nabla_2 \circ b = f$. The first equation

gives rise to a diagram:

$$\begin{array}{ccc}
X & \xrightarrow{b} & 2 + 1 \\
\downarrow \text{!} & \dashrightarrow & \downarrow \cong \\
1 & \xrightarrow{\kappa_1} & 1 + 2 \\
\parallel & \lrcorner & \downarrow \text{id+!} \\
1 & \xrightarrow{\kappa_1} & 1 + 1
\end{array}
\quad \nabla_1$$

Via this dashed map we know that $b = \kappa_1 \circ \kappa_1 \circ \text{!}: X \rightarrow 1 \rightarrow 1 + 2 \cong 2 + 1$, and so:

$$f = \nabla_2 \circ b = [[\kappa_2, \kappa_1], \kappa_2] \circ \kappa_1 \circ \kappa_1 \circ \text{!} = \kappa_2 \circ \text{!} = 0.$$

For the second statement in the theorem, we concentrate on $X = 1$. We already know that the composition operation \cdot from Proposition 3.1 preserves 0. Hence we only need to show that it preserves \odot in each coordinate separately. Assume $x_1 \perp x_2$, for $x_1, x_2: 1 \rightarrow 2$, via bound $b: 1 \rightarrow 3$ satisfying $\nabla_i \circ b = x_i$.

We do the first coordinate first. We need to show $y \cdot x_1 \perp y \cdot x_2$, for which we can use as bound $c = [b, \kappa_2] \circ y: 1 \rightarrow 2 \rightarrow 3$, since:

$$\begin{array}{ll}
\nabla_1 \circ c = [\nabla_1 \circ b, \nabla_1 \circ \kappa_2] \circ y & \nabla_2 \circ c = [\nabla_2 \circ b, \nabla_2 \circ \kappa_2] \circ y \\
= [x_1, \kappa_2] \circ y & = [x_2, \kappa_2] \circ y \\
= y \cdot x_1 & = y \cdot x_2.
\end{array}$$

Now we get the required equation:

$$\begin{aligned}
y \cdot x_1 \odot y \cdot x_2 &= (! + \text{id}) \circ c = [(! + \text{id}) \circ b, (! + \text{id}) \circ \kappa_2] \circ y \\
&= [x_1 \odot x_2, \kappa_2] \circ y = y \cdot (x_1 \odot x_2).
\end{aligned}$$

The preservation of \odot in the second coordinate proceeds along the same line, using the bound $d = [[(\kappa_1 + \text{id}) \circ y, (\kappa_2 + \text{id}) \circ y], \kappa_2] \circ b: 1 \rightarrow 3 \rightarrow 3$. \square

As already briefly mentioned, stochastic matrices, in the form of the Kleisli category $\mathcal{Kl}(\mathcal{D})$ of the distribution monad \mathcal{D} , form an important example of a convex category. A general result to this effect will appear as Proposition 6.2, but it may be useful to see what the operation \odot defined in (5) amounts to in this case. Assume we have two orthogonal probabilities $f, g: 1 \rightarrow \mathcal{D}(2)$. Like in Remark 3.2 (1) they can be written as convex sums $f = x\mathbf{1} + (1-x)\mathbf{2}$ and $g = y\mathbf{1} + (1-y)\mathbf{2}$, for $x, y \in [0, 1]$. Orthogonality $f \perp g$ means that there is a bound $b: 1 \rightarrow \mathcal{D}(3)$, with $\mathcal{D}(\nabla_1) \circ b = f$ and $\mathcal{D}(\nabla_2) \circ b = g$. If we write $b = b_1\mathbf{1} + b_2\mathbf{2} + b_3\mathbf{3}$, for $b_i \in [0, 1]$ satisfying $b_0 + b_1 + b_2 = 1$, we get:

$$x = b_1 \quad 1 - x = b_2 + b_3 \quad y = b_2 \quad 1 - y = b_1 + b_3.$$

In particular, $x + y = b_1 + b_2 = 1 - b_3 \leq 1$. Hence x, y are summable (*i.e.* orthogonal) in $[0, 1]$. The sum $f \otimes g: 1 \rightarrow \mathcal{D}(2)$ can now be described as in (5), and yields the expected sum of convex combinations:

$$f \otimes g = (! + \text{id})(b) = (b_1 + b_2)\mathbf{1} + b_3\mathbf{2} = (x + y)\mathbf{1} + (1 - (x + y))\mathbf{2}.$$

The next result shows that one can form convex sums of maps into 2, in a convex category.

Definition 5.3 *Suppose, in an arbitrary convex category, we have n ‘probabilities’ $x_i: X \rightarrow 2$ which are orthogonal and add up to 1, together with n maps $f_i: 1 \rightarrow Y$. The x_i ’s have a bound $b: X \rightarrow n + 1$ with $\nabla_i \circ b = x_i$. Because $\otimes_i x_i = 1$, this b factors as $b = \kappa_1 \circ c$, for $c: X \rightarrow n$ —using what was called bound restriction, via a pullback like in (6) in the proof above. Hence we can now have a convex sum, written for instance as:*

$$x \star f = \otimes_i (x_i \cdot f_i) = \left(X \xrightarrow{c} n \xrightarrow{[f_1, \dots, f_n]} Y \right).$$

The notation \otimes is justified by the following result.

