Affine Monads and Side-Effect-Freeness

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Outline

Context: effectus theory and side-effects

Affine monads

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Main results

Conclusions

Quantum

► Quantum computation and logic is a fascinating area
► ‘Hot topic’, because of all the buzz about quantum computers
► Potentially large impact, esp. in security
  • existing (public key) algorithms are vulnerable
  • new research area: “post-quantum crypto”
► New challenges for existing concepts in (theoretical) CS
  • three overlapping areas: physics, math, CS
  • John Baez: category theory is “Rosetta Stone”
► Strong coalgebraic flavour
  • “states” play an important role
  • quantum observations can have a side-effect (state-change)
Logic, side-effects, and commutativity

Consider the logical equivalence $\equiv$ of:

$\text{it's raining} \land \text{Ichiro is sleeping} \equiv \text{Ichiro is sleeping} \land \text{it's raining}$

Conjunction $\land$ is obviously commutative.

Compare this to:

$\text{there are 5 eggs in the basket} \land \text{Ichiro is making an omelette}$

$\equiv \text{Ichiro is making an omelette} \land \text{there are 5 eggs in the basket}$

- If predicates can have side-effects, commutativity is no longer obvious. Conjunction should be used as 'and-then'.
- This plays an important role in the quantum world — also in imperative programming where $\land$ (and $\land\land$) are not commutative.

Effectus theory

- Own (group’s) work has led to a new categorical notion: effectus.
  - it’s a certain kind of category, with 0, +, 1, some pullbacks, and some jointly monic maps.
  - its predicates form effect modules, its states are convex sets, and together they form a 'state-and-effect' triangle.

- An effectus is an abstract model for quantum computation and logic.
  - probabilistic computation forms a special "commutative" subclass.
  - Boolean computation is a further "idempotent" subclass.

- Side-effects are part of the formalism, via instruments.
  - For each predicate $p$ on $X$, there is an instrument map:
    $$X \xrightarrow{\text{instr}_p} X + X$$
  - It is called side-effect-free if $\nabla \circ \text{instr}_p = \text{id}$, where $\nabla = [\text{id}, \text{id}]$.

- We have: in the probabilistic and Boolean case, instruments are side-effect-free, but not in the quantum case.

Overview: subclasses of effectuses (ArXiv, 1512.05813)

- general 'non-commutative' effectuses,
  - von Neumann algebras $\text{vNA}^{\text{op}}$,
- commutative effectuses
  - commutative von Neumann algebras, $\mathcal{K}(D),\mathcal{K}(G),\ldots$,
- Boolean effectuses
  - Sets, extensive categories.

Characterising subclasses

Theorem (See Effectus Intro paper on ArXiv)

The Boolean effectuses are precisely the extensive categories (with 1).

Wild conjecture

Commutative effectuses are Kleisli categories of a commutative monad

Examples: $\mathcal{K}(D),\mathcal{K}(G),\mathcal{K}(E)$

$\mathcal{K}(R) \cong \text{CCstar}^{op}$ (commutative $C^*$-algebras)
Main question underlying the CMCS paper

How are effectus properties and monad properties connected?

▶ Is there a relation between commutativity in effectuses and commutativity of monads?
▶ Is side-effect-freeness related to some property of a monad
  • being ‘affine’ is a candidate — that is, $T(1) \cong 1$

These questions have “good” answers
▶ they are first steps towards the wild conjecture

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Setting

▶ We work in a distributive category $C$
  • with finite products $(1, \times)$ and coproducts $(0, +)$.
  • where $\times$ distributes over $+$
▶ We assume a monad $T : C \to C$
▶ The monad is strong if there is a strength map
  $\text{st}_1 : T(X) \times Y \to T(X \times Y)$ suitably commuting with other structure
  • by swapping we get $\text{st}_2 : X \times T(Y) \to T(X \times Y)$
▶ The monad is commutative if the following diagram commutes:

```
\begin{align*}
T(X) \times T(Y) & \xrightarrow{\mu} T(X \times Y) \\
T(X) \times T(Y) & \xrightarrow{\text{st}_1} T(X \times Y) \\
T(X) \times T(Y) & \xrightarrow{\text{st}_2} T(X \times Y)
\end{align*}
```

