Richard V. Kadison (1925)

- Student of Marshall Stone, PhD Univ. Chicago 1956
- Famous ao. for the Kadison-Ringrose books *Fundamentals of the Theory of Operators* (I and II), from 1983 and 1986

**Outline**

**Introduction & overview**
- Preliminaries on double duals of vector spaces
- Order unit spaces, and the easy half of Kadison duality
- Kadison duality, the difficult half

**Own contribution in these lectures**
- No new mathematical results!
- Systematic presentation of results and proofs — in categorical terminology — but without historical details
- Proofs of auxiliary results are scattered around in the literature, and collected here
- Possibly there is some novelty in the different adjoint relationships and resulting monads between:
  - order unit spaces
  - sets, and convex sets, and compact Hausdorff spaces, and convex compact Hausdorff spaces

**Relevant / used literature**
- **Original article**
  - R. Kadison, A representation theory for commutative topological algebra, Memoirs of the AMS, 7, 1951
- **Additional books/papers**
  - Alfsen, Compact Convex Sets and Boundary Integrals, 1971
  - Nagel, Order unit and base norm spaces, in LNP 29, 1974
  - Asimow and Ellis, Convexity Theory and its Applications in Functional Analysis, 1980
  - Alfsen and Shultz, State spaces of operator algebras: basic theory, orientations and *C*-products, 2001
  - Paulsen and Tomforde, Vector Spaces with an order unit, Indiana Univ. Math. Journ., 2009

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**Lectures on Kadison Duality**

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Radboud University Nijmegen

Oxford University, May 20-22, 2014

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**Complete order unit spaces**

\[ \text{complete order unit spaces} \overset{\text{pp}}{\cong} \text{convex compact Hausdorff spaces} \]

- Essentials published in 1951 as Memoirs AMS — one half: representation of order unit spaces
- At the heart of the duality between states and observables/effects/predicates in quantum foundations
- Crying out for a modern, categorical description

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**Preliminaries on double duals of vector spaces**

**Order unit spaces, and the easy half of Kadison duality**

**Kadison duality, the difficult half**

**Appendix on *C*-algebras and effect modules**

**Effect algebras & modules**

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Lectures on Kadison Duality

Basic states-and-effects adjunction (BJ, Ifip TCS’10)

Kadison duality, in relation to $C^*$-algebras

With Mandermaker it was shown [QPL’11] that this adjunction restricts to an equivalence

$$\text{EMod}^{\text{op}} \cong \text{CCH}_{\text{sep}}^{\text{op}}$$

between Banach effect modules and convex compact Hausdorff spaces

The proof used Kadison duality:

$$\text{EMod}^{\text{op}} \cong \text{BOUS}^{\text{op}} \cong \text{CCH}_{\text{sep}}^{\text{op}}$$

Kadison, assumed

proven

Heisenberg predicate transformers

Schrödinger state transformers

where $C^*_\text{UP}$ is the category of $C^*$-algebras with positive unital (UP) maps.

The effects of a $C^*$-algebra $A$ can equivalently be described as:

- $[0,1]_A = \{ a \in A \mid 0 \leq a \leq 1 \}$
- UP-maps $C^2 \rightarrow A$, so maps $A \rightarrow 1 + 1$ in $(C^*_\text{UP})^{\text{op}}$

The states are the maps $A \rightarrow C$

Relevant categories — details follow

- Sets — sets and functions
- Conv — convex sets and affine functions
- $CH$ — compact Hausdorff spaces and continuous maps
- $CCH$ — convex compact Hausdorff spaces and affine continuous maps
  - subcategory $\text{CCH}_{\text{sep}}^{\text{op}} \rightarrow \text{CCH}$ where points can be separated by maps to $[0,1]$
  - equivalently, convex compact subspaces of locally convex space
- $\text{EMod}$ — effect modules
  - subcategory $\text{BEMod} \rightarrow \text{EMod}$ of Banach/complete modules
- $\text{OUS}$ — order unit spaces
  - subcategory $\text{BOUS} \rightarrow \text{OUS}$ of Banach/complete spaces

Frequently dualities arise by “homming into” a dualising object $D$, as in:

- Set-theoretically there is a “double dual” map:
  $$X \xrightarrow{\text{dd}} D(D^X)$$
  given by $\text{dd}(x)(f) = f(x)$

- In order to make this an isomorphism one restricts the maps
  $$X \rightarrow D$$
  and also $D^X \rightarrow D$

This gives a representation of $X$ inside the function spaces.
Weakly open and weakly continuous

**Definition** For $\Omega \subseteq V^*$ the weak topology $\sigma(V, \Omega)$ on $V$ is the weakest topology that makes each $\omega : V \to K$ (where $\omega(v) = \omega_0$) continuous.