Lemma 5.4 *Suppose in the previous definition $X = 1$ and $Y = 2$, so that we can form multiplications $x_i \cdot f_i: 1 \rightarrow 2$. Then indeed, $x \star f$ is an actual sum $\otimes_i x_i \cdot f_i$.*

Proof. We shall do the proof for $n = 2$, and prove that an arbitrary sum $x \cdot f_1 \otimes x^\perp \cdot f_2$ exists. We take as bound $b = [(\kappa_1 + \text{id}) \circ f_1, (\kappa_2 + \text{id}) \circ f_2] \circ x: 1 \rightarrow 2 \rightarrow 3$. Then:

$$\begin{aligned} \nabla_1 \circ b &= [[\kappa_1, \kappa_2] \circ f_1, [\kappa_2, \kappa_2] \circ f_2] \circ x \\ &= [f_1, \kappa_2 \circ [\text{id}, \text{id}] \circ f_2] \circ x \\ &= [f_1, \kappa_2] \circ x \quad \text{since } [\text{id}, \text{id}] \circ f_2: 1 \rightarrow 2 \rightarrow 1 \text{ is identity} \\ &= x \cdot f_1 \\ \nabla_2 \circ b &= [[\kappa_2, \kappa_2] \circ f_1, [\kappa_1, \kappa_2] \circ f_2] \circ x \\ &= [\kappa_2, f_2] \circ [\kappa_2, \kappa_1] \circ x^\perp \\ &= x^\perp \cdot f_2 \\ (! + \text{id}) \circ b &= [(\text{id} + \text{id}) \circ f_1, (\text{id} + \text{id}) \circ f_2] \circ x \\ &= [f_1, f_2] \circ x \\ &= x \star f, \end{aligned}$$

where \star is as described in Definition 5.3, since $\kappa_1 \circ x$ is a bound for x, x^\perp . \square

It turns out that the effect algebra structure on homsets of maps $X \rightarrow 2$ is preserved under precomposition. This means that we obtain an indexed category of effect algebras, which has a generic object (classifier) by construction. In this indexed category the fibres are effect algebras. They have some basic logical structure $\otimes, 0, 1, (-)^\perp$ that generalises both probabilities and propositions. Such logical modeling is not pursued here any further.

Proposition 5.5 *Let \mathbb{A} be a convex category. For each map $h: X \rightarrow Y$, precomposition with h , as in:*

$$\mathbb{A}(Y, 2) \xrightarrow{h^* = (-) \circ h} \mathbb{A}(X, 2)$$

preserves the effect algebra structure. This yields an indexed category

$$\mathbb{A}^{op} \longrightarrow \mathbf{EA} \quad \text{by} \quad X \longmapsto \mathbb{A}(X, 2).$$

The fibres over X are by construction the same as maps $X \rightarrow 2$ in the base category. Hence $2 \in \mathbb{A}$ forms a generic object (or classifier of predicates, see [13]).

Proof. Clearly, $h^*(0) = 0 \circ h = \kappa_2 \circ ! \circ h = \kappa_2 \circ ! = 0$, and similarly $h^*(1) = 1$. Now assume $f \perp g$ for $f, g: Y \rightarrow 2$, via a bound $b: Y \rightarrow 3$. Then $b \circ h: X \rightarrow 3$ is trivially also a bound for $f \circ h$ and $g \circ h$, and thus $h^*(f \otimes g) = (! + \text{id}) \circ b \circ h = h^*(f) \otimes h^*(g)$. \square

Remark 5.6 (1) *Finite coproducts $(+, 0)$ in an arbitrary category are called disjoint and universal when the coprojections κ_i are monic and form pull-back squares as on the left below, and additionally that in a square as on the right below, the induced map $Z_1 + Z_2 \rightarrow Z$ is an isomorphism.*

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \downarrow \lrcorner & & \downarrow \kappa_2 \\ X & \xrightarrow{\kappa_1} & X + Y \end{array} \qquad \begin{array}{ccccc} Z_1 & \longrightarrow & Z & \longleftarrow & Z_2 \\ \downarrow \lrcorner & & \downarrow & & \lrcorner \downarrow \\ X & \xrightarrow{\kappa_1} & X + Y & \xleftarrow{\kappa_2} & Y \end{array}$$

In this setting one can prove that diagrams of the form below are pullbacks.

$$\begin{array}{ccc} X & \xrightarrow{\kappa_1} & X + Z \\ f \downarrow \lrcorner & & \downarrow f+g \\ Y & \xrightarrow{\kappa_1} & Y + Z \end{array} \qquad \begin{array}{ccc} X + W & \xrightarrow{f+\text{id}} & Y + W \\ \text{id}+g \downarrow \lrcorner & & \downarrow \text{id}+g \\ X + Z & \xrightarrow{f+\text{id}} & Y + Z \end{array}$$

The three diagrams in Definition 5.1 are special instances. Since disjoint and universal coproducts are quite common (e.g. in every topos), the crucial condition in Definition 5.1 is the first one (requiring that the ∇_i from (3) are jointly monic).

(2) *Recall the action $*$: $\text{Hom}(1, 2) \times \text{Hom}(X, 2) \rightarrow \text{Hom}(X, 2)$ from Remark 3.2. In a convex category with tensors distributing over $+$, this $*$ is a bihomomorphism of effect algebras. This requires that we reformulate the action via a tensor, like in (2). As a result, $\text{Hom}(X, 2)$ becomes an “effect*

module” over $\mathbf{Hom}(1, 2)$. Such a module is the same as a “convex effect algebra”, see [20].

6 Convex monads

This section introduces convex monads and relates them to effect monoids via an adjunction. This extends the adjunction in [14] between convex functors and effect algebras. It is similar to the adjunction between semirings and additive monads in [7].