Affineness

Definition

The monad $T$ is called affine if $T(1) \cong 1$

Examples

▶ The non-empty powerset monad $P_\emptyset$ on $\text{Sets}$
▶ The distribution monad $\mathcal{D}$ on $\text{Sets}$
▶ The Giry monad $\mathcal{G}$ on $\text{Meas}$
▶ The expectation monad $\mathcal{E} = \text{EMod}([0,1][\cdot], [0,1])$ on $\text{Sets}$
▶ The Radon monad $\mathcal{R} = \text{Stat}(\mathcal{C}(-))$ on $\text{CH}$

Note: if $T$ is affine, then 1 is final in $K\ell(T)$.
**Affine submonad**

Assuming enough pullbacks, the affine submonad $T_a \mapsto T$ is defined via:

$$
\begin{align*}
T_a(X) & \xrightarrow{1} 1 \\
\downarrow & \\
T(X) & \xrightarrow{T(1)} T(1)
\end{align*}
$$

**Lemma (Lindner 1979)**

- **This** $T_a$ is an affine monad, and $T_a \mapsto T$ is a monad map
- **in fact**, $T_a$ is the greatest affine submonad
- **if** $T$ is strong / commutative then so is $T_a$

**Affine submonad examples**

- The affine submonad of powerset is non-empty powerset
  \[
  \mathcal{P}_a(X) = \{ U \subseteq X \mid \mathcal{P}(U)(U) = \{\ast\} \}
  = \{ U \subseteq X \mid \{x \mid x \in U\} = \{\ast\} \}
  = \{ U \subseteq X \mid \{\ast \mid x \in U\} = \{\ast\} \}
  = \{ U \subseteq X \mid U = \emptyset \}
  \]
- The affine submonad of multiset monad $\mathcal{M}_{\mathbb{R}_{\geq 0}}$ is distribution $\mathcal{D}$

  We now restrict to formal sums $\varphi = \sum_i r_i x_i$ with:
  \[
  1 = \mathcal{M}(\varphi) = \sum_i r_i
  \]

**Causal maps**

Write:

$$
\tau_X \overset{\text{def}}{=} \left( X \xrightarrow{1} 1 \xrightarrow{\eta} T(1) \right)
$$

**Definition**

A map $f : X \rightarrow T(Y)$ is called **causal** if $\tau_Y \circ f = \tau_X$, where $\circ$ is Kleisli composition.

**Lemma**

A map $X \rightarrow T(Y)$ is causal iff it factors as $X \rightarrow T_a(Y)$ via the affine submonad $T_a$

**Example:** maps $X \rightarrow \mathcal{D}(Y)$ are causal maps $X \rightarrow \mathcal{M}_{\mathbb{R}_{\geq 0}}(Y)$.

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Predicates

- We shall work in the Kleisli category $K(T)$.
- A predicate on $X$ is a Kleisli map $X \to T(1+1)$ in $C$.
- There are truth and false predicates:
  - $\top = (X \to \top$ T(2)$)$
  - $\perp = (X \to \top$ T(2)$)$

- There is also negation / orthosupplement $\neg p = (X \to \top$ T(2)$)$.

- In many (probabilistic) examples, predicates are maps $X \to [0,1]$. 

Instruments

For a predicate $p : X \to 1+1$ we define an instrument $\text{instr}_p : X \to X + X$ in $K(T)$ as:

$$\text{instr}_p = \left( X \xrightarrow{p} T(2) \times X \xrightarrow{n_1} T(2 \times X) \xrightarrow{\sim} T(1+1) \right)$$

- We have $(! + !) \circ \text{instr}_p = p$
- The instrument is called side-effect-free if:

$$X \xrightarrow{\text{instr}_p} X + X \xrightarrow{\nabla \cdot \text{instr}_p} X$$

Lemma

If $T$ is affine, then each instrument is side-effect-free.

Instrument example: powerset

- Take a predicate $p : X \to P(2) \cong 4$
- Then $\text{instr}_p : X \to P(X + X)$ is:

$$\text{instr}_p(x) = \{ \kappa_1 x \mid 1 \in p(x) \} \cup \{ \kappa_2 x \mid 0 \in p(x) \}$$

- These instruments are not side-effect-free:

$$(\nabla \cdot \text{instr}_p)(x) = \{ x \mid 1 \in p(x) \text{ or } 0 \in p(x) \} = \begin{cases} \{x\} & \text{if } p(x) \neq \emptyset \\ \emptyset & \text{if } p(x) = \emptyset. \end{cases}$$

- For the (affine) non-empty powerset the case $p(x) = \emptyset$ does not occur, so we get side-effect-freeness.

