Effectus Theory, and Beyond

Amsterdam Quantum Logic Workshop

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Outline

Background

A crash course on effect algebras and effect modules

Effectuses

Effectuses from biproduct categories with truth

Beyond effectuses

Conclusions

Where we are, sofar

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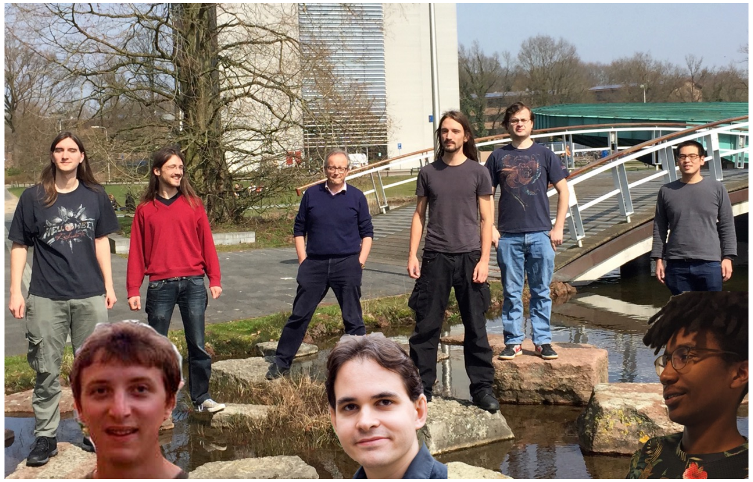
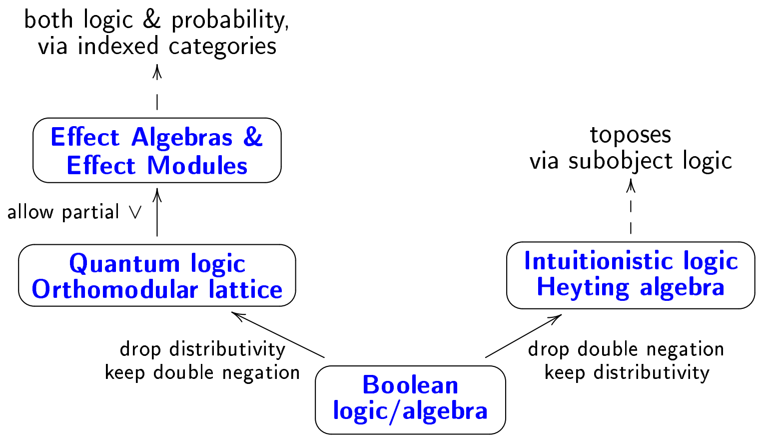
About this talk

- Overview of quantum logic research at Nijmegen
- Performed within context of ERC Advanced Grant Quantum Logic, Computation, and Security
  - Running period: 1 May 2013 – 1 May 2018
- Focus on categorical axiomatisation
  - esp. differences/similarties with probabilistic and classical computing
- Not all ongoing work is covered in this presentation
  - focus on what we call an effectus

Group picture



From Boolean to intuitionistic & quantum logic



Aha-moments in categorical logic

effectus theory (2010s)

- focus on maps $X \to 1+1$
- they form an effect module

topos theory (1970s)

- focus on subobjects $A \rightarrowtail X$
- they form a Heyting algebra

Example (without knowing yet what an effectus is)

The opposite $\mathbf{Rng}^{\mathrm{op}}$ of the category of rings (with unit) is an effectus, with:

$$R \xrightarrow{\text{predicate}} 1+1 \qquad \text{in } \mathbf{Rng}^{\mathrm{op}}$$
$$\overline{\overline{\mathbb{Z}\times\mathbb{Z} \longrightarrow R}} \qquad \text{in } \mathbf{Rng}$$
$$\overline{\overline{\text{idempotent } e \in R, \text{ so } e^2 = e}}$$