Definition 6.1 *Let \mathbb{A} be a category with final object 1 and with finite coproducts for which the three squares in Definition 5.1 are pullbacks (e.g. because the coproducts are disjoint and universal, see Remark 5.6). A functor $F: \mathbb{A} \rightarrow \mathbb{A}$ will be called convex if it satisfies the following three requirements.*

- (1) $F(1) \xrightarrow{\cong} 1$;
- (2) For each n , the n -tuple of maps $F(\nabla_i): F(n+1) \rightarrow F(2)$ is jointly monic—with ∇_i described in (3);
- (3) F preserves the three pullbacks from Definition 5.1.

A monad $T = (T, \eta, \mu)$ on \mathbb{A} is called convex if the functor $T: \mathbb{A} \rightarrow \mathbb{A}$ is convex.

We shall write $\mathbf{CnvFun}(\mathbb{A}) \hookrightarrow \mathbb{A}^{\mathbb{A}}$ and $\mathbf{CnvMnd}(\mathbb{A}) \hookrightarrow \mathbf{Mnd}(\mathbb{A})$ for the full categories of convex functors and monads on \mathbb{A} . In the special case where $\mathbb{A} = \mathbf{Sets}$ we write $\mathbf{CnvFun} = \mathbf{CnvFun}(\mathbf{Sets})$ and $\mathbf{CnvMnd} = \mathbf{CnvMnd}(\mathbf{Sets})$.

A functor F satisfying the first requirement $F(1) \cong 1$ is sometimes called *affine*, see e.g. [17,12]. The notion of convex functor was introduced (and used) in [14]; here we focus on convex monads. The following observation is fundamental.

Proposition 6.2 *The Kleisli categories $\mathcal{Kl}(T)$ and $\mathcal{Kl}_{\mathbb{N}}(T)$ of a convex monad T are convex categories. In particular, the homset of maps $1 \rightarrow T(2)$ is an effect monoid (by Theorem 5.2). This yields a functor:*

$$\begin{aligned} \mathbf{CnvMnd}(\mathbb{A}) &\longrightarrow \mathbf{EMon} \\ T &\longmapsto \mathbb{A}(1, T(2)) = \mathcal{Kl}(T)(1, 2) = \mathcal{Kl}_{\mathbb{N}}(T)(1, 2). \end{aligned}$$

Proof. Assume T is a convex monad on a category \mathbb{A} . The first requirement $T(1) \cong 1$ in Definition 6.1 guarantees that the object 1 is final in the Kleisli categories. The other two requirements follow because coproducts in a Kleisli category are inherited from \mathbb{A} via the functor $\mathcal{J}: \mathbb{A} \rightarrow \mathcal{Kl}(T)$, that

sends a map f in \mathbb{A} to $\mathcal{J}(f) = \eta \circ f$ in $\mathcal{Kl}(T)$, see Section 2. It satisfies $\mathcal{J}(f) \bullet g = \mu \circ T(\mathcal{J}(f)) \circ g = T(f) \circ g$, where \bullet is Kleisli composition. Co-projections in $\mathcal{Kl}(T)$ are of the form $\mathcal{J}(\kappa_i)$; cotuples in $\mathcal{Kl}(T)$ are as in \mathbb{A} . Hence $[\mathcal{J}(f), \mathcal{J}(g)] = \mathcal{J}([f, g])$. In particular, ∇_i from (3) in $\mathcal{Kl}(T)$ can be described as $\mathcal{J}(\nabla_i)$. Thus, $\mathcal{J}(\nabla_i) \bullet f = \mathcal{J}(\nabla_i) \bullet g$ implies $T(\nabla_i) \circ f = T(\nabla_i) \circ g$, and thus $f = g$. Hence the $\mathcal{J}(\nabla_i)$ are jointly monic in $\mathcal{Kl}(T)$ because the $T(\nabla_i)$ are jointly monic in \mathbb{A} . Similarly, preservation by T of the pullbacks in Definition 5.1 makes them pullbacks in $\mathcal{Kl}(T)$ and $\mathcal{Kl}_{\mathbb{N}}(T)$.

We briefly check functoriality. If $\sigma: T \Rightarrow S$ is a map of monads, then $\sigma_2 \circ (-): \mathbb{A}(1, T(2)) \rightarrow \mathbb{A}(1, S(2))$ is a map of effect monoids. One has:

$$\sigma_2 \circ 0 = \sigma_2 \circ \mathcal{J}(\kappa_2) = \sigma_2 \circ \eta \circ \kappa_2 = \eta \circ \kappa_2 = \mathcal{J}(\kappa_2) = 0.$$

Similarly, $\sigma_2 \circ 1 = 1$. Next, if $f_1 \perp f_2$ for $f_1, f_2: 1 \rightarrow T(2)$, say via a bound $b: 1 \rightarrow T(3)$ satisfying $\mathcal{J}(\nabla_i) \bullet b = f_i$, then it is not hard to see that $\sigma_3 \circ b: 1 \rightarrow S(3)$ is a bound for the $\sigma_2 \circ f_i$. Hence:

$$\begin{aligned} \sigma_2 \circ (f_1 \oplus f_2) &= \sigma_2 \circ ((! + \text{id}) \bullet b) \\ &= \sigma_2 \circ (\mathcal{J}(! + \text{id}) \bullet b) \\ &= \sigma_2 \circ T(! + \text{id}) \circ b \\ &= S(! + \text{id}) \circ \sigma_3 \circ b \\ &= (! + \text{id}) \bullet (\sigma_3 \circ b) \\ &= (\sigma_2 \circ f_1) \oplus (\sigma_2 \circ f_2). \end{aligned}$$