Thus, **subbasic opens** of this topology are of the form
$$V^\circ = \{ v \in V \mid \omega(v) \in S \}$$
for $\omega \in \Omega$ and $S \subseteq K$ open.

We write the **continuous dual** as
$$V^* = \{ f : V \to K \mid f \text{ is linear and continuous} \}$$

**Question:** can we characterise maps in $V^\circ$?

The weak-* topology

Recall the double dual map $dd : V \to V^{**}$ with $dd(v)(\omega) = \omega(v)$.

**Definition** The image $dd(V) \subseteq V^{**}$ yields the weak-* topology on $V^{**}$.

Often it is written as $\sigma(V^{**}, V)$ instead of $\sigma(V^{**}, dd(V))$.

It’s subbasic opens are:
$$dd^{-1}(S) = \{ \omega \in V^{**} \mid dd(v)(\omega) = \omega(v) \in S \}$$
for $v \in V$ and $S \subseteq K$ open.

### Examples

- In Stone duality, one proves for a Boolean algebra $B$, $B \cong \text{Cont}(\text{BA}(B, 2), 2)$.
- Here the plan is to prove for an order unit space $V$, $V \cong \text{AffineCont}(\text{PositiveUnit}(V, \mathbb{R}), \mathbb{R})$.

This injection is an isomorphism iff $V$ is complete (Banach).

It is obtained via restriction of the double dual map $V \to V^{**}$.

### Algebraic duals and linear combinations

**Proposition** Let $V \in \text{Vec}_K$. $\omega \in V^*$ is a linear combination
$$\omega = \sum_i k_i \omega_i, \omega_i \in V^*$$
if and only if $\bigcap_i \ker(\omega_i) \subseteq \ker(\omega)$.

**Proof** (only-if) is easy; for (if) use induction on $n$, with base case $k = K$.

**Lemma** If $\ker(\omega) \subseteq \ker(\rho)$ for $\omega, \rho \in V^*$, then $\rho = k \cdot \omega$, for some scalar $k \in K$.

**Proof** If $\omega \neq 0$, use $V/\ker(\omega) \to K$, so quotient $V/\ker(\omega)$ is one-dimensional. Choose base vector $b \in V$ such that $V/\ker(\omega) = K \cdot \{ b + \ker(\omega) \}$. Take $k = \frac{1}{\omega(b)}$. Using $\ker(\omega) \subseteq \ker(\rho)$ we have a well-defined map $\mathbb{P} : V/\ker(\omega) \to K$, namely $\mathbb{P}(v + \ker(\omega)) = \rho(v)$. Pick $v$ and write $v + \ker(\omega) = s \cdot b + \ker(\omega)$, to show $\rho(v) = k \cdot \omega(v)$. □

### Weak topology on a vector space

**Definition** For $\Omega \subseteq V^*$ the **weak topology** $\sigma(V, \Omega)$ on $V$ is the weakest topology that makes each $\omega : V \to K$ (where $\omega(v) = \omega_0$) continuous.

Thus, **subbasic opens** of this topology are of the form
$$V^{\circ} = \{ v \in V \mid \omega(v) \in S \}$$
for $\omega \in \Omega$ and $S \subseteq K$ open.

