Lectures on Kadison Duality

Bart Jacobs

Institute for Computing and Information Sciences – Digital Security
Radboud University Nijmegen

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Outline

Richard V. Kadison (1925)

- Student of Marshal Stone, PhD Univ. Chicago 1956
- Famous ao. for the Kadison-Ringrose books *Fundamentals of the Theory of Operators* (I and II), from 1983 and 1986

Kadison duality

$$\begin{pmatrix}\text{complete order}\\ \text{unit spaces}\end{pmatrix}^{\mathrm{op}} \simeq \begin{pmatrix}\text{convex compact}\\ \text{Hausdorff spaces}\end{pmatrix}$$

- Essentials published in 1951 as Memoirs AMS — one half: representation of order unit spaces
- At the heart of the duality between states and observables/effects/predicates in quantum foundations
- Crying out for a modern, categorical description

Own contribution in these lectures

- No new mathematical results!
- Systematic presentation of results and proofs — in categorical terminology — but without historical details
- Proofs of auxiliary results are scattered around in the literature, and collected here
- Possibly there is some novelty in the different adjoint relationships and resulting monads between:
  1. order unit spaces
  2. sets, and convex sets, and compact Hausdorff spaces, and convex compact Hausdorff spaces

Relevant / used literature

- Original article
  - R. Kadison, A representation theory for commutative topological algebra, Memoirs of the AMS, 7, 1951
- Additional books/papers
  - Ellis, The duality of partially ordered normed vector spaces, J. London Math. Soc., 1964.
  - Alfsen, Compact Convex Sets and Boundary Integrals, 1971
  - Nagel, Order unit and base norm spaces, in LNP 29, 1974
  - Asimow and Ellis, Convexity Theory and its Applications in Functional Analysis, 1980
  - Kadison and Ringrose, Fundamentals of the Theory of Operator Algebras. Volume I. Elementary Theory, 1983
  - Alfsen and Shultz, State spaces of operator algebras: basic theory, orientations and C^*-products, 2001
  - Paulsen and Tomforde, Vector Spaces with an order unit, Indiana Univ. Math. Journ., 2009

Prerequisites

Basic knowledge of:

1. linear algebra
2. topology
3. category theory
   - categories, functors, adjunctions
   - mainly used for organising mathematical structures and the transformations between them

Relevant categories — details follow

- **Sets** — sets and functions
- **Conv** — convex sets and affine functions
- **CH** — compact Hausdorff spaces and continuous maps
- **CCH** — convex compact Hausdorff spaces and affine continuous maps
  - subcategory $\mathbf{CCH}_{\mathrm{sep}} \hookrightarrow \mathbf{CCH}$ where points can be separated by maps to $[0,1]$
  - equivalently, convex compact subspaces of locally convex space
- **EMod** — effect modules
  - subcategory $\mathbf{BEMod} \hookrightarrow \mathbf{EMod}$ of Banach/complete modules
- **OUS** — order unit spaces
  - subcategory $\mathbf{BOUS} \hookrightarrow \mathbf{OUS}$ of Banach/complete spaces

Basic states-and-effects adjunction (BJ, Ifip TCS'10)

$$\begin{tikzcd} \mathbf{EMod}^{\mathrm{op}} \arrow[r, bend left, "\mathbf{EMod}(-,[0,1])"] \arrow[r, phantom, "\top"] & \mathbf{Conv} \arrow[l, bend left, "\mathbf{Conv}(-,[0,1])"] \end{tikzcd}$$

With Mandemaker it was shown [QPL'11] that this adjunction restricts to an equivalence

$$\mathbf{BEMod}^{\mathrm{op}} \simeq \mathbf{CCH}_{\mathrm{sep}}$$

between Banach effect modules and convex compact Hausdorff spaces

The proof used Kadison duality:

$$\begin{tikzcd} \mathbf{BEMod}^{\mathrm{op}} \arrow[r, phantom, "\simeq"] & \mathbf{BOUS}^{\mathrm{op}} \arrow[r, phantom, "\simeq"] & \mathbf{CCH}_{\mathrm{sep}} \\ & & \end{tikzcd}$$

proven

Kadison, assumed

Kadison duality, in relation to C^*-algebras

$$\begin{tikzcd} \overbrace{\mathbf{BEMod}^{\mathrm{op}} \simeq \mathbf{BOUS}^{\mathrm{op}}}^{\text{Heisenberg: predicate transformers}} \simeq \overbrace{\mathbf{CCH}_{\mathrm{sep}}}^{\text{Schrödinger: state transformers}} \\ (\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}} \arrow[ul, "\text{effects}"] \arrow[u, "\text{self-adjoints}"] \arrow[ur, "\text{states}"'] \end{tikzcd}$$

where $\mathbf{Cstar}_{\mathrm{UP}}$ is the category of C^*-algebras with positive unital (UP) maps.

The effects of a C^*-algebra A can equivalently be described as:

- $[0,1]_A = \{a \in A \mid 0 \leq a \leq 1\}$
- UP-maps $\mathbb{C}^2 \to A$, so maps $A \to 1+1$ in $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$

The states are the maps $A \to \mathbb{C}$

Relevant structures/categories

$$\begin{tikzcd} & \mathbf{BEMod} \simeq \mathbf{BOUS} \simeq (\mathbf{CCH}_{\mathrm{sep}})^{\mathrm{op}} \arrow[dl] \arrow[dr] & \\ \mathbf{Ban} \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & & \mathbf{OUS} \arrow[d] \arrow[dll] \\ \mathbf{NVect} \arrow[u, bend left] \arrow[dr] & & \mathbf{OVect} \arrow[dl] \\ & \mathbf{Vect} & \end{tikzcd}$$

norm

order

Dualities, via dualising objects

Frequently dualities arise by "homming into" a dualising object D, as in:

$$\begin{tikzcd} \mathbf{A}^{\mathrm{op}} \arrow[r, bend left, "\mathrm{Hom}(-,D)"] \arrow[r, phantom, "\simeq"] & \mathbf{B} \arrow[l, bend left, "\mathbf{Conv}(-,D)"] \end{tikzcd}$$

- Set-theoretically there is a "double dual" map:

$$X \xrightarrow{\mathrm{dd}} D^{(D^X)} \quad \text{given by} \quad \mathrm{dd}(x)(f) = f(x)$$

- In order to make this an isomorphism one restricts the maps

$$X \to D \qquad \text{and also} \qquad D^X \to D$$

This gives a representation of X inside the function spaces.

Examples

- In Stone duality, one proves for a Boolean algebra B,

$$B \xrightarrow{\cong} \mathrm{Cont}\Big(\mathrm{BA}(B,2),\, 2\Big) \cong \mathrm{Clopens}\Big(\mathrm{Ultrafilters}(B)\Big)$$

- Here the plan is to prove for an order unit space V,

$$V \rightarrowtail \mathrm{AffineCont}\Big(\mathrm{PositiveUnital}(V,\mathbb{R}),\, \mathbb{R}\Big)$$

This injection is an isomorphism iff V is complete (Banach).