Finally we check that composition of the effect monoid, as defined in Proposition 3.1 and interpreted in $\mathcal{Kl}(T)$, is preserved by $\sigma_2 \circ (-)$. This follows because σ is a map of monads, and thus commutes appropriately with the units η and multiplications μ : for $f, g: 1 \rightarrow T(2)$,

$$\begin{aligned} \sigma_2 \circ (f \cdot g) &= \sigma_2 \circ ([g, \kappa_2] \bullet f) \\ &= \sigma_2 \circ \mu \circ T([g, \mathcal{J}(\kappa_2)]) \circ f \\ &= \mu \circ \sigma_{S(2)} \circ T(\sigma_2) \circ T([g, \eta \circ \kappa_2]) \circ f \\ &= \mu \circ S([\sigma_2 \circ g, \sigma_2 \circ \eta \circ \kappa_2]) \circ \sigma_2 \circ f \\ &= [\sigma_2 \circ g, \mathcal{J}(\kappa_2)] \bullet (\sigma_2 \circ f) \\ &= (\sigma_2 \circ f) \cdot (\sigma_2 \circ g). \end{aligned} \quad \square$$

The following construction gives an important class of examples of convex functors on the category of sets. It generalises the construction of the distribution functor (monad) \mathcal{D} from the unit interval $[0, 1]$ to an arbitrary effect algebra (monoid).

Definition 6.3 For an effect algebra E define a functor $\mathcal{D}_E: \mathbf{Sets} \rightarrow \mathbf{Sets}$ by:

$$\mathcal{D}_E(X) = \{\varphi: X \rightarrow E \mid \text{supp}(\varphi) \text{ is finite and orthogonal, and } \bigotimes_{x \in E} \varphi(x) = 1\}.$$

For a function $f: X \rightarrow Y$ one gets $\mathcal{D}_E(f): \mathcal{D}_E(X) \rightarrow \mathcal{D}_E(Y)$ by:

$$\mathcal{D}_E(f)(\varphi)(y) = \bigotimes_{x \in f^{-1}(y)} \varphi(x).$$

Whenever convenient, we write a distribution $\varphi \in \mathcal{D}_E(X)$ with finite support $\text{supp}(\varphi) = \{x_1, \dots, x_n\}$ and $\varphi(x_i) = s_i \in E$ as a formal convex sum $\varphi = \bigotimes_i s_i x_i$, like in Section 2 for the special case $E = [0, 1]$.

Proposition 6.4 Functors \mathcal{D}_E are convex, and satisfy $\mathcal{D}_E(2) \cong E$. If E is an effect monoid, then \mathcal{D}_E is a convex monad on \mathbf{Sets} . The mapping $E \mapsto \mathcal{D}_E$ yields functors $\mathbf{EA} \rightarrow \mathbf{CnvFun}$ and $\mathbf{EMon} \rightarrow \mathbf{CnvMnd}$.

Proof. We begin by describing what the sets $\mathcal{D}_E(1)$ and $\mathcal{D}_E(2)$ are. An element $\varphi \in \mathcal{D}_E(1)$ is a map $\varphi: \{1\} \rightarrow E$ with $\bigotimes_{x \in \{1\}} \varphi(x) = 1$. Hence φ is completely determined as $\varphi(1) = 1$. Thus $\mathcal{D}_E(1) \cong 1$, making \mathcal{D}_E an affine functor.

An element $\varphi \in \mathcal{D}_E(2)$ is a map $\varphi: \{1, 2\} \rightarrow E$ satisfying $\varphi(1) \perp \varphi(2)$ and $\varphi(1) \otimes \varphi(2) = 1$. Hence $\varphi(2) = \varphi(1)^\perp$, so that φ is determined by $\varphi(1) \in E$. Thus $\mathcal{D}_E(2) \cong E$.

If we have two elements $\varphi, \psi \in \mathcal{D}_E(n+1)$ satisfying $\mathcal{D}_E(\nabla_i)(\varphi) = \mathcal{D}_E(\nabla_i)(\psi)$, for $1 \leq i \leq n$, then $\varphi(i) = \mathcal{D}_E(\nabla_i)(\varphi)(1) = \mathcal{D}_E(\nabla_i)(\psi)(1) = \psi(i)$. But then $\varphi = \psi$, as required in point 2 in Definition 6.1, since the remaining value at $n+1$ is determined by the others:

$$\varphi(n+1) = \left(\varphi(1) \otimes \dots \otimes \varphi(n)\right)^\perp = \left(\psi(1) \otimes \dots \otimes \psi(n)\right)^\perp = \psi(n+1).$$

We turn to point 3 in Definition 6.1 and check that the functor \mathcal{D}_E preserves the three pullbacks in Definition 5.1. For the first one, assume $\varphi \in \mathcal{D}_E(n+1)$ satisfies $\mathcal{D}_E(! + \text{id})(\varphi) = \mathcal{D}_E(\kappa_1)(*)$, where $\kappa_1: 1 \rightarrow 1+1$ and $*$ is the single element $* = \lambda x. 1 \in \mathcal{D}_E(1)$. This means that $\varphi(1) \otimes \dots \otimes \varphi(n) = \mathcal{D}_E(! + \text{id})(\varphi)(1) = \mathcal{D}_E(\kappa_1)(*)(1) = 1$, and thus $\varphi(n+1) = 0$. Hence there is a unique element $\varphi' \in \mathcal{D}_E(n)$ with $\mathcal{D}_E(\kappa_1)(\varphi') = \varphi$, namely $\varphi'(i) = \varphi(i)$ for $1 \leq i \leq n$.

