Affine Monads and Side-Effect-Freeness

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Outline

Where we are, sofar

Quantum

- Quantum computation and logic is a fascinating area
- "Hot topic", because of all the buzz about quantum computers
- Potentially large impact, esp. in security
  - existing (public key) algorithms are vulnerable
  - new research area: "post-quantum crypto"
- New challenges for existing concepts in (theoretical) CS
  - three overlapping areas: physics, math, CS
  - John Baez: category theory is "Rosetta Stone"
- Strong coalgebraic flavour
  - "states" play an important role
  - quantum observations can have a side-effect (state-change)

Logic, side-effects, and commutativity

Consider the logical equivalence $\equiv$ of:

it's raining $\wedge$ Ichiro is sleeping $\equiv$ Ichiro is sleeping $\wedge$ it's raining

Conjunction $\wedge$ is obviously commutative

Compare this to:

there are 5 eggs in the basket $\wedge$ Ichiro is making an omelette

$\overset{??}{\equiv}$ Ichiro is making an omelette $\wedge$ there are 5 eggs in the basket

- If predicates can have side-effects, commutativity is no longer obvious. Conjunction should be used as 'and-then'
- This plays an important role in the quantum world — and also in imperative programming where & (and &&) are not commutative

Effectus theory

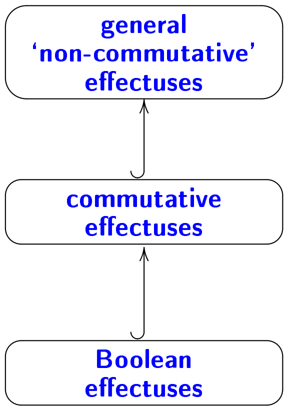
- Own (group's) work has led to a new categorical notion: effectus
  - it's a certain kind of category, with $0, +, 1$, some pullbacks, and some jointly monic maps
  - its predicates form effect modules, its states are convex sets, and together they form a "state-and-effect" triangle
- An effectus is an abstract model for quantum computation and logic
  - probabilistic computation forms a special "commutative" subclass
  - Boolean computation is a further "idempotent" subclass
- Side-effects are part of the formalism, via instruments

  For each predicate p on X, there is an instrument map:

  $$X \xrightarrow{\mathrm{instr}_p} X + X$$

  It is called side-effect-free if $\nabla \circ \mathrm{instr}_p = \mathrm{id}$, where $\nabla = [\mathrm{id}, \mathrm{id}]$.
- **We have**: in the probabilistic and Boolean case, instruments are side-effect-free, but not in the quantum case!

Overview: subclasses of effectuses (ArXiv, 1512.05813)



von Neumann algebras $\mathbf{vNA}^{\mathrm{op}}$

commutative von Neumann algebras, $\mathcal{K}\ell(\mathcal{D}), \mathcal{K}\ell(\mathcal{G}), \ldots$

Sets, extensive categories

Characterising subclasses

Theorem (See Effectus Intro paper on ArXiv)

The Boolean effectuses are precisely the extensive categories (with 1).

Wild conjecture

Commutative effectuses are Kleisli categories of a commutative monad

Examples: $\mathcal{K}\ell(\mathcal{D})$ $\mathcal{K}\ell(\mathcal{G})$ $\mathcal{K}\ell(\mathcal{E})$
$\mathcal{K}\ell(\mathcal{R}) \simeq \mathbf{CCstar}^{\mathrm{op}}$ (commutative C^*-algebras) ...

Main question underlying the CMCS paper

How are effectus properties and monad properties connected?

- Is there a relation between commutativity in effectuses and commutativity of monads?
- Is side-effect-freeness related to some property of a monad
  - being "affine" is a candidate — that is, $T(1) \cong 1$

These questions have "good" answers

- they are first steps towards the *wild conjecture*

Where we are, sofar

Setting

- We work in a distributive category **C**
  - with finite products $(1, \times)$ and coproducts $(0, +)$,
  - where $\times$ distributes over $+$
- We assume a monad $T\colon \mathbf{C} \to \mathbf{C}$
- The monad is strong if there is a strength map $\mathrm{st}_1\colon T(X) \times Y \to T(X \times Y)$ suitably commuting with other structure
  - by swapping we get $\mathrm{st}_2\colon X \times T(Y) \to T(X \times Y)$
- The monad is commutative if the following diagram commutes:

$$
\begin{array}{ccccccc}
 & \overset{\mathrm{st}_1}{\nearrow} & T(X \times T(Y)) & \xrightarrow{T(\mathrm{st}_2)} & T^2(X \times Y) & \overset{\mu}{\searrow} & \\
T(X) \times T(Y) & & & & & & T(X \times Y) \\
 & \underset{\mathrm{st}_2}{\searrow} & T(T(X) \times Y) & \xrightarrow[T(\mathrm{st}_1)]{} & T^2(X \times Y) & \underset{\mu}{\nearrow} &
\end{array}
$$

Affineness

Definition

The monad T is called affine if $T(1) \cong 1$

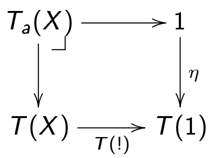
Examples

- The non-empty powerset monad $\mathcal{P}_+$ on **Sets**
- The distribution monad $\mathcal{D}$ on **Sets**
- The Giry monad $\mathcal{G}$ on **Meas**
- The expectation monad $\mathcal{E} = \mathbf{EMod}([0,1]^{(-)}, [0,1])$ on **Sets**
- The Radon monad $\mathcal{R} = \mathit{Stat}(C(-))$ on **CH**

Note: if T is affine, then 1 is final in $\mathcal{K}\ell(T)$.