It is obtained via restriction of the double dual map $V \to V^{**}$

Overview over part I

For a vector space V, over $K = \mathbb{R}$ or $K = \mathbb{C}$, there is the algebraic dual:

$$V^* = \{\omega\colon V \to K \mid \omega \text{ is linear}\}.$$

It comes with the double dual map $\mathrm{dd}\colon V \to V^{**}$, with $\mathrm{dd}(v)(\omega) = \omega(v)$. It is linear & natural & iso if $\dim V < \infty$.

If W is a normed vector space there is the norm dual:

$$W^{\#} = \{\omega\colon W \to K \mid \omega \text{ is linear and bounded}\}.$$

Both V^* and $W^{\#} \hookrightarrow W^*$ carry a "weak-* topology"

Main results in part I: the double dual map restricts to

$$V \xrightarrow{\text{surj}} V^{*\prime} \qquad\qquad W \xrightarrow{\cong} W^{\#\prime}$$

where $(-)'$ describes the continuous dual

Algebraic duals and linear combinations

Proposition Let $V \in \mathbf{Vect}_K$. $\omega \in V^*$ is a linear combination $\omega = \sum_i k_i\omega_i$ of $\omega_1, \ldots, \omega_n \in V^*$ if and only if $\bigcap_i \ker(\omega_i) \subseteq \ker(\omega)$.

Proof (only-if) is easy; for (if) use induction on n, with base case:

Lemma If $\ker(\omega) \subseteq \ker(\rho)$ for $\omega, \rho \in V^*$, then $\rho = k \cdot \omega$, for some scalar $k \in K$.

Proof If $\omega \neq 0$, use $V/\ker(\omega) \rightarrowtail K$, so quotient $V/\ker(\omega)$ is one-dimensional. Choose base vector $b \in V$ such that $V/\ker(\omega) = K \cdot (b + \ker(\omega))$. Take $k = \frac{\rho(b)}{\omega(b)}$. Using $\ker(\omega) \subseteq \ker(\rho)$ we have a well-defined map $\overline{\rho}\colon V/\ker(\omega) \to K$, namely $\overline{\rho}(v + \ker(\omega)) = \rho(v)$. Pick v and write $v + \ker(\omega) = s \cdot b + \ker(\omega)$, to show $\rho(v) = k \cdot \omega(v)$. $\square$

Weak topology on a vector space

Definition For $\Omega \subseteq V^*$ the weak topology $\sigma(V,\Omega)$ on V is the weakest topology that makes each $(\omega\colon V \to K) \in \Omega$ continuous.

Thus, subbasic opens of this topology are of the form

$$\omega^{-1}(S) = \{v \in V \mid \omega(v) \in S\}$$

for $\omega \in \Omega$ and $S \subseteq K$ open.

We write the continuous dual as:

$$V' = \{f\colon V \to K \mid f \text{ is linear and continuous}\}$$

Question: can we characterise maps in V' ?

Weakly open and weakly continuous

Lemma Consider the weak topology $\sigma(V,\Omega)$, for $\Omega \subseteq V^*$

1. basic opens around $0 \in V$ are of the form:

$$\{x \in V \mid \forall i \leq n.\, |\omega_i(x)| < r\}$$

for $\omega_1, \ldots, \omega_n \in \Omega$ and $r > 0$.

2. If $f\colon V \to K$ is weakly continuous, then:

$$|f(v)| \;\leq\; b \cdot \bigvee_i |\omega_i(v)|$$

for certain $\omega_1, \ldots, \omega_n \in \Omega$ and $b \geq 0$.

The weak-* topology

Recall the double dual map $\mathrm{dd}\colon V \to V^{**}$ with $\mathrm{dd}(v)(\omega) = \omega(v)$.

Definition The image $\mathrm{dd}(V) \subseteq V^{**}$ yields the weak-* topology on V^*.

Often it is written as $\sigma(V^*, V)$ instead of $\sigma(V^*, \mathrm{dd}(V))$.

It's subbasic opens are:

$$\mathrm{dd}(v)^{-1}(S) = \{\omega \in V^* \mid \mathrm{dd}(v)(\omega) = \omega(v) \in S\}$$

for $v \in V$ and $S \subseteq K$ open.

Double dual surjectivity theorem

Theorem $V \twoheadrightarrow V^{*\prime}$

That is: for each weakly continuous $\rho\colon V^* \to K$ there is a $v \in V$ with $\rho = \mathrm{dd}(v)$.

Proof There are $b \geq 0$ and $v_1, \ldots, v_n \in V$ with, for all $\omega \in V^*$,

$$|\rho(\omega)| \leq b \cdot \bigvee_i |\mathrm{dd}(v_i)(\omega)| = b \cdot \bigvee_i |\omega(v_i)|.$$

Then $\ker(\mathrm{dd}(v_1)) \cap \cdots \cap \ker(\mathrm{dd}(v_n)) \subseteq \ker(\rho)$. Thus we can write ρ as linear combination $\rho = k_1\mathrm{dd}(v_1) + \cdots + k_n\mathrm{dd}(v_n)$. But then $\rho = \mathrm{dd}(k_1 v_1 + \cdots + k_n v_n)$ by linearity of dd. $\square$

Normed vector spaces

A normed vector space V has a norm $\| - \|\colon V \to \mathbb{R}_{\geq 0}$ satisfying:

- $\|v\| = 0$ iff $v = 0$;
- $\|k \cdot v\| = |k| \cdot \|v\|$;
- $\|v + w\| \leq \|v\| + \|w\|$

A linear map $f\colon V \to W$ between normed V, W is bounded if there is an $s \in \mathbb{R}_{\geq 0}$ with $\|f(v)\| \leq s \cdot \|v\|$, for all $v \in V$.

We write $\mathbf{NVect}_K$ for the resulting category

The operator norm is defined for such a bounded f as:

$$\|f\|_{\mathrm{op}} = \bigwedge\{s \in \mathbb{R}_{\geq 0} \mid \forall v \in V.\, \|f(v)\| \leq s \cdot \|v\|\}.$$

Each homset $\mathrm{Hom}(V, W)$ is then itself a normed vector space, with $\|f(v)\| \leq \|f\|_{\mathrm{op}} \cdot \|v\|$.

Notation

For a normed space V write the norm dual:

$$V^{\#} = \{\omega\colon V \to K \mid \omega \text{ is linear and bounded}\}.$$

For $r \geq 0$ write the r-ball as:

$$V_{\leq r} = \{v \in V \mid \|v\| \leq r\}$$

These two can be combined in:

$$\left(V^{\#}\right)_{\leq 1} = \{\omega \in V^{\#} \mid \|\omega\|_{\mathrm{op}} \leq 1\}.$$

Hahn-Banach extension & separation results

Proposition Let V be a normed vector space over the reals

1. Let $S \subseteq V$ be a linear subspace with a linear map $\omega\colon S \to \mathbb{R}$ satisfying $|\omega(x)| \leq \|x\|$, for all $x \in S$. Then there is an $\omega' \in (V^{\#})_{\leq 1}$ with $\omega'|_S = \omega$.
2. For each $v \in V$ there is an $\omega \in (V^{\#})_{\leq 1}$ with $\omega(v) = \|v\|$.
3. For $v \neq v'$ in V there is an $\omega \in (V^{\#})_{\leq 1}$ with $\omega(v) \neq \omega(v')$.
4. For each closed linear subspace $S \subseteq V$ and each $v \notin S$ there is an $\omega \in V^{\#}$ with $\omega|_S = 0$ and $\omega(v) = 1$.