Preservation of the second pullback is left to the reader. For the third one, assume $\varphi \in \mathcal{D}_E(n+1)$ and $\psi \in \mathcal{D}_E(1+m)$ satisfying $\mathcal{D}_E(! + \text{id})(\varphi) = \mathcal{D}_E(\text{id} + !)(\psi)$. This means:

$$\varphi(1) \otimes \dots \otimes \varphi(n) = \psi(1) \quad \varphi(n+1) = \psi(2) \otimes \dots \otimes \psi(m+1).$$

The $\chi \in \mathcal{D}_E(n+m)$ that we are looking for must satisfy $\varphi = \mathcal{D}_E(\text{id}+!)(\chi)$ and $\psi = \mathcal{D}_E(!+\text{id})(\chi)$. That is:

$$\begin{aligned}\varphi(i) &= \chi(i), \text{ for } 1 \leq i \leq n, \quad \varphi(n+1) = \chi(n+1) \otimes \cdots \otimes \chi(n+m) \\ \psi(1) &= \chi(1) \otimes \cdots \otimes \chi(n) \quad \psi(j+1) = \chi(n+j-1), \text{ for } 2 \leq j \leq m+1.\end{aligned}$$

Hence there is a precisely one choice for such a χ , so that \mathcal{D}_E applied the last pullback in (4) is again a pullback.

Finally we have to check that the mapping $E \mapsto \mathcal{D}_E$ is functorial. Given a map $g: E \rightarrow D$ in **EA**, there is a natural transformation $\sigma: \mathcal{D}_E \Rightarrow \mathcal{D}_D$ given by $\sigma_X(\varphi) = g \circ \varphi$, that is well-defined and natural because g is a homomorphism.

If E is an effect monoid, we have to prove that \mathcal{D}_E is a monad. The unit and multiplication are essentially as in Section 2 for $[0, 1]$, but formulated in the effect monoid E . Thus the unit $\eta: X \rightarrow \mathcal{D}_E(X)$ is $\eta(x) = 1x$, whereas the multiplication $\mu: \mathcal{D}_E^2(X) \rightarrow \mathcal{D}_E(X)$ is given by $\mu(\bigoplus_i s_i \varphi_i)(x) = \bigoplus_i s_i \cdot \varphi_i(x)$. The latter expression involves an actual sum in E , which is well-defined since: $\bigoplus_x \bigoplus_i s_i \cdot \varphi_i(x) = \bigoplus_i \bigoplus_x s_i \cdot \varphi_i(x) = \bigoplus_i s_i \cdot (\bigoplus_x \varphi_i(x)) = \bigoplus_i s_i \cdot 1 = \bigoplus_i s_i = 1$. It is not difficult to check that the unit and multiplication laws hold. Moreover, a map of effect monoids $E \rightarrow D$ induces a map of monads $\mathcal{D}_E \Rightarrow \mathcal{D}_D$.

The main result in this section is then the adjointness of these functors between effect monoids and convex monads.

Theorem 6.5 *The functor $\mathbf{EMon} \rightarrow \mathbf{CnvMnd}$ from Proposition 6.4 given by $E \mapsto \mathcal{D}_E$ is left adjoint to the functor $T \mapsto T(2)$ from Proposition 6.2.*

Proof. For an effect monoid E and a convex monad T on **Sets** we have to prove that there is a bijective correspondence:

$$\begin{array}{ccc} E \xrightarrow{f} T(2) & \text{in } \mathbf{EMon} \\ \hline \mathcal{D}_E \xrightarrow{\bar{\sigma}} T & \text{in } \mathbf{CnvMnd} \end{array}$$

The upward direction is easy: one maps $\sigma: \mathcal{D}_E \Rightarrow T$ to:

$$\bar{\sigma} = \left(E \xrightarrow{\cong} \mathcal{D}_E(2) \xrightarrow{\sigma_2} T(2) \right)$$

It is not hard to see that this is a map of effect monoids.

The other direction requires more work. So suppose we have $f: E \rightarrow T(2)$ in **EMon**. We have to define a natural transformation $\bar{f}: \mathcal{D}_E \Rightarrow T$. So assume $\varphi \in \mathcal{D}_E(X)$, say with $\text{supp}(\varphi) = \{x_1, \dots, x_n\}$. The elements $\varphi(x_i) \in E$ are pairwise orthogonal, and thus so are $f(\varphi(x_i)) \in T(2)$. This means that there

is a (unique) bound $\beta \in T(n+1)$ with $T(\nabla_i)(\beta) = f(\varphi(x_i))$, and also:

$$T(! + \text{id})(\beta) = \bigotimes_i f(\varphi(x_i)) = f(\bigotimes_i \varphi(x_i)) = f(1) = 1 = T(\kappa_1)(*).$$

Since this sum is 1, we can use bound restriction as used previously in (6) and write $\beta = T(\kappa_1)(\beta')$, for a unique $\beta' \in T(n)$. Finally, we put:

$$\begin{aligned} \bar{f}_X(\varphi) &= \left(T(n \xrightarrow{[x_1, \dots, x_n]} X)(\beta') \right) \in T(X) \\ &= (f \circ \varphi) \star x \\ &= \bigotimes_i f(\varphi(x_i)) \cdot x_i, \quad \text{as in Definition 5.3.} \end{aligned}$$

Checking naturality is left to the reader. We proceed with showing that \bar{f} is a monad map. The equation $\bar{f} \circ \eta = \eta$ follows because $\eta(x) = 1x \in \mathcal{D}_E(X)$ and the associated restricted bound is $\eta_1 \in T(1)$. Hence we get $\bar{f}(\eta(x)) = T(x)(\eta_1) = \eta(x)$ by naturality.