Hence the predicates on $R \in \mathbf{Rng}^{\mathrm{op}}$ are its idempotents

- These idempotents $e \in R$ form an effect algebra, with:

  truth 1 falsum 0 orthocomplement $e^{\perp} = 1 - e$

  Additionally there is a partial sum $e \ovee d = e + d$ if $ed = 0 = de$.
- If R is commutative, then the idempotents form a Boolean algebra!
  (this case is well-known/studied, eg. in sheaf theory for commutative rings)

Origin of ‘effectus’: two papers

New Directions paper

- full title: *New Directions in Categorical Logic, for Classical, Probabilistic and Quantum Logic*, 69 pages
- See arxiv.org/abs/1205.3940, different versions 2012 - 2014
- Introduces four successive assumptions (and elaborates them)

States of Convex Sets, FoSSaCS’15, with B&B Westerbaan

- Name ‘effectus’ introduced there
- for category satisfying **Assumption I** from *New Directions*
- this forms a basic building block, with many instances

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Effect algebras, definition

Effect algebras axiomatise the unit interval $[0,1]$ with its (partial!) addition $+$ and “negation” $x^\perp = 1 - x$.

Definition

A Partial Commutative Monoid (PCM) consists of a set M with zero $0 \in M$ and partial operation $\ovee \colon M \times M \to M$, which is suitably commutative and associative.

One writes $x \perp y$ if $x \ovee y$ is defined.

Definition

An effect algebra is a PCM in which each element x has a unique ‘orthosuplement’ $x^\perp$ with $x \ovee x^\perp = 1 \ (= 0^\perp)$
Additionally, $x \perp 1 \Rightarrow x = 0$ must hold.

Effect algebras, observations

- There is then a partial order, via $x \leq y$ iff $y = x \ovee z$, for some z
- Each Boolean algebra is an effect algebra, with:

$$x \perp y \text{ iff } x \wedge y = 0, \qquad \text{and then} \qquad x \ovee y = x \vee y$$

- In fact, each orthomodular lattice is an effect algebra (in the same way)
- Frequently occurring form: unit intervals:

$$[0,1]_G = \{x \in G \mid 0 \leq x \leq 1\}$$

in an ordered Abelian group with order unit $1 \in G$.

  - $x^\perp = 1 - x$
  - $x \perp y$ iff $x + y \leq 1$, and in that case $x \ovee y = x + y$.

Homomorphisms of effect algebras

Definition

A homomorphism of effect algebras $f\colon X \to Y$ satisfies:

- $f(1) = 1$
- if $x \perp x'$ then both $f(x) \perp f(x')$ and $f(x \ovee x') = f(x) \ovee f(x')$.

This yields a category **EA** of effect algebras.

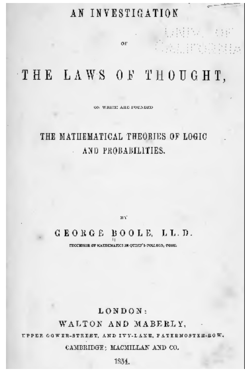
Example:

- A probability measure yields a map $\Sigma_X \to [0,1]$ in **EA**
- Recall the indicator (characteristic) function $\mathbf{1}_U\colon X \to [0,1]$, for a subset $U \subseteq X$.
  - It gives a map of effect algebras:

$$\mathcal{P}(X) \xrightarrow{\mathbf{1}_{(-)}} [0,1]^X$$

Naturality of partiality

George Boole in 1854 thought of disjunction as a partial operation


AN INVESTIGATION
OF
THE LAWS OF THOUGHT,
THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES.
BY
GEORGE BOOLE, LL.D.
LONDON:
WALTON AND MABERLY,
CAMBRIDGE: MACMILLAN AND CO.
1854.

"Now those laws have been determined from the study of instances, in all of which it has been a necessary condition, that the classes or things added together in thought should be mutually exclusive. The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common." (p.66)

Effect modules

Effect modules are effect algebras with a scalar multiplication, with scalars not from $\mathbb{R}$ or $\mathbb{C}$, but from $[0,1]$.