Affine submonad

Assuming enough pullbacks, the affine submonad $T_a \rightarrowtail T$ is defined via:



Lemma (Lindner 1979)

- *This T_a is an affine monad, and $T_a \rightarrowtail T$ is a monad map*
  - *in fact, T_a is the greatest affine submonad*
- *if T is strong / commutative then so is T_a*

Affine submonad examples

- The affine submonad of powerset is non-empty powerset

$$\begin{aligned}\mathcal{P}_a(X) &= \{U \subseteq X \mid \mathcal{P}(!)(U) = \{*\}\} \\ &= \{U \subseteq X \mid \{!(x) \mid x \in U\} = \{*\}\} \\ &= \{U \subseteq X \mid \{* \mid x \in U\} = \{*\}\} \\ &= \{U \subseteq X \mid U \neq \emptyset\}\end{aligned}$$

- The affine submonad of multiset monad $\mathcal{M}_{\mathbb{R}_{\geq 0}}$ is distribution $\mathcal{D}$

  We now restrict to formal sums $\varphi = \sum_i r_i | x_i \rangle$ with:

$$1 = \mathcal{M}(!)(\varphi) = \sum_i r_i$$

Causal maps

Write:

$$\bar{\top}_X \stackrel{\text{def}}{=} \Big(X \xrightarrow{!_X} 1 \xrightarrow{\eta_1} T(1) \Big)$$

Definition

A map $f : X \to T(Y)$ is called causal if $\bar{\top}_Y \bullet f = \bar{\top}_X$, where $\bullet$ is Kleisli composition.

Lemma

A map $X \to T(Y)$ is causal iff it factors as $X \to T_a(Y)$ via the affine submonad T_a

Example: maps $X \to \mathcal{D}(Y)$ are causal maps $X \to \mathcal{M}_{\mathbb{R}_{\geq 0}}(Y)$.

Where we are, sofar

Predicates

- We shall work in the Kleisli category $\mathcal{K}\ell(T)$
- A predicate on X is a Kleisli map $X \to 2 = 1 + 1$
  - that is, a map $X \to T(1+1)$ in $\mathbf{C}$
- There are truth and false predicates:

$$\mathbf{1} = \Big(X \to 1 \xrightarrow{\kappa_1} 2 \xrightarrow{\eta} T(2)\Big) \qquad \mathbf{0} = \Big(X \to 1 \xrightarrow{\kappa_2} 2 \xrightarrow{\eta} T(2)\Big)$$

- There is also negation / orthosupplement

$$p^{\perp} = \Big(X \xrightarrow{p} T(1+1) \xrightarrow[\cong]{T([\kappa_2,\kappa_1])} T(1+1)\Big)$$

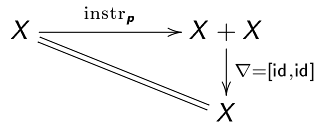
Note: $p^{\perp\perp} = p$ and $\mathbf{1}^{\perp} = \mathbf{0}$ and $\mathbf{0}^{\perp} = \mathbf{1}$
- In many (probabilistic) examples, predicates are maps $X \to [0,1]$

Instruments

For a predicate $p \colon X \to 1 + 1$ we define an instrument $\text{instr}_p \colon X \to X + X$ in $\mathcal{K}\ell(T)$ as:

$$\text{instr}_p = \Big(X \xrightarrow{\langle p, \text{id}\rangle} T(2) \times X \xrightarrow{\text{st}_1} T(2 \times X) \xrightarrow{\cong} T(X+X)\Big)$$

- We have $(! + !) \bullet \text{instr}_p = p$
- The instrument is called side-effect-free if:



Lemma

If T is affine, then each instrument is side-effect-free

Instrument example: powerset

- Take a predicate $p \colon X \to \mathcal{P}(2) \cong 4$
- Then $\text{instr}_p \colon X \to \mathcal{P}(X + X)$ is:

$$\text{instr}_p(x) = \{\kappa_1 x \mid 1 \in p(x)\} \cup \{\kappa_2 x \mid 0 \in p(x)\}$$

- These instruments are not side-effect-free:

$$(\nabla \bullet \text{instr}_p)(x) = \{x \mid 1 \in p(x) \text{ or } 0 \in p(x)\} = \begin{cases} \{x\} & \text{if } p(x) \neq \emptyset \\ \emptyset & \text{if } p(x) = \emptyset. \end{cases}$$

- For the (affine) non-empty powerset the case $p(x) = \emptyset$ does not occur, so we get side-effect-freeness.