The double dual, in the normed case

Lemma The double dual map $\mathrm{dd}\colon V \to V^{**}$ restricts to $\mathrm{dd}\colon V \to V^{\#\#}$

Proof We have $\|\mathrm{dd}(v)\|_{\mathrm{op}} \leq \|v\|$ since:

$$|\mathrm{dd}(v)(\omega)| = |\omega(v)| \leq \|\omega\|_{\mathrm{op}} \cdot \|v\| = \|v\| \cdot \|\omega\|_{\mathrm{op}}. \qquad \square$$

Definition The weak-* topology on $V^{\#}$ is induced by $\mathrm{dd}(V) \subseteq V^{\#\#}$.

Main results about the weak-* topology

Theorem For a normed vector space V over $\mathbb{R}$,

1. Each ball $(V^{\#})_{\leq r} \subseteq V^{\#}$ is compact Hausdorff in the weak-* topology — aka. Banach-Alaoglu
2. $\mathrm{dd}\colon V \to V^{\#\#}$ is an isometry: $\|\mathrm{dd}(v)\|_{\mathrm{op}} = \|v\|$.

Proof sketch $(V^{\#})_{\leq r}$ is a closed subset of the compact space $D = \prod_{v \in V}[-r \cdot \|v\|, r \cdot \|v\|]$.

We saw $\|\mathrm{dd}(v)\|_{\mathrm{op}} \leq \|v\|$. Then $(\geq)$ follows because there is an $\omega \in (V^{\#})_{\leq 1}$ with $\omega(v) = \|v\|$. $\square$

Double dual isomorphism theorem

Theorem $V \xrightarrow{\cong} V^{\#\prime}$

That is: for each weakly continuous $\rho\colon V^{\#} \to K$ there is a unique $v \in V$ with $\rho = \mathrm{dd}(v)$.

Proof Existence we already know. Uniqueness follows because dd is an isometry, and thus injective. $\square$

Complete normed spaces

Definition A Banach space is a complete normed vector space: every Cauchy sequence has a limit — using distance $d(v, w) = \|v - w\|$.

We write $\mathbf{Ban}_K \hookrightarrow \mathbf{NVect}_K$ for the full subcategory.

Proposition The standard metric completion construction yields a left adjoint:

$$\begin{tikzcd} \mathbf{Ban}_K \arrow[d, hook] \arrow[d, bend left, phantom, "\dashv"] \\ \mathbf{NVect}_K \end{tikzcd}$$

Recall: we have done the left leg, so far

$$\begin{tikzcd} & \mathbf{BEMod} \simeq \mathbf{BOUS} \simeq (\mathbf{CCH}_{\mathrm{sep}})^{\mathrm{op}} \arrow[dl] \arrow[dr] & \\ \mathbf{Ban} \arrow[d, hook] \arrow[d, bend left, phantom, "\dashv"] & & \mathbf{OUS} \arrow[dll] \arrow[d] \\ \mathbf{NVect} \arrow[dr] & & \mathbf{OVect} \arrow[dl] \\ & \mathbf{Vect} & \end{tikzcd}$$

norm order

With main result, for a normed vector space V,

$$V \xrightarrow[\cong]{\mathrm{dd}} V^{\#\prime} = \{\rho\colon V^{\#} \to \mathbb{R} \mid \rho \text{ is weakly continuous}\}$$

where:

- $V^{\#} = \{\omega\colon V \to \mathbb{R} \mid \omega \text{ is linear and bounded}\}$
- the weak-* topology on $V^{\#}$ is the least one that makes all point-evaluation maps $\mathrm{dd}(v)\colon V^{\#} \to \mathbb{R}$ continuous

Order on a vector space

Definition A vector space V is ordered if it carries a partial order $\leq$ satisfying:

1. $v \leq v'$ implies $v + w \leq v' + w$, for all $w \in V$;
2. $v \leq v'$ implies $r \cdot v \leq r \cdot v'$ for all $r \in \mathbb{R}_{\geq 0}$.

We write $\mathbf{OVect}_K \hookrightarrow \mathbf{Vect}_K$ for the subcategory of ordered vector spaces with monotone linear maps between them.

(Aside: in a Riesz space there is additionally a join $\vee$)

For a linear map $f\colon V \to W$ it is equivalent to say:

- f is monotone: $v \leq v'$ implies $f(v) \leq f(v')$
- f is positive: $0 \leq v$ implies $0 \leq f(v)$

Unit in an ordered space

Definition A strong unit in an ordered vector space V is a positive element $1 \in V$ with:

- for each $v \in V$ there is an $n \in \mathbb{N}$ with $-n \cdot 1 \leq v \leq n \cdot 1$

An Archimedean unit is a strong unit with:

- $\big[\forall r > 0.\, v \leq r \cdot 1\big]$ implies $v \leq 0$

An order unit space is a ordered vector space over $\mathbb{R}$ with an Archimedean unit.

The category $\mathbf{OUS} \hookrightarrow \mathbf{OVect}_{\mathbb{R}}$ has as maps that are

- linear, positive/monotone, and unital (preserve 1)
- we call them simple Positive-Unital (UP)

How order unit spaces may arise

$$\begin{tikzcd} \mathbf{Sets} \arrow[dr, "\ell^\infty"] & & \mathbf{CH} \arrow[dl, "\mathrm{C}"'] \\ & \mathbf{OUS}^{\mathrm{op}} & \\ \mathbf{Conv} \arrow[ur, "\mathrm{A}^\infty"] & & \mathbf{CCH} \arrow[ul, "\mathrm{AC}"'] \end{tikzcd}$$

We shall see that all these functors have (right) adjoints, given by "states"

Recall that **Conv** is the category of convex sets and affine maps

- in a convex set X, finite convex sums $\sum_i r_i x_i$ exist, of elements $x_i \in X$ and $r_i \in [0, 1]$ with $\sum_i r_i = 1$
- affine functions preserve such convex sums

ℓ^∞: Sets $\to$ OUS^{op}

- For a set X, write $\ell^\infty(X)$ for the bounded maps $X \to \mathbb{R}$:
$$\ell^\infty(X) = \{\phi\colon X \to \mathbb{R} \mid \exists b.\, \forall x.\, |\phi(x)| \leq b\}$$
- Vector space operations on $\ell^\infty(X)$ are inherited pointwise from $\mathbb{R}$
- The order is also pointwise: $\phi \leq \phi'$ iff $\phi(x) \leq \phi'(x)$ for all x
- The unit $1\colon X \to \mathbb{R}$ in $\ell^\infty(X)$ is the "constant 1" map
  - This unit is strong because maps in $\ell^\infty(X)$ are bounded
  - It's Archimedean since $\mathbb{R}$ is Archimedean
- For a function $f\colon X \to Y$ we get $(-) \circ f\colon \ell^\infty(Y) \to \ell^\infty(X)$.