(Or more generally from an "effect monoid", ie. effect algebra with multiplication)

Definition

An effect module M is a effect algebra with an action $[0,1] \times M \to M$ that is a "bihomomorphism"

A map of effect modules is a map of effect algebras that commutes with scalar multiplication.

We get a category **EMod** $\hookrightarrow$ **EA**.

Effect modules, main examples

Probabilistic examples

- Fuzzy predicates $[0,1]^X$ on a set X, with scalar multiplication

$$r \cdot p \stackrel{\text{def}}{=} x \mapsto r \cdot p(x)$$

- Measurable predicates $\mathrm{Hom}(X, [0,1])$, for a measurable space X, with the same scalar multiplication
- Continuous predicates $\mathrm{Hom}(X, [0,1])$, for a topological space X

Quantum examples

- Effects $\mathcal{E}(H)$ on a Hilbert space: operators $A\colon H \to H$ satisfying $0 \le A \le I$, with scalar multiplication $(r, A) \mapsto rA$.
- Effects in a C^*-algebra A: positive elements below the unit:

$$[0,1]_A = \{a \in A \mid 0 \le a \le 1\}.$$

This one covers the previous illustrations.

Basic adjunction, between effects and states

Theorem By "homming into $[0,1]$" one gets an adjunction:

$$\mathbf{EMod}^{\mathrm{op}} \underset{\mathrm{Hom}(-,[0,1])}{\overset{\mathrm{Hom}(-,[0,1])}{\rightleftarrows}} \mathbf{Conv} \qquad (\top)$$

This adjunction restricts to an equivalence of categories between:

- Banach effect modules, which have a complete norm
  (or equivalently, complete order unit spaces)
- convex compact Hausdorff spaces

This is called **Kadison duality**

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Effectus

An **effectus** is a category with finite coproducts $(0, +)$ and 1 such that

- these diagrams are pullbacks:

$$\begin{array}{ccc} A+X & \xrightarrow{\mathrm{id}+g} & A+Y \\ {\scriptstyle f+\mathrm{id}}\downarrow & & \downarrow{\scriptstyle f+\mathrm{id}} \\ B+X & \xrightarrow[\mathrm{id}+g]{} & B+Y \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & A \\ {\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\ A+X & \xrightarrow[\mathrm{id}+f]{} & A+Y \end{array}$$

- these arrows are jointly monic:

$$X+X+X \underset{\mathbb{X}=[\kappa_2,\kappa_1,\kappa_2]}{\overset{\mathbb{IV}=[\kappa_1,\kappa_2,\kappa_2]}{\rightrightarrows}} X+X$$

Perspective:

$$\left(\begin{array}{c}\text{disjoint and universal}\\ \text{coproducts}\end{array}\right) \Longrightarrow \left(\text{effectus}\right) \Longrightarrow \left(\begin{array}{c}\text{disjoint}\\ \text{coproducts}\end{array}\right)$$

Internal logic

effectus	**meaning**
objects X	types
arrows $X \xrightarrow{f} Y$	programs
1 (final object)	singleton/unit type
$1 \xrightarrow{\omega} X$	state
$X \xrightarrow{p} 1+1$	predicate
$1 \xrightarrow{\omega} X \xrightarrow{p} 1+1$	validity
$1 \to 1+1$	scalar
$f_*(\omega) = f \circ \omega$	state transformation
$f^*(q) = q \circ f$	predicate transformation