Instrument example: state monad

- Consider $T(X) = (S \times X)^S$, for a fixed set of states S
- A predicate is a map $p \colon X \to (S + S)^S$
- The associated instrument $\text{instr}_p \colon X \to (S \times X + S \times X)^S$ is:

$$\text{instr}_p(x)(s) = \begin{cases} \kappa_1(s', x) & \text{if } p(x)(s) = \kappa_1 s' \\ \kappa_2(s', x) & \text{if } p(x)(s) = \kappa_2 s' \end{cases}$$

This instrument incorporates the side-effects of the predicate p

Intermezzo: quantum instruments

- In the quantum model $\mathbf{vNA}^{\mathrm{op}}$ everything is turned around
- A predicate in a von Neumann algebra A is a $p \in A$ with $0 \leq p \leq 1$
- The associated instrument is a function $\mathrm{instr}_p \colon A \oplus A \to A$, given by:

$$\mathrm{instr}_p(x, y) = \sqrt{p} \cdot x \cdot \sqrt{p} \; + \; \sqrt{1-p} \cdot y \cdot \sqrt{1-p}.$$

- Side-effect-freeness means $\mathrm{instr}_p \circ \Delta = \mathrm{id}$
- **Important:** commutative vNA's are side-effect-free:

$$\begin{aligned}
(\mathrm{instr}_p \circ \Delta)(x) &= \mathrm{instr}_p(x, x) \\
&= \sqrt{p} \cdot x \cdot \sqrt{p} \; + \; \sqrt{1-p} \cdot x \cdot \sqrt{1-p} \\
&= \sqrt{p} \cdot \sqrt{p} \cdot x \; + \; \sqrt{1-p} \cdot \sqrt{1-p} \cdot x \\
&= p \cdot x + (1-p) \cdot x \\
&= x.
\end{aligned}$$

Where we are, sofar

Strong affineness

- If T is affine, then predicates give side-effect-free instruments
- For a bijective correspondence we need a stronger property

Definition

A (strong) monad T is called strongly affine if the following squares are pullbacks

$$\begin{array}{ccc}
T(X) \times Y & \xrightarrow{\pi_2} & Y \\
{\scriptstyle \mathrm{st}_1} \downarrow & & \downarrow {\scriptstyle \eta_Y} \\
T(X \times Y) & \xrightarrow[T(\pi_2)]{} & T(Y)
\end{array}$$

(Strongly affine implies affine)

Strongly affine (counter)examples

- The standard affine monad examples $\mathcal{P}_+$, $\mathcal{D}$, $\mathcal{G}$, $\mathcal{E}$ and $\mathcal{R}$ are also strongly affine
  - (proofs are not entirely trivial)
- (Kenta Cho) The monad $\mathcal{D}_\pm$ is affine but not strongly affine
  - $\mathcal{D}_\pm(X)$ contains $\sum_i r_i | x_i \rangle$ with $r_i \in \mathbb{R}$ and $\sum_i r_i = 1$
  - In this monad $\mathcal{D}_\pm$ there is interference: positive and negative factors can cancel each other out

Strongly affine monads and instruments

Theorem (I)

If T is strongly affine, then there is a bijective corrrespondence

$$\frac{\text{predicates}}{\text{side-effect-free instruments}}$$

More precisely, the correspondence is between maps in $\mathcal{K}\ell(T)$,

$$\frac{X \xrightarrow{p} 2}{X \xrightarrow[f]{} X+X \quad \text{with } \nabla \bullet f = id}$$

Relating commutativity

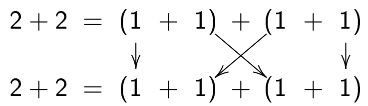
Theorem (II)

If the monad T is commutative, then instruments commute — giving commutativity in an effectus-theoretic sense.

More precisely, for predicates $p, q \colon X \to 2$ we have:

$$\begin{array}{ccccc} X & \xrightarrow{instr_p} & X+X & \xrightarrow{q+q} & 2+2 \\ \| & & & & \cong \downarrow [\kappa_1+\kappa_1, \kappa_2+\kappa_2] \\ X & \xrightarrow[instr_q]{} & X+X & \xrightarrow[p+p]{} & 2+2 \end{array}$$

The isomorphism on the right can be illustrated as:



Where we are, sofar

Final remarks

- Quantum theory forms a rich source of inspiration for program semantics and logic — and for coalgebra in particular
- Recent formalisation in terms of effectuses
  - framework deals with side-effects of observations
  - Boolean and probabilitistic computation given by subclasses
- Characterising the commutative (probabilistic and side-effect-free) fragment is an open challenge
  - Kleisli categories of suitable monads play an important role
- This CMCS paper clarifies the role of strong affiness and of commutativity of the monad