A^∞: Conv $\to$ OUS^{op}

- For an affine set $X \in$ **Conv** we take
$$\mathrm{A}^\infty(X) = \{\phi\colon X \to \mathbb{R} \mid \phi \text{ is bounded and affine}\}$$
- OUS-structure is like in $\ell^\infty(X)$
  - affine maps are closed under (pointwise) vector operations

C: CH $\to$ OUS^{op} and AC: CCH $\to$ OUS^{op}

- For a compact Hausdorff space $X \in$ **CH** take
$$\mathrm{C}(X) = \{\phi\colon X \to \mathbb{R} \mid \phi \text{ is continuous}\}$$
Such maps ϕ are automatically bounded.
  - OUS-structure is obtained pointwise, like before
- Similarly, for a convex compact Hausdorff space X,
$$\mathrm{AC}(X) = \{\phi\colon X \to \mathbb{R} \mid \phi \text{ is affine continuous}\}$$

Aside: the ℓ^∞ and C constructions yield C^*-algebras, but A^∞ and AC do not, in the convex case.

- affine maps are not closed under multiplication

States and effects in OUS

- A state in an OUS V is a UP-map $\omega\colon V \to \mathbb{R}$
  - we write $\mathrm{Stat}(V) = \mathrm{Hom}(V, \mathbb{R})$
  - $\mathbb{R}$ is initial in **OUS**, so a state is a point $1 \to V$ in **OUS**^{op}
  - Example: $\mathrm{Stat}(\mathbb{R}^n) = \mathcal{D}(n) =$ probability distributions on n
- An effect is an element $v \in V$ with $0 \leq v \leq 1$
  - we write $[0,1]_V = \{v \in V \mid 0 \leq v \leq 1\}$
  - an effect corresponds to a map $\mathbb{R}^2 \to V$
  - and thus to a map $V \to 1+1$ in **OUS**^{op}
  - example: $[0,1]_{\mathbb{R}^n} = [0,1]^n$
- The Born rule gives the validity probability $\omega \models v$
  - obtained by function application $(\omega \models v) = \omega(v) \in [0,1]$
  - more abstractly, by composition in **OUS**^{op}
$$\begin{tikzcd} 1 \arrow[r, "\text{state}"] \arrow[dr, "\text{probability}"'] & V \arrow[d, "\text{effect}"] \\ & 1+1 \end{tikzcd}$$

Aside about OUS-effect module relationship

- The mapping $V \mapsto [0,1]_V$ yields an equivalence of categories:
$$\textbf{OUS} \simeq \textbf{AEMod}$$
where **AEMod** $\hookrightarrow$ **EMod** is the full subcategory of Archimedean effect modules
- By imposing (suitable) completeness requirements it restricts to:
$$\textbf{BOUS} \simeq \textbf{BEMod}$$
as discussed in the introduction (with 'B' for 'Banach')

Convexity for OUS-maps

Lemma For $V, W \in$ **OUS** the homset $\mathrm{Hom}(V, W)$ is convex. Pre- and post-composition $(-) \circ g$ and $h \circ (-)$ are affine.

Proof The key point is that $f = \sum_i r_i f_i\colon V \to W$ is unital again:
$$f(1) = \textstyle\sum_i r_i \cdot f_i(1) = \sum_i r_i \cdot 1 = (\sum_i r_i) \cdot 1 = 1 \cdot 1 = 1. \quad \square$$

Corollary Taking states yields a functor:
$$\textbf{OUS}^{\mathrm{op}} \xrightarrow{\ \mathrm{Stat}\ } \textbf{Conv}$$

Adjunctions **Sets** $\leftrightarrows$ **OUS**^{op} and **Conv** $\leftrightarrows$ **OUS**^{op}

Proposition The states functor is right adjoint in:

$$\begin{tikzcd} \mathbf{OUS}^{\mathrm{op}} \arrow[d, bend right, "\mathrm{Stat}"'] \\ \mathbf{Sets} \arrow[u, bend right, "\ell^\infty"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{OUS}^{\mathrm{op}} \arrow[d, bend right, "\mathrm{Stat}"'] \\ \mathbf{Conv} \arrow[u, bend right, "\mathrm{A}^\infty"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

Proof By swapping arguments we get bijective correspondences:

$$\frac{V \xrightarrow{f} \ell^\infty(X)}{X \xrightarrow[g]{} \mathrm{Stat}(V)} \quad \begin{array}{l}\text{in } \mathbf{OUS} \\ \text{in } \mathbf{Sets}\end{array} \qquad \frac{V \xrightarrow{f} \mathrm{A}^\infty(Y)}{Y \xrightarrow[g]{} \mathrm{Stat}(V)} \quad \begin{array}{l}\text{in } \mathbf{OUS} \\ \text{in } \mathbf{Conv}\end{array}$$

Relating ℓ^∞ and A^∞

These adjunctions allow some abstract categorical reasoning.

Proposition $\mathrm{A}^\infty \circ \mathcal{D} \cong \ell^\infty$, where $\mathcal{D}$ is the (discrete probability) distribution monad on **Sets**

Proof We have $\mathbf{Conv} = \mathcal{EM}(\mathcal{D})$ and so the result follows by composition and uniqueness of adjoints in:

$$\begin{tikzcd} \mathbf{OUS}^{\mathrm{op}} \arrow[d, bend right, "\mathrm{Stat}"'] \arrow[dd, bend right=60, "\mathrm{Stat}"] \\ \mathbf{Conv} \arrow[u, bend right, "\mathrm{A}^\infty"] \arrow[u, phantom, "\dashv"] \arrow[d, bend right, "\mathrm{forget}"'] \\ \mathbf{Sets} \arrow[u, bend right, "\mathcal{D}"] \arrow[u, phantom, "\dashv"] \arrow[uu, bend right=60, "\ell^\infty"] \end{tikzcd}$$

Order unit spaces are normed

Definition Each $V \in \mathbf{OUS}$ carries a norm defined by:

$$\|v\|_{\mathrm{ou}} = \bigwedge\{r \in \mathbb{R}_{\geq 0} \mid -r \cdot 1 \leq v \leq r \cdot 1\}.$$

Example The induced norm on $\ell^\infty(X) \in \mathbf{OUS}$ is the supremum (aka. uniform) norm:

$$\|\phi\|_\infty = \bigvee\{|\phi(x)| \mid x \in X\}$$

It exists since ϕ is bounded. Similarly for $\mathrm{A}^\infty(X)$, $\mathrm{C}(X)$, $\mathrm{AC}(X)$
(The spaces are actually complete)

Lemma Each map f in **OUS** satisfies $\|f\|_{\mathrm{op}} = 1$. We get a functor:

$$\mathbf{OUS} \longrightarrow \mathbf{NVect}_{\mathbb{R}}$$

In particular, $\mathrm{Stat}(V) \subseteq (V^\#)_{\leq 1}$

Weak-* topology on states

$\mathrm{Stat}(V) \subseteq V^\#$ inherits the weak-* topology, with subbasic opens $\mathrm{dd}(v)^{-1}(S) = \{\omega \in \mathrm{Stat}(V) \mid \omega(v) \in S\}$, for $v \in V$, $S \subseteq \mathbb{R}$ open.