$$f_*(\omega) \models q \;=\; \omega \models f^*(q)$$

Examples of states and predicates in an effectus

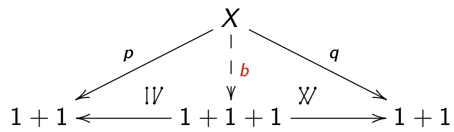
	State	Predicate	Validity	Scalars
	$1 \xrightarrow{\omega} X$	$X \xrightarrow{p} 1+1$	$\omega \models p$	$1 \to 1+1$
classical **Sets**	element $\omega \in X$	subset $p \subseteq X$	$\omega \in p$	$\{0,1\}$
probabilistic $\mathcal{K}\ell(\mathcal{D})$	discrete distribution $\omega \equiv \sum_i s_i \lvert x_i \rangle$	fuzzy predicates $X \xrightarrow{p} [0,1]$	$\sum_i s_i p(x_i)$	$[0,1]$
probabilistic $\mathcal{K}\ell(\mathcal{G})$	probability measure $\Sigma_X \xrightarrow{\phi} [0,1]$	measurable predicates $X \xrightarrow{p} [0,1]$	$\int p d\phi$	$[0,1]$
quantum $\mathbf{vN}^{\mathrm{op}}$	normal state $\omega\colon X \to \mathbb{C}$	effect $0 \le p \le I$ in X	$\omega(p)$	$[0,1]$

Effect structure on predicates $X \to 1+1$

- We get some logical structure for free:

$$1 = (X \xrightarrow{\kappa_1 \circ !} 1+1) \quad 0 = (X \xrightarrow{\kappa_2 \circ !} 1+1) \quad p^\perp = (X \xrightarrow{p} 1+1 \xrightarrow[\cong]{[\kappa_2,\kappa_1]} 1+1)$$

  Then $p^{\perp\perp} = p$, $0^\perp = 1$, $1^\perp = 0$.
- Define $p \perp q$, for $p, q\colon X \to 1+1$ if there is a bound b in:



  In that case put $p \ovee q = (\nabla + \mathrm{id}) \circ b\colon X \to 1+1$.
- Predicates $1 \to 1+1$ on 1 will be called scalars
  - they carry a monoid structure $p \cdot q = [p, \kappa_2] \circ q$
  - it is commutative in presence of distributive tensors

The structure of predicates and states

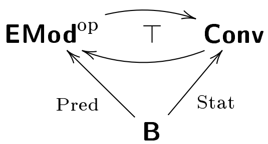
Theorem

Let **B** be an effectus. Then:

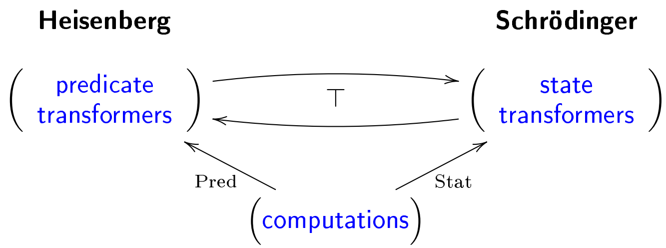
(1) The predicates $X \to 1+1$ form an effect module

(2) The states $1 \to X$ form a convex set

Predicate transformers f^* and state transformers f_* preserve this structure.



General picture: "state-and-effect triangles"



- The traditional distinction in program semantics between predicate transformers and state transformers also exists in the quantum world
- It corresponds to the different approaches of Heisenberg (matrix mechanics) and Schrödinger (wave equation, for pure state changes)

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- General construction, to obtain an effectus from a biproduct category "with truth"
  - new, unpublished work
- Inspired by CP*-construction, yielding an effectus
  - only needed: these biproducts with truth
  - outcome of discussion between: Aleks Kissinger, Bram & Bas Westerbaan, Kenta Cho (in Barbados)
- Many examples of effectuses arise via this construction
  - (own observations)

Biproducts and truth (or 'counits', or 'trace')

Recall that a category with finite biproducts $(0, \oplus)$ is enriched over commutative monoids.

Definition

A **biproduct category with truth** has:

- finite biproducts $(0, \oplus)$
- a special object I with "truth maps" $\mathbf{1}_X : X \to I$ satisfying:
  - $\mathbf{1}_I = \mathrm{id} : I \to I$
  - coprojections commute with truth: $\mathbf{1} \circ \kappa_i = \mathbf{1}$
  - positivity: $\mathbf{1} \circ f = 0 \Longrightarrow f = 0$
  - cancellation: $f + g = \mathbf{1} = f + h \Longrightarrow g = h$.