Instrument example: state monad

- Consider $T(X) = (S \times X)^S$, for a fixed set of states $S$
- A predicate is a map $p : X \to (S + S)^S$
- The associated instrument $\text{instr}_p : X \to (S \times X + S \times X)^S$ is:

$$\text{instr}_p(x)(s) = \begin{cases} \kappa_1(s', x) & \text{if } p(x)(s) = \kappa_1 s' \\ \kappa_2(s', x) & \text{if } p(x)(s) = \kappa_2 s'. \end{cases}$$

This instrument incorporates the side-effects of the predicate $p$. 
Intermezzo: quantum instruments

▶ In the quantum model \( vNA^{pp} \) everything is turned around
▶ A predicate in a von Neumann algebra \( A \) is a \( p \in A \) with \( 0 \leq p \leq 1 \)
▶ The associated instrument is a function \( \text{instr}_p : A \otimes A \to A \), given by:

\[
\text{instr}_p(x, y) = \sqrt{p} \cdot x \cdot \sqrt{p} + \sqrt{1-p} \cdot y \cdot \sqrt{1-p}.
\]
▶ Side-effect-freeness means \( \text{instr}_p \circ \Delta = \text{id} \)
▶ Important: commutative \( vNA \)’s are side-effect-free:

\[
\begin{align*}
    (\text{instr}_p \circ \Delta)(x) &= \text{instr}_p(x, x) \\
    &= \sqrt{p} \cdot x \cdot \sqrt{p} + \sqrt{1-p} \cdot x \cdot \sqrt{1-p} \\
    &= \sqrt{p} \cdot \sqrt{p} \cdot x + \sqrt{1-p} \cdot \sqrt{1-p} \cdot x \\
    &= p \cdot x + (1-p) \cdot x \\
    &= x.
\end{align*}
\]

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Strong affineness

▶ If \( T \) is affine, then predicates give side-effect-free instruments
▶ For a bijective correspondence we need a stronger property

Definition

A (strong) monad \( T \) is called **strongly affine** if the following squares are pullbacks:

\[
\begin{array}{ccc}
T(X) \times Y & \xrightarrow{\pi_2} & Y \\
\downarrow \pi_1 & & \downarrow \text{id}_Y \\
T(X \times Y) & \xrightarrow{T(\pi_2)} & T(Y)
\end{array}
\]

(Strongly affine implies affine)

Strongly affine (counter)examples

▶ The standard affine monad examples \( P_+, D, G, E \) and \( R \) are also strongly affine
   • (proofs are not entirely trivial)
▶ (Kenta Cho) The monad \( D_\pm \) is affine but not strongly affine
   • \( D_\pm(X) \) contains \( \sum_i r_i |x_i \rangle \) with \( r_i \in \mathbb{R} \) and \( \sum_i r_i = 1 \)
   • In this monad \( D_\pm \) there is interference: positive and negative factors can cancel each other out
Strongly affine monads and instruments

Theorem (I)

If $T$ is strongly affine, then there is a bijective correspondence

- predicates
- side-effect-free instruments

More precisely, the correspondence is between maps in $\mathcal{K}(T)$.

\[
\begin{align*}
X & \xrightarrow{p} 2 \\
X & \xrightarrow{f} X + X \quad \text{with } \nabla \bullet f = id
\end{align*}
\]

Relating commutativity

Theorem (II)

If the monad $T$ is commutative, then instruments commute — giving commutativity in an effectus-theoretic sense.

More precisely, for predicates $p, q : X \to 2$ we have:

\[
\begin{align*}
X \xrightarrow{\text{inst}_p} X + X & \xrightarrow{q + q} 2 + 2 \\
X \xrightarrow{\text{inst}_q} X + X & \xrightarrow{p + p} 2 + 2
\end{align*}
\]

The isomorphism on the right can be illustrated as:

\[
\begin{align*}
2 + 2 & = (1 + 1) + (1 + 1) \\
\downarrow & \quad \downarrow \\
2 + 2 & = (1 + 1) + (1 + 1)
\end{align*}
\]

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Final remarks

- Quantum theory forms a rich source of inspiration for program semantics and logic — and for coalgebra in particular
- Recent formalisation in terms of effectuses
  - framework deals with side-effects of observations
  - Boolean and probabilistic computation given by subclasses
- Characterising the commutative (probabilistic and side-effect-free) fragment is an open challenge
  - Kleisli categories of suitable monads play an important role
- This CMCS paper clarifies the role of strong affiness and of commutativity of the monad