Proposition

1. This weak-* topology on $\mathrm{Stat}(V)$ is compact Hausdorff
2. Alternative basic opens are, for $v \in V$ and $s \in \mathbb{Q}$,
$$\Box_s(v) = \{\omega \in \mathrm{Stat}(V) \mid \omega(v) > s\},$$
3. For each $f \colon V \to W$ in **OUS**, the map $(-) \circ f \colon \mathrm{Stat}(W) \to \mathrm{Stat}(V)$ is continuous

Proof $\mathrm{Stat}(V) \subseteq (V^\#)_{\leq 1}$ is a closed subset of a compact Hausdorff space. The rest is easy $\square$

Adjunctions **CH** $\leftrightarrows$ **OUS**^{op} and **CCH** $\leftrightarrows$ **OUS**^{op}

Proposition The states functor is also right adjoint in:

$$\begin{tikzcd} \mathbf{OUS}^{\mathrm{op}} \arrow[d, bend right, "\mathrm{Stat}"'] \\ \mathbf{CH} \arrow[u, bend right, "\mathrm{C}"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{OUS}^{\mathrm{op}} \arrow[d, bend right, "\mathrm{Stat}"'] \\ \mathbf{CCH} \arrow[u, bend right, "\mathrm{AC}"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

As a result, $\mathrm{C} \circ \mathcal{U} \cong \ell^\infty$ and $\mathrm{AC} \circ \mathcal{E} \cong \ell^\infty$ in:

$$\begin{tikzcd} \mathbf{OUS}^{\mathrm{op}} \arrow[d, bend right, "\mathrm{Stat}"'] \arrow[dd, bend right=60, "\mathrm{Stat}"] \\ \mathbf{CH} \arrow[u, bend right, "\mathrm{C}"] \arrow[u, phantom, "\dashv"] \arrow[d, bend right, "\mathrm{forget}"'] \\ \mathbf{Sets} \arrow[u, bend right, "\mathcal{U}"] \arrow[u, phantom, "\dashv"] \arrow[uu, bend right=60, "\ell^\infty"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{OUS}^{\mathrm{op}} \arrow[d, bend right, "\mathrm{Stat}"'] \arrow[dd, bend right=60, "\mathrm{Stat}"] \\ \mathbf{CCH} \arrow[u, bend right, "\mathrm{AC}"] \arrow[u, phantom, "\dashv"] \arrow[d, bend right, "\mathrm{forget}"'] \\ \mathbf{Sets} \arrow[u, bend right, "\mathcal{E}"] \arrow[u, phantom, "\dashv"] \arrow[uu, bend right=60, "\ell^\infty"] \end{tikzcd}$$

Here $\mathcal{U}$ = ultrafilter, and $\mathcal{E}$ = expectation.

Summarising adjunction diagram, with induced monads

$$\begin{tikzcd} \mathbf{Sets} \arrow[loop left, "\mathcal{E}"] \arrow[dr, shift left, "\ell^\infty"] & & \mathbf{CH} \arrow[loop right, "\mathcal{R}"] \arrow[dl, shift left, "\mathrm{Stat}"] \\ & \mathbf{OUS}^{\mathrm{op}} \arrow[ul, shift left, "\mathrm{Stat}"] \arrow[ur, shift left, "\mathrm{C}"] \arrow[dl, shift left, "\mathrm{A}^\infty"] \arrow[dr, shift left, "\mathrm{AC}"] & \\ \mathbf{Conv} \arrow[loop below, "\text{unused}"] \arrow[ur, shift left, "\mathrm{Stat}"] & & \mathbf{CCH} \arrow[loop below, "\text{see below}"] \arrow[ul, shift left, "\mathrm{Stat}"] \end{tikzcd}$$

Here $\mathcal{R}$ is the Radon monad, with (Furber & Jacobs, Calco'13):

$$\mathcal{K}\ell(\mathcal{R}) \simeq (\mathbf{CCstar}_{\mathrm{UP}})^{\mathrm{op}}$$

We continue with the lower-right adjunction $\mathbf{OUS}^{\mathrm{op}} \rightleftarrows \mathbf{CCH}$.

The easy half of Kadison duality

Theorem The adjunction $\mathbf{OUS}^{\mathrm{op}} \rightleftarrows \mathbf{CCH}$ restricts to $\mathbf{OUS}^{\mathrm{op}} \rightleftarrows \mathbf{CCH}_{\mathrm{sep}}$, with separation of points

The adjunction's unit, for $X \in \mathbf{CCH}_{\mathrm{sep}}$, is the evaluation map

$$X \xrightarrow{\mathrm{ev}} \mathrm{Stat}(\mathrm{AC}(X)) \quad \text{given by} \quad \mathrm{ev}(x)(\phi) = \phi(x)$$

This unit ev is an isomorphism.

Proof It is easy to see that ev is continuous and affine. It is injective by the separation property: if $\mathrm{ev}(x)(\phi) = \phi(x) = \phi(y) = \mathrm{ev}(y)(\phi)$ for all $\phi \in \mathrm{Stat}(X)$, then $x = y$.
For surjectivity we prove that the image $\mathrm{ev}(X) \subseteq \mathrm{Stat}(\mathrm{AC}(X))$ is weak-* dense.

Proof, continued, with auxiliary result

Lemma For each $\phi \in \mathrm{AC}(X)$ and for each state $\omega \in \mathrm{Stat}(\mathrm{AC}(X))$ there is an $x \in X$ with $\omega(\phi) = \phi(x)$.

Proof The image $\phi(X) \subseteq \mathbb{R}$ is compact, and thus closed and bounded, and also convex. Hence it is a closed interval: $\phi(X) = [s_1, s_2] \subseteq \mathbb{R}$. Take the middle $m = \frac{1}{2}(s_1 + s_2) \in [s_1, s_2]$ and radius $r = \frac{1}{2}(s_2 - s_1)$, so that $\phi(X) = \{t \in \mathbb{R} \mid |m - t| \leq r\}$.
The function $\phi - m \cdot 1 \in \mathrm{AC}(X)$ has r as upperbound:

$$\|\phi - m \cdot 1\|_\infty = \bigvee_{x \in X} \big|(\phi - m \cdot 1)(x)\big| = \bigvee_{x \in X} \big|\phi(x) - m\big| \leq r$$

because $\phi(x) \in \phi(X)$. For $\omega \in \mathrm{Stat}(\mathrm{AC}(X))$, since $\|\omega\|_{\mathrm{op}} = 1$,

$$\big|\omega(\phi) - m\big| = \big|\omega(\phi - m \cdot 1)\big| \leq \|\phi - m \cdot 1\|_\infty \leq r$$

But then $\omega(\phi) \in \phi(X)$, and thus $\omega(\phi) = \phi(x)$, for some $x \in X$. $\square$

Proof, continued, denseness

Recall that we still need to prove $\mathrm{ev}(X) \subseteq \mathrm{Stat}(\mathrm{AC}(X))$ is dense

Assume a non-empty subbasis open of $\mathrm{Stat}(\mathrm{AC}(X))$, namely:

$$\square_s(\phi) = \{\omega \mid \omega(\phi) > s\}$$

for some $\phi \in \mathrm{AC}(X)$ and $s \in \mathbb{Q}$. We must show $\square_s(\phi) \cap \mathrm{ev}(X) \neq \emptyset$.

So let $\omega(\phi) > s$, using that $\square_s(\phi) \neq \emptyset$. By the Lemma, there is an $x \in X$ with $\omega(\phi) = \phi(x) = \mathrm{ev}(x)(\phi)$. But then $\mathrm{ev}(x) \in \square_s(\phi)$. $\square$

Where do we stand?