Call a map f causal (or counital) if it commutes with truth: $\mathbf{1} \circ f = \mathbf{1}$.

Main result

Theorem

In a biproduct category with truth, the subcategory of causal maps is an effectus.

Proof: elementary categorical reasoning.

Motivating example: $\mathrm{CP}^*(\mathbf{B})$, for a dagger compact category $\mathbf{B}$ with biproducts

But there are many more examples.

Algebraic examples: Abelian groups

Consider the following categories of Abelian groups (cf. Goodearl)

	Objects	Morphisms
Ab	Abelian groups	group homomorphisms
OAb	partially ordered Abelian groups	positive/monotone group homomorphisms
OUAb	ordered Abelian groups with order unit	positive & unital group homomorphisms
$\mathbf{OUAb}_P$	ordered Abelian groups with order unit	(only) positive group homomorphisms

Idea: $(\mathbf{OUAb}_P)^{\mathrm{op}}$ is a biproduct category with truth, and $\mathbf{OUAb}^{\mathrm{op}}$ is the associated category of causal maps.

Abelian group example

- $\mathbf{Ab}, \mathbf{OAb}, \mathbf{OUAb}_P$ have biproducts (but **OUAb** not)
  - the opposite categories then also have biproducts
- There are truth maps $\mathbf{1}: G \to \mathbb{Z}$ in $(\mathbf{OUAb}_P)^{\mathrm{op}}$
  - $\mathbf{1}: \mathbb{Z} \to G$ in $\mathbf{OUAb}_P$
  - it simply points to the order unit element $1 \in G$
  - it satisfies the requirements
- The causal maps $G \to H$ commuting with truth are precisely the unital ones, preserving the order units
  - hence $\mathbf{OUAb}^{\mathrm{op}}$ is an effectus

Algebraic examples: variations

The same approach can be used for similar example:

- $\mathbf{OUS}^{\mathrm{op}}$ where **OUS** is the category of order unit spaces with positive unital maps
- $(\mathbf{Cstar}_{PU})^{\mathrm{op}}$, the category of C^*-algebras with positive unital maps

In each case, the larger category of (only) positive maps has biproducts and truth.

Intermezzo: additive monads

Definition

Let **B** be a category with finite products $(1, \times)$ and coproducts $(0, +)$.
A monad T on **B** is called additive if:

$$T(0) \cong 1 \qquad \text{and} \qquad T(X+Y) \cong T(X) \times T(Y)$$

Theorem (Coumans & BJ) TFAE

- T is additive
- $\mathcal{K}\ell(T)$ has finite biproducts
- $\mathcal{EM}(T)$ has finite biproducts

Examples: Powerset, multiset $\mathcal{M}(X) = \{\phi : X \to S \mid \mathrm{supp}(\varphi) \text{ is finite}\}$

Kleisli example: multiset monad

- Take multiset monad $\mathcal{M}(X)$ with maps $X \to \mathbb{R}_{\geq 0}$
  - $\mathcal{K}\ell(\mathcal{M})$ has biproducts
- But also truth maps $X \to 1$ in $\mathcal{K}\ell(\mathcal{M})$
  - in **Sets** they are functions $X \to \mathcal{M}(1) \cong \mathbb{R}_{\geq 0}$
  - they are the "contant 1" functions
- The resulting subcategory of $\mathcal{K}\ell(\mathcal{M})$ of causal maps is: ...
  - the Kleisli category $\mathcal{K}\ell(\mathcal{D})$ of the distribution monad $\mathcal{D}$
  - $\varphi \in \mathcal{D}(X)$ are $\varphi\colon X \to \mathbb{R}_{\geq 0}$ with $\sum_x \varphi(x) = 1$
  - important example of effectus

Other Kleisli examples: Expectation, Radon, Giry

A similar construction can be used for other monads, eg.