For the multiplication equation $\bar{f} \circ \mu = \mu \circ \bar{f} \circ \mathcal{D}_E(\bar{f})$ we assume $\Phi = \bigotimes_{i \leq n} s_i \varphi_i \in \mathcal{D}_E^2(X)$. For convenience we assume that all of the φ_i have the same support $\{x_1, \dots, x_m\}$; this can always be achieved by adding 0 values. We write $\varphi_i = \bigotimes_{j \leq m} t_{ij} x_j$.

Since $\bigotimes_i f(s_i) = 1$, there is a restricted bound $a \in T(n)$ with $T(\nabla_i \circ \kappa_1)(a) = f(s_i)$. Similarly, the n equations $\bigotimes_j f(t_{ij}) = 1$ yield n restricted bounds $a_i \in T(m)$ with $T(\nabla_j \circ \kappa_1)(a_i) = f(t_{ij})$. The essence of the proof of the μ -equation is the following claim.

$$\begin{aligned} b \stackrel{\text{def}}{=} \mu\left(T([a_1, \dots, a_n])(a)\right) \in T(m) \quad \text{is a (restricted) bound} \\ \text{for the } m \text{ sums} \quad \bigotimes_i f(s_i) \cdot f(t_{ij}) \in T(2). \end{aligned} \tag{7}$$

With this claim we are done, since $\mu(\Phi)(x_j) = \bigotimes_i s_i \cdot t_{ij}$, and so:

$$\begin{aligned} \bar{f}(\mu(\Phi)) &= T([x_1, \dots, x_m])(b) \\ &= \left(T([x_1, \dots, x_m]) \circ \mu \circ T([a_1, \dots, a_n]) \right)(a) \\ &= \left(\mu \circ T^2([x_1, \dots, x_m]) \circ T([a_1, \dots, a_n]) \right)(a) \\ &= \left(\mu \circ T\left(T([x_1, \dots, x_m]) \circ a_1, \dots, T([x_1, \dots, x_m]) \circ a_n\right) \right)(a) \\ &= \left(\mu \circ T([\bar{f}(\varphi_1), \dots, \bar{f}(\varphi_n)]) \right)(a) \\ &= \mu\left(\bar{f}\left(s_1 \bar{f}(\varphi_1) \otimes \dots \otimes s_n \bar{f}(\varphi_n)\right)\right) \\ &= \left(\mu \circ \bar{f} \circ \mathcal{D}_E(\bar{f}) \right)(\Phi). \end{aligned}$$

So what remains is a proof of the claim (7). This is easy, since for each $j \leq m$

we have:

$$\begin{aligned}
& T(\nabla_j \circ \kappa_1) \left(\mu \left(T([a_1, \dots, a_n]) \right) (a) \right) \\
&= \left(T(\nabla_j \circ \kappa_1) \circ \mu \circ T([a_1, \dots, a_n]) \right) (a) \\
&= \left(\mu \circ T \left([T(\nabla_j \circ \kappa_1) \circ a_1, \dots, T(\nabla_j \circ \kappa_1) \circ a_n] \right) \right) (a) \\
&= \left(\mu \circ T([f(t_{1j}), \dots, f(t_{nj})]) \right) (a) \\
&= \bigotimes_i f(s_i) \cdot f(t_{ij}), \quad \text{by Lemma 5.4.} \quad \square
\end{aligned}$$

7 Triangles of adjunctions

Having established the adjunction between convex monads and effect monoids we proceed to obtain “triangles of adjunctions” like in [7]. This additionally involves Lawvere theories. We refer to [7] for the precise description of the categories **Law** of Lawvere theories, and **SMLaw** of symmetric monoidal Lawvere theories. Roughly, a Lawvere theory is a category with natural numbers $n \in \mathbb{N}$ as objects in which addition (of numbers) yields finite coproducts, with 0 being initial. A symmetric monoidal Lawvere theory additionally has multiplication (of numbers) as tensor structure, with 1 as tensor unit. Here we will use the full subcategories **CnvLaw** \hookrightarrow **Law** and **CnvSMLaw** \hookrightarrow **SMLaw** of convex (symmetric monoidal) Lawvere theories in which the object 1 is final. Every Lawvere theory \mathbb{L} contains the subcategory $\aleph_0 \hookrightarrow \mathbb{L}$, where $\aleph_0 \hookrightarrow \mathbf{Sets}$ is the full category with $n \in \mathbb{N}$ as objects, considered as n -element set.

7.1 Non-commutative case

At this stage we focus on effect monoids that are not commutative. Then we have the following situation.

Theorem 7.1 *The adjunction from Theorem 6.5 is part of a triangle of adjunctions between effect monoids, convex monads and convex Lawvere theories:*

$$\begin{array}{ccc}
& \mathbf{EMon} & \\
\mathcal{D}_{(-)} \swarrow & & \searrow \text{Hom}(1,2) \\
\mathbf{CnvMnd} & & \mathbf{CnvLaw} \\
& \mathcal{T} & \\
& \perp & \\
& \mathcal{K}\mathcal{L}_{\mathbb{N}} &
\end{array}$$

$\mathcal{K}\mathcal{L}_{\mathbb{N}}(-)(1,2)$ (arrow from **CnvMnd** to **EMon**)
 $\mathcal{K}\mathcal{L}_{\mathbb{N}}\mathcal{D}_{(-)}$ (arrow from **EMon** to **CnvLaw**)
 \dashv (arrow from **EMon** to **CnvMnd**)
 \dashv (arrow from **EMon** to **CnvLaw**)
 $\mathcal{K}\mathcal{L}_{\mathbb{N}}$ (arrow from **CnvMnd** to **CnvLaw**)

Proof. Much of this triangle is already known: the adjunction on the left,

between effect monoids and convex monads is from Theorem 6.5, where indeed, the right adjoint is $T \mapsto \mathcal{Kl}_{\mathbb{N}}(T)(1, 2)$, see Proposition 6.2.