- Our aim to prove an equivalence $\mathbf{BOUS}^{\mathrm{op}} \simeq \mathbf{CCH}_{\mathrm{sep}}$
- We have adjoint functors in two directions:

$$\begin{tikzcd} \mathbf{OUS}^{\mathrm{op}} \arrow[r, bend left, "\mathrm{Stat}=\mathrm{Hom}(-,\mathbb{R})"] \arrow[r, phantom, "\top"] & \mathbf{CCH}_{\mathrm{sep}} \arrow[l, bend left, "\mathrm{AC}"] \end{tikzcd}$$

- The unit of this adjunction is an isomorphism: for $X \in \mathbf{CCH}_{\mathrm{sep}}$,

$$X \xrightarrow[\cong]{\mathrm{ev}} \mathrm{Stat}(\mathrm{AC}(X))$$

- The counit is the (restricted) double dual map, for $V \in \mathbf{OUS}$,

$$V \xrightarrow{\mathrm{dd}} \mathrm{AC}(\mathrm{Stat}(V))$$

- Next, we will prove two things:
  - this dd is injective — in fact an isometry
  - it is also surjective — via the Krein-Šmulian Theorem

Basic results about order unit spaces

Lemma Let V be an order unit space with, norm $\| - \|$ and $v \in V$

1. one can write $v = v_+ - v_-$, for positive vectors $v_+, v_- \in V$;
2. $-\|v\| \cdot 1 \leq v \leq \|v\| \cdot 1$;
3. if $f(1) = 0$ for a positive linear map $f \colon V \to W$, then $f = 0$;
4. each positive linear functional $\rho \colon V \to \mathbb{R}$ is bounded, with $\|\rho\|_{\mathrm{op}} = \rho(1)$;
5. for each non-zero positive linear map $\rho \colon V \to \mathbb{R}$ there is a unique $r > 0$ such that $r\rho$ is a state, namely $r = \frac{1}{\rho(1)}$.
6. if $-w \leq v \leq w$ then $\|v\| \leq \|w\|$.

Extension & separation results for order unit spaces

Proposition Let V be an order unit space.

1. If $S \subseteq V$ is a linear subspace containing 1, then each state $\omega \colon S \to \mathbb{R}$ extends to a state $\omega' \colon V \to \mathbb{R}$ with $\omega'|_S = \omega$
2. Each $\rho \in V^\#$ can be written as $\rho = \rho_+ - \rho_-$ where $\rho_+, \rho_- \in V^\#$ are positive and $\|\rho_+\|_{\mathrm{op}} + \|\rho_-\|_{\mathrm{op}} = \|\rho\|_{\mathrm{op}}$
3. $v \geq 0$ iff $\omega(v) \geq 0$ for all $\omega \in \mathrm{Stat}(V)$
4. If $v \neq v'$, then there is an $\omega \in \mathrm{Stat}(V)$ with $\omega(v) \neq \omega(v')$
5. $\|v\| = \bigvee\{|\omega(v)| \mid \omega \in \mathrm{Stat}(V)\}$

The counit of $\mathbf{OUS}^{\mathrm{op}} \rightleftarrows \mathbf{CCH}_{\mathrm{sep}}$ is an isometry

The "double dual" counit map

$$V \xrightarrow{\mathrm{dd}} \mathrm{AC}(\mathrm{Stat}(V))$$

- is an isometry: $\|\mathrm{dd}(v)\|_{\mathrm{op}} = \|v\|$ — and thus injective
- preserves and reflects the order

Proof of the second point

$$\begin{aligned} \mathrm{dd}(v) \leq \mathrm{dd}(v') &\iff \mathrm{dd}(v)(\omega) \leq \mathrm{dd}(v')(\omega), \text{ for all } \omega \\ &\qquad \text{since the order is pointwise} \\ &\iff \omega(v) \leq \omega(v'), \text{ for all } \omega \\ &\iff v \leq v' \qquad \text{by the previous result} \end{aligned}$$

□

The counit is an isomorphism

Theorem (Kadison) The counit $\mathrm{dd}\colon V \rightarrowtail \mathrm{AC}(\mathrm{Stat}(V))$ is an isomorphism iff V is complete (Banach)

Proof The (only if) part is easy, since $\mathrm{AC}(\mathrm{Stat}(V))$ is complete.

The (if) part requires much more work.

Strategy: extend $\phi \in \mathrm{AC}(\mathrm{Stat}(V))$ to a weakly continuous $\overline{\phi}\colon V^{\#} \to \mathbb{R}$ in:

$$\begin{tikzcd} \mathrm{Stat}(V) \arrow[r, hook] \arrow[dr, "\text{affine continuous } \phi"'] & V^{\#} \arrow[d, dotted, "\text{weakly continous } \overline{\phi}"] \\ & \mathbb{R} \end{tikzcd}$$

Then use the isomorphism $V \xrightarrow{\cong} V^{\#\prime}$

Proof, step 1, assuming $\phi\colon \mathrm{Stat}(V) \to \mathbb{R}$

Recall that each $\rho \in V^{\#}$ can be written as difference $\rho = \rho_1 - \rho_2$ of positive maps $\rho_1, \rho_2 \geq 0$.

If $\rho_i \neq 0$, then we can write $\rho_i = r_i \omega_i$ for $\omega_i \in \mathrm{Stat}(V)$, where $r_i = \rho_i(1)$ and $\omega_i = \frac{1}{\rho_i(1)}\rho_i$.

Then define:

$$\overline{\phi}(\rho) = r_1\phi(\omega_1) - r_2\phi(\omega_2) = \rho_1(1)\phi(\tfrac{1}{\rho_1(1)}\rho_1) - \rho_2(1)\phi(\tfrac{1}{\rho_2(1)}\rho_2)$$

One now proves that this $\overline{\phi}\colon V^{\#} \to \mathbb{R}$

- is well-defined, ie. independent of choice of ρ_1, ρ_2
- is linear — and also bounded
- extends $\phi\colon \mathrm{Stat}(V) \to \mathbb{R}$

These verifications use that ϕ is affine

Auxiliary result

We rely on:

Theorem (Krein-Šmulian) If V is a complete order unit space, then $f\colon V^{\#} \to \mathbb{R}$ is weakly continuous if its restriction to $(V^{\#})_{\leq r} \to \mathbb{R}$ is weakly continuous, for each $r \geq 0$.

- The general result is formulated for Banach spaces (not $V^{\#}$)
- The proof is non-trivial and uninsightful, see for instance (Dunford & Schwartz 1957), (Conway 1990), (Pedersen 1989), (Asimov & Ellis 1980).

Proof, step 2, given $\overline{\phi}\colon V^{\#} \to \mathbb{R}$

We need to prove that the restriction $\overline{\phi}\colon (V^{\#})_{\leq r} \to \mathbb{R}$ is weakly continuous, for an arbitrary $r > 0$.