- $\mathcal{E}(X) = \mathbf{OUS}(\mathbb{R}^X, \mathbb{R})$
- Consider variation $\mathcal{E}_P(X) = \mathbf{OUS}_P(\mathbb{R}^X, \mathbb{R})$
  - $\mathbf{OUS}_P$ is order unit spaces with positive maps only
- $\mathcal{E}_P$ is additive monad, since $\mathbf{OUS}_P$ has biproducts:

$$\begin{array}{rcl} \mathcal{E}_P(X+Y) & = & \mathbf{OUS}_P(\mathbb{R}^{X+Y},\, \mathbb{R}) \\ & \cong & \mathbf{OUS}_P(\mathbb{R}^X \times \mathbb{R}^Y,\, \mathbb{R}) \\ & \cong & \mathbf{OUS}_P(\mathbb{R}^X,\, \mathbb{R}) \times \mathbf{OUS}_P(\mathbb{R}^Y,\, \mathbb{R}) \\ & = & \mathcal{E}_P(X) \times \mathcal{E}_P(Y). \end{array}$$

- $\mathcal{K}\ell(\mathcal{E}_P)$ has truth, and $\mathcal{K}\ell(\mathcal{E})$ is effectus of causal maps

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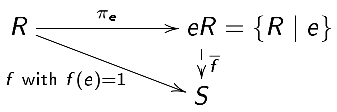
Additional structure

- Partial maps in an effectus, via the Kleisli category of the lift monad $(-) + 1$
  - Kenta Cho has characterised such categories of partial maps
  - "finitely partially additive categories with truth"
  - he obtained a 2-equivalence
- Monoidal effectuses, with distributive tensors
  - obvious extension, but not heavily studied so far
- Effectuses with measurement, via quotient & comprehension
  - active, exciting research topic
  - many examples, but full (categorical) picture still unclear

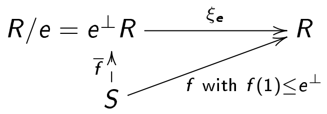
Comprehension & quotient example

- Consider the effectus $\mathbf{CRng}^{\mathrm{op}}$ of commutative rings
  - Let $e \in R$ be idempotent/predicate
- The principal ideal eR comes with $\pi_e = e(-)\colon R \to eR$, which is universal



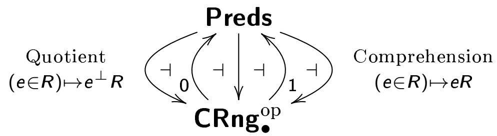
This behaves like comprehension $\pi_e\colon \{R \mid e\} \to R$ in $\mathbf{CRng}^{\mathrm{op}}$

- The ring $e^{\perp}R$ comes with subunital inclusion $\xi_e\colon e^{\perp}R \to R$ which is also universal:

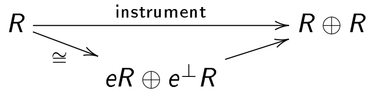


This $\xi\colon R \to R/e$ behaves like quotient map, in an opposite category.

Categorical picture: chain of adjunctions



- The base category $\mathbf{CRng}_{\bullet}^{\mathrm{op}}$ is the Kleisli category of lift
  - it contains rings and subunital maps, not necessarily preserving 1
- The measurement instrument map from *New Directions* involves ring decomposition



- In the non-commutative case of von Neumann algebras the situation is similar, but more complicated/subtle.

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Main points

- Effectus is a basic notion of the "Nijmegen school"
  - weak axioms, but suprisingly rich logical structure
  - many examples: classical / probabilistic / quantum
  - esp. state-and-effect triangles
  - (but also in new approach to contextuality, see ICALP'15)
- Powerful construction method via biproduct categories with truth
  - establishes links between Nijmegen and Oxford schools
  - "stronger entanglement of research"
- Beyond effectuses there are intruiging research questions about the relation between:
  - measurement
  - comprehension and quotients
  - decomposition / sheaf theory