On the right of the triangle, we know from Theorem 5.2 that the homset of maps $1 \rightarrow 2$ in a convex category is an effect monoid. In the reverse direction we get a functor $\mathcal{Kl}_{\mathbb{N}}\mathcal{D}_{(-)}$ from effect monoids to convex Lawvere theories by composition. This adjunction can be obtained also by composition, once we know the functor $\mathcal{T}: \mathbf{CnvLaw} \rightarrow \mathbf{CnvMnd}$ in the adjunction at the bottom. So this is what we will concentrate on.

There is a standard adjunction between monads and Lawvere theories that is used in [7]. The functor $\mathcal{T}: \mathbf{Law} \rightarrow \mathbf{Mnd}$ involved can be described abstractly as a Kan extension, and more concretely as a quotient: for a Lawvere theory \mathbb{L} and set X ,

$$\mathcal{T}(\mathbb{L})(X) = \left(\coprod_{k \in \mathbb{N}} \mathbb{L}(1, k) \times X^k \right) / \sim,$$

where \sim is the least equivalence relation such that, for each $f: k \rightarrow m$ in $\mathfrak{N}_0 \hookrightarrow \mathbb{L}$,

$$\kappa_m(f \circ g, v) \sim \kappa_k(g, v \circ f) \quad \text{where } g \in \mathbb{L}(1, k) \text{ and } v \in X^m.$$

For the reasoning below we use the following characterisation of \sim .

$$\begin{aligned} \kappa_k(g, v) \sim \kappa_\ell(h, w) &\iff \exists p: 1 \rightarrow m \text{ in } \mathbb{L} \exists g': m \rightarrow k, h': m \rightarrow \ell \text{ in } \mathfrak{N}_0 \\ &\text{such that } g = g' \circ p, h = h' \circ p, v \circ g' = w \circ h'. \end{aligned}$$

A direct consequence is that if $\kappa_k(g, \text{id}) = \kappa_k(h, \text{id})$, then $g = h$.

All we have to do now is show that $\mathcal{T}(\mathbb{L})$ is a convex monad in case \mathbb{L} is a convex Lawvere theory. We fix a Lawvere theory \mathbb{L} and abbreviate the induced monad as $T = \mathcal{T}(\mathbb{L})$.

- (1) $T(1) \cong 1$ because: $T(1) = \left(\coprod_k \mathbb{L}(1, k) \times 1^k \right) / \sim \cong \left(\coprod_k \mathbb{L}(1, k) \right) / \sim \cong 1$, since for each $g: 1 \rightarrow k$ in \mathbb{L} we have $!: k \rightarrow 1$ in $\mathfrak{N}_0 \hookrightarrow \mathbb{L}$ satisfying $\kappa_k(g) \sim \kappa_1(! \circ g) = \kappa_1(!)$ because 1 is final.
- (2) The maps $T(\nabla_i): T(n+1) \rightarrow T(2)$ are jointly monic. Suppose we have $\kappa_k(g, v), \kappa_\ell(h, w) \in T(n+1)$ with $\kappa_k(g, \nabla_i \circ v) \sim \kappa_\ell(h, \nabla_i \circ w)$, for each i . Since the ∇_i, v, w are all maps in \mathfrak{N}_0 we get $\kappa_2(\nabla_i \circ v \circ g, \text{id}) = \kappa_2(\nabla_i \circ w \circ h, \text{id})$, and thus $\nabla_i \circ v \circ g = \nabla_i \circ w \circ h$, as observed above. Now we can use that the ∇_i are jointly monic in \mathbb{L} and conclude that $v \circ g = w \circ h$. But then we are done, since $\kappa_k(g, v) \sim \kappa_{n+1}(v \circ g, \text{id}) = \kappa_{n+1}(w \circ h, \text{id}) \sim \kappa_\ell(h, w)$.
- (3) We still need to prove that T preserves the three pullbacks from Definition 5.1. We shall do so for the first one. So assume we have a map α

making the outer diagram below commute.

$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & T(n+1) \\
 \downarrow ! & & \downarrow T(!+\text{id}) \\
 T(n) & \xrightarrow{T(\kappa_1)} & T(n+1) \\
 \downarrow ! & & \downarrow T(!+\text{id}) \\
 T(1) & \xrightarrow{T(\kappa_1)} & T(2)
 \end{array}$$

Writing $\alpha(a) = \kappa_{\ell_a}(g_a, v_a)$ we have $\kappa_{\ell_a}(g_a, (! + \text{id}) \circ v_a) \sim \kappa_1(\text{id}, \kappa_1)$, for each $a \in A$. This yields $\kappa_2((! + \text{id}) \circ v_a \circ g_a, \text{id}) \sim \kappa_2(\kappa_1, \text{id})$, and thus $(! + \text{id}) \circ v_a \circ g_a = \kappa_1$ by the above observation about \sim . The fact that we have a pullback in \mathbb{L} now gives a unique $h_a: 1 \rightarrow n$ with $\kappa_1 \circ h_a = v_a \circ g_a$. Then we can define $\beta(a) = \kappa_n(h_a, \text{id}) \in T(n)$. It yields $(T(\kappa_1) \circ \beta)(a) = \kappa_n(h_a, \kappa_1) \sim \kappa_{n+1}(\kappa_1 \circ h_a, \text{id}) = \kappa_{n+1}(v_a \circ g_a, \text{id}) \sim \kappa_{\ell_a}(g_a, v_a) = \alpha(a)$. \square

7.2 Commutative case

For the notion of commutative monad we refer to [7].