Assume a weakly converging net $\nu_\alpha \in (V^{\#})_{\leq r}$ with $\lim \nu = \rho \in (V^{\#})_{\leq r}$. We need to write all maps as difference of positive parts

$$\nu_\alpha = \rho_\alpha^+ - \rho_\alpha^- \qquad \rho = \rho^+ - \rho^-$$

Next we need to scale them to states. These scaling factors and states form a new net, in a compact space:

$$\mu_\alpha \in [-r, r] \times \mathrm{Stat}(V) \times [-r, r] \times \mathrm{Stat}(V)$$

Thus there is a converging subnet. With this subnet one shows $\overline{\phi}(\lim \nu) = \lim \overline{\phi}(\nu)$. □

Finally, Kadison duality

Theorem There is an equivalence of categories

$$\mathbf{BOUS}^{\mathrm{op}} \simeq \mathbf{CCH}_{\mathrm{sep}}$$

Main occurrence: relating observables and states in C^*-algebras:

$$\begin{tikzcd} \mathbf{BOUS}^{\mathrm{op}} \arrow[r, phantom, "\simeq"] & \mathbf{CCH}_{\mathrm{sep}} \\ & (\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}} \arrow[ul, bend left, "\text{self-adjoints}"] \arrow[ur, bend right, "\text{states}"'] \end{tikzcd}$$

which gives the same picture as eg. in the non-deterministic case

$$\begin{tikzcd} (\mathbf{CL}_{\bigwedge})^{\mathrm{op}} \arrow[r, phantom, "\simeq"] & \mathbf{CL}_{\bigvee} \\ & \mathcal{K}\ell(\mathcal{P}) \arrow[ul, bend left, "2^{(-)}"] \arrow[ur, bend right, "\mathcal{P}"'] \end{tikzcd}$$

Such triangles form a general pattern for predicate and state transformer semantics.

A cousin-duality: Yosida

Recall: A Riesz space is an ordered vector space with binary joins $\vee$ — and thus also $\wedge$.

Write **BARsz** $\hookrightarrow$ **BOUS** for the category of "Banach-Archimedean Riesz spaces"

Theorem (Yosida/Stone) There is an equivalence of categories:

$$\begin{tikzcd} \mathbf{BARsz}^{\mathrm{op}} \arrow[r, bend left, "\mathrm{Stat}=\mathrm{Hom}(-,\mathbb{R})"] \arrow[r, phantom, "\simeq"] & \mathbf{CH} \arrow[l, bend left, "\mathrm{C}"] \end{tikzcd}$$

Kadison and Yosida

The following diagram commutes

$$\begin{tikzcd} \mathbf{CCH}_{\mathrm{sep}} \arrow[r, "\simeq"] \arrow[d, shift left, "\mathrm{forget}"] \arrow[d, phantom, "\dashv"] & \mathbf{BOUS}^{\mathrm{op}} \\ \mathbf{CH} \arrow[u, bend left, "\mathcal{R}"] \arrow[r, phantom, "\simeq"] & \mathbf{BARsz}^{\mathrm{op}} \arrow[u, "\mathrm{forget}"'] \end{tikzcd}$$

As a result, there is a left adjoint to the forgetful functor in:

$$\begin{tikzcd} \mathbf{BOUS} \arrow[d, shift right] \arrow[d, phantom, "\dashv"] \\ \mathbf{BARsz} \arrow[u, shift right, "\mathrm{forget}"'] \end{tikzcd}$$

Maps between C^*-algebras

- Traditionally, almost always, C^*-algebras are used with $*$-homomorphisms as maps
- However, there are good reasons to consider them with the (weaker) unital positive (UP) maps
  - probabilistic nature of UP-maps, in the commutative case
  - relation to order unit spaces and effect algebras
- The goal here is to explain some of the differences — and to promote UP-maps a bit
- (I don't wish to distinguish positive and completely positive maps at this stage)

Maps between C^*-algebras, in detail

A linear map $f\colon A \to B$ between C^*-algebras is called:

- unital (**U**), if $f(1) = 1$
  (variation: subunital means $0 \leq f(1) \leq 1$)
- positive (**P**), if $a \geq 0 \Rightarrow f(a) \geq 0$
  (where $a \geq 0$ means $a = x^*x$, for some x)
- multiplicative (**M**), if $f(a \cdot a') = f(a) \cdot f(a')$
- involutive (**I**), if $f(a^*) = f(a)^*$

FACTS UP $\Rightarrow$ I and MIU $\Rightarrow$ UP

We use categories $\mathbf{Cstar}_{\mathrm{MIU}}$ and $\mathbf{Cstar}_{\mathrm{UP}} \hookrightarrow \mathbf{Cstar}_{\mathrm{MIU}}$

MIU-maps are usally called $*$-homomorphisms; they are the "standard" maps in C^*-algebra theory

- eg. Gelfand duality: $\mathbf{CH} \simeq (\mathbf{CCstar}_{\mathrm{MIU}})^{\mathrm{op}}$

Some characteristic differences between MIU and UP

- Functors from Hilbert spaces to C^*-algebras:

$$\left(\begin{array}{c}\text{Hilbert spaces}\\ \text{with unitary maps}\end{array}\right) \longrightarrow (\mathbf{Cstar}_{\mathrm{MIU}})^{\mathrm{op}}$$

$$\left(\begin{array}{c}\text{Hilbert spaces}\\ \text{with isometries}\end{array}\right) \longrightarrow (\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$$

- Projections and effects arise via different maps:

$$\frac{\text{MIU-maps } \mathbb{C}^2 \to A}{\text{projections in } A} \qquad \frac{\text{UP-maps } \mathbb{C}^2 \to A}{\text{effects in } A}$$

MIU-maps example

For $n, m \in \mathbb{N}$, there is a bijective correspondence:

$$\frac{\text{MIU-maps } \mathbb{C}^n \longrightarrow \mathbb{C}^m}{\text{functions } m \longrightarrow n}$$

Essentially, this is the finite-dimensional version of Gelfand duality:

$$\mathbf{FinSets} \simeq (\mathbf{FdCCstar}_{\mathrm{MIU}})^{\mathrm{op}}$$

MIU-maps example, continued

Proof of the correspondence.

- Each $f\colon m \to n$ obviously gives $(-) \circ f\colon \mathbb{C}^n \to \mathbb{C}^m$. It preserves the (pointwise) structure.
- Assume $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ is a MIU map. Write the standard base vectors as $|\,i\rangle = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^n$.

  Since $|\,i\rangle \cdot |\,i\rangle = |\,i\rangle$, we get $\varphi(|\,i\rangle) \cdot \varphi(|\,i\rangle) = \varphi(|\,i\rangle)$, so that $\varphi(|\,i\rangle) = (r_{i1}, \ldots, r_{im}) \in \mathbb{C}^m$ consists of $r_{ij} \in \{0, 1\}$.

  Since $\sum_i |\,i\rangle = 1 \in \mathbb{C}^n$, we get $\sum_i \varphi(|\,i\rangle) = \varphi(1) = 1 \in \mathbb{C}^m$, so that $\sum_i r_{ij} = 1$, for each $j \leq m$.

  But then: for each $j \leq m$ there is precisely one $i \leq n$ with a $r_{ij} = 1$. This yields a function $m \to n$. $\square$

UP-maps example

For $n, m \in \mathbb{N}$, there is a bijective correspondence:

$$\begin{array}{c} \text{UP-maps} \quad \mathbb{C}^n \longrightarrow \mathbb{C}^m \\ \hline\hline \text{functions} \quad m \longrightarrow \mathcal{D}(n) \end{array}$$

where $\mathcal{D}$ is the distribution monad.