Theorem 7.2 *The triangle of adjunctions from Theorem 7.1 restricts to:*

$$\begin{array}{ccc}
 & \mathbf{CEMon} & \\
 \mathcal{D}_{(-)} \swarrow & & \searrow \mathbf{Hom}(1,2) \\
 \mathbf{CnvCMnd} & & \mathbf{CnvSMLaw} \\
 \swarrow \perp & \mathcal{Kl}_{\mathbb{N}}(-)(1,2) & \searrow \perp \\
 & \mathcal{T} & \\
 \swarrow \perp & \mathcal{Kl}_{\mathbb{N}}\mathcal{D}_{(-)} & \searrow \perp \\
 & \mathcal{Kl}_{\mathbb{N}} &
 \end{array}$$

where we recall that **CEMon** is the category of commutative effect monoids, **CnvCMnd** is the category of commutative convex monads, and **CnvSMLaw** is the category of convex symmetric monoidal Lawvere theories.

Proof. Much of this restriction follows directly from [7]: commutativity of the monoid is in one-one correspondence with commutativity of the monad, which again corresponds to symmetric monoidal structure in Lawvere theories. Finally, proposition 3.1 (2) shows that in presence of tensors \otimes scalar multiplication on homsets $\mathbf{Hom}(1, 2)$ becomes commutative. \square

Acknowledgements

Thanks to Ana Sokolova for discussion and comments.

References

- [1] S. Abramsky and B. Coecke. A categorical semantics of quantum protocols. In *Logic in Computer Science*, pages 415–425. IEEE, Computer Science Press, 2004.
- [2] S. Abramsky and B. Coecke. A categorical semantics of quantum protocols. In K. Engesser, Dov M. Gabbai, and D. Lehmann, editors, *Handbook of Quantum Logic and Quantum Structures*, pages 261–323. North Holland, Elsevier, Computer Science Press, 2009.
- [3] M.A. Arbib and E.G. Manes. *Algebraic Approaches to Program Semantics*. Texts and Monogr. in Comp. Sci. Springer, Berlin, 1986.
- [4] M. Barr and Ch. Wells. *Toposes, Triples and Theories*. Springer, Berlin, 1985. Revised and corrected version available from URL: www.cwru.edu/artsci/math/wells/pub/ttt.html.
- [5] F. Bartels, A. Sokolova, and E. de Vink. A hierarchy of probabilistic system types. *Theor. Comp. Sci.*, 327(1-2):3–22, 2004.
- [6] F. Borceux. *Handbook of Categorical Algebra*, volume 50, 51 and 52 of *Encyclopedia of Mathematics*. Cambridge Univ. Press, 1994.
- [7] D. Coumans and B. Jacobs. Scalars, monads and categories, 2010. arxiv.org/abs/1003.0585.
- [8] A. Dvurečenskij and S. Pulmannová. *New Trends in Quantum Structures*. Kluwer Acad. Publ., Dordrecht, 2000.
- [9] D. J. Foulis and M.K. Bennett. Effect algebras and unsharp quantum logics. *Found. Physics*, 24(10):1331–1352, 1994.
- [10] D. J. Foulis, R.J. Greechie, and M.K. Bennett. The transition to unigroups. *Int. Journ. Theor. Physics*, 37(1):45–63, 1998.
- [11] C. Heunen and B. Jacobs. Quantum logic in dagger kernel categories. *Order*, 27(2):177–212, 2010. doi:10.1007/s11083-010-9145-5.
- [12] B. Jacobs. Semantics of weakening and contraction. *Ann. Pure & Appl. Logic*, 69(1):73–106, 1994.
- [13] B. Jacobs. *Categorical Logic and Type Theory*. North Holland, Amsterdam, 1999.
- [14] B. Jacobs. Convexity, duality, and effects. In C.S. Calude and V. Sassone, editors, *IFIP Theoretical Computer Science 2010*, number 82(1) in IFIP Adv. in Inf. and Comm. Techn., pages 1–19. Springer, Boston, 2010.
- [15] B. Jacobs and J. Mandemaker. Coreflections in algebraic quantum logic. In *Quantum Physics and Logic (QPL)*, 2010.

- [16] G.M. Kelly and M. L. Laplaza. Coherence for compact closed categories. *Journ. of Pure & Appl. Algebra*, 19:193–213, 1980.
- [17] A. Kock. Bilinearity and cartesian closed monads. *Math. Scand.*, 29:161–174, 1971.
- [18] E.G. Manes. *Algebraic Theories*. Springer, Berlin, 1974.
- [19] S. Mac Lane. *Categories for the Working Mathematician*. Springer, Berlin, 1971.
- [20] S. Pulmannová and S. Gudder. Representation theorem for convex effect algebras. *Commentationes Mathematicae Universitatis Carolinae*, 39(4):645–659, 1998. Available from <http://dml.cz/dmlcz/119041>.