This gives "probabilistic" Gelfand duality, in the finite case:

$$\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \simeq \big(\mathbf{FdCCstar}_{\mathrm{UP}}\big)^{\mathrm{op}}$$

where $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \hookrightarrow \mathcal{K}\ell(\mathcal{D})$ is the full subcategory with numbers $n \in \mathbb{N}$ as objects.

Thus, $\mathbf{FdCCstar}_{\mathrm{UP}}$ is the Lawvere theory of the distribution monad

UP-maps example, continued

Proof of the correspondence.

- Each $f\colon m \to \mathcal{D}(n)$ gives a map $\mathbb{C}^n \to \mathbb{C}^m$ by:

$$v \longmapsto \lambda j \leq m.\ \sum_{i \leq n} f(j)(i) \cdot v(i)$$

- Assume $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ is a UP map. The base vector $|\,i\rangle \in \mathbb{C}^n$ is positive, and so $\varphi(|\,i\rangle) = (r_{i1}, \ldots, r_{im}) \in \mathbb{C}^m$ consists of positive (real) numbers r_{ij}

  As before, $\sum_i \varphi(|\,i\rangle) = \varphi(1) = 1 \in \mathbb{C}^m$, so for each $j \leq m$ we have $\sum_i r_{ij} = 1$.

  Thus we get the required map $m \to \mathcal{D}(n)$. $\square$

Beyond the finite-dimensional case

In (Furber & Jacobs, CALCO'13) it is shown that:

- There is a Radon monad $\mathcal{R}\colon \mathbf{CH} \to \mathbf{CH}$ on the category $\mathbf{CH}$ of compact Hausdorff spaces
- This monad $\mathcal{R}$ is given by states of the C^*-algebra $C(X)$:

$$\mathcal{R}(X) = \mathrm{Hom}_{\mathrm{UP}}(C(X), \mathbb{C})$$

- We then get::

$$\mathcal{K}\ell(\mathcal{R}) \simeq \big(\mathbf{CCstar}_{\mathrm{UP}}\big)^{\mathrm{op}}$$

Effect algebras, definition

Effect algebras axiomatise the unit interval $[0, 1]$ with its (partial!) addition $+$ and "negation" $x \mapsto 1 - x$.

A Partial Commutative Monoid (PCM) consists of a set M with zero $0 \in M$ and partial operation $\ovee\colon M \times M \to M$, which is suitably commutative and associative.

One writes $x \perp y$ if $x \ovee y$ is defined.

An effect algebra is a PCM in which each element x has a unique 'orthosuplement' $x^\perp$ with $x \ovee x^\perp = 1\ (= 0^\perp)$
Additionally, $x \perp 1 \Rightarrow x = 0$ must hold.

There is then a partial order, via $x \leq y$ iff $y = x \ovee z$, for some z.

Effect algebras, main examples

1. Projections / closed subspaces on a Hilbert space form an effect algebra; $P^\perp$ is orthocomplement:

$$\langle x \,|\, y \rangle = 0 \qquad \text{for all } x \in P, y \in P^\perp$$

2. Orthomodular lattices are effect algebras, with $\ovee$ as join $x \vee y$ only for elements with $x \perp y$, i.e. $x \leq y^\perp$
3. Each Boolean algebra is an effect algebra: it is a distributive orthomodular lattice, in which $x \perp y$ iff $x \wedge y = 0$.

   In particular, the Boolean algebra of measurable subsets of a measure space forms an effect algebra, where $U \ovee V$ is defined if $U \cap V = \emptyset$, and is then equal to $U \cup V$.

Homomorphisms of effect algebras

Definition

A homomorphism of effect algebras $f\colon X \to Y$ satisfies:

- $f(1) = 1$
- if $x \perp x'$ then both $f(x) \perp f(x')$ and $f(x \owedge x') = f(x) \owedge f(x')$.

This yields a category **EA** of effect algebras.

Examples:

- There is a functor $\mathbf{Meas}^{\mathrm{op}} \to \mathbf{EA}$, via $(A, \Sigma_X) \mapsto \Sigma_X$
  (This forms an adjunction, via homming into $2 = \{0, 1\}$)
- A probability measure yields a map $\Sigma_X \to [0, 1]$ in **EA**.

Effect modules

Effect modules are effect algebras with a scalar multiplication, with scalars not from $\mathbb{R}$ or $\mathbb{C}$, but from $[0, 1]$.
(Or more generally from an "effect monoid", ie. effect algebra with multiplication)

Definition

An effect module M is a effect algebra with an action $[0, 1] \times M \to M$ that is a "bihomomorphism"

A map of effect modules is a map of effect algebras that commutes with scalar multiplication.

We get a category $\mathbf{EMod} \hookrightarrow \mathbf{EA}$.

Effect modules, main examples

Probabilistic examples

- Fuzzy predicates $[0, 1]^X$ on a set X, with scalar multiplication
  $$r \cdot p \stackrel{\text{def}}{=} \lambda x \in X.\, r \cdot p(x).$$
- Measurable predicates $\mathrm{Hom}(X, [0, 1])$, for a measurable space X, with the same scalar multiplication.

Quantum examples

- Effects $\mathcal{E}(H)$ on a Hilbert space: operators $A\colon H \to H$ satisfying $0 \leq A \leq I$, with scalar multiplication $(r, A) \mapsto rA$.
- Effects in a C^*-algebra A: positive elements below the unit:
  $$[0, 1]_A = \{a \in A \mid 0 \leq a \leq 1\}.$$

This one covers the previous three illustrations.

Full & faithful functors

Since positive unital maps are determined by what they do on positive elements we get full & faitful functors

$$\mathbf{Cstar}_{\mathrm{UP}} \xrightarrow{\text{effects}} \mathbf{EMod} \qquad \mathbf{Cstar}_{\mathrm{UP}} \xrightarrow{\text{self-adjoints}} \mathbf{OUS}$$

Together they form the state-effect diagram:

$$\begin{tikzcd}
\mathbf{BEMod}^{\mathrm{op}} \arrow[r, phantom, "\simeq"] & \mathbf{BOUS}^{\mathrm{op}} \arrow[r, phantom, "\simeq"] & \mathbf{CCH}_{\mathrm{sep}} \\
& (\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}} \arrow[ul, "\text{effects}=\mathrm{Hom}(-,1+1)"] \arrow[u, "\text{self-adjoints}"] \arrow[ur, "\text{states}=\mathrm{Hom}(1,-)"']
\end{tikzcd}$$

Observation about predicates, and sneak-preview

There is a general construction that yields effect algebra/module structure on predicates $X \to 1 + 1$ in a category.

The examples below discuss maps $X \to 1 + 1$ as predicates.

- Obviously in **Sets**
- In $\mathcal{K}\ell(\mathcal{D})$ one gets fuzzy predicates $[0, 1]^X$
- in $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$ on gets effects
- in rings $\mathbf{Rng}^{\mathrm{op}}$ one gets idempotents $e^2 = e$, forming a Boolean algebra — used in the sheaf theory of rings
- in distributive lattices $\mathbf{DLat}^{\mathrm{op}}$ one gets elements with complements, ie. x with x' satisfying $x \wedge x' = 0, x \vee x' = 1$.

Thanks for your attention. Questions/remarks?