We write the **continuous dual** as
$$V^* = \{ f : V \to K \mid f \text{ is linear and continuous} \}$$

**Question:** can we characterise maps in $V^\circ$?
For each closed linear subspace \( V \) of \( V^* \), there is an \( f = \dd(\nu) \in V^** \) with \( f \) linear and bounded. For \( r \geq 0 \) write the \( r \)-ball as:
\[
V_r \subseteq V \mid \| v \| \leq r
\]
These two can be combined in:
\[
(V^*)_r \subseteq \{ \omega \in V^* \mid \| \omega \|_{op} \leq 1 \}
\]

We write \( \nu = V \leq K \mid \omega \) is linear and bounded. Each ball \( \nu \in V^* \), with \( \| \nu \|_{op} \leq 1 \), is weakly continuous and each \( \nu \notin S \) there is an \( \omega \in V^* \) with \( \| \omega \| = 1 \).

A normed vector space \( V \) has a norm \( \| - \| : V \to \mathbb{R}_{\geq 0} \) satisfying:

1. \( \| v \| = 0 \) iff \( v = 0 \);
2. \( \| k \cdot v \| = \| k \| \cdot \| v \| \);
3. \( \| v + w \| \leq \| v \| + \| w \| \)

A linear map \( f : V \to W \) is bounded if \( \| f(v) \| \leq s \cdot \| v \| \) for all \( v \in V \).
We write \( V_{\text{Norm}} \) for the resulting category.

The operator norm is defined for such a bounded \( f \) as:
\[
\| f \|_{op} = \bigwedge \{ s \in \mathbb{R}_{>0} \mid \forall v \in V, \| f(v) \| \leq s \cdot \| v \| \}
\]
Each homomorphism \( \text{Hom}(V, W) \) is then itself a normed vector space, with \( \| f \| \leq \| f \|_{op} \).

\[ \| f \|_{op} = \bigwedge \{ s \in \mathbb{R}_{>0} \mid \forall v \in V, \| f(v) \| \leq s \cdot \| v \| \} \]

Each homomorphism \( \text{Hom}(V, W) \) is then itself a normed vector space, with \( \| f \| \leq \| f \|_{op} \).

The double dual in the normed case

**Theorem** For a normed vector space \( V \) over \( \mathbb{R} \):

1. Each ball \( \{ V^* \}_{c,1} \subseteq V^* \) is compact Hausdorff in the weak-* topology — aka. Banach-Alaoglu.
2. \( \dd : V \to V^{**} \) is an isometry: \( \| \dd(\nu) \|_{op} = \| \nu \| \).

**Proof sketch** \( \{ V^* \}_{c,1} \) is a closed subset of the compact space \( D = \prod_{v \in V} [0, \| v \|] \).
We saw \( \| \dd(\nu) \|_{op} = \| \nu \| \). Then \( (2) \) follows because there is an \( \omega \in (V^*)_{c,1} \) with \( \omega(v) = \| v \| \).
Double dual isomorphism theorem

**Theorem** $V \overset{\sim}{\longrightarrow} V^{**}$

That is, for each weakly continuous $\rho: V^* \to K$ there is a unique $v \in V$ with $\rho = \Delta v$.

**Proof** Existence we already know. Uniqueness follows because $\Delta$ is an isometry, and thus injective.

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**Definition** A Banach space is a complete normed vector space: every Cauchy sequence has a limit — using distance $d(v, w) = \|v - w\|$.

We write $\text{Ban}_K \hookrightarrow \text{NVec}_K$ for the full subcategory.

**Proposition** The standard metric completion construction yields a left adjoint:

$$\text{Ban}_K \xleftarrow{\sim} \text{NVec}_K$$

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**Recall:** we have done the left leg, so far

Order on a vector space

**Definition** A vector space $V$ is ordered if it carries a partial order $\leq$ satisfying:

1. $v \leq v'$ implies $v + w \leq v' + w$, for all $w \in V$.
2. $v \leq v'$ implies $r \cdot v \leq r \cdot v'$ for all $r \in \mathbb{R}_+$.

We write $\text{OVect}_\leq \hookrightarrow \text{Vec}_\leq$ for the subcategory of ordered vector spaces with monotone linear maps between them.

(Aside: in a Riesz space there is additionally a join $\vee$)

For a linear map $f: V \to W$ it is equivalent to say:

- $f$ is monotone: $v \leq v'$ implies $f(v) \leq f(v')$
- $f$ is positive: $0 \leq v$ implies $0 \leq f(v)$

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**Unit in an ordered space**

**Definition** A strong unit in an ordered vector space $V$ is a positive element $1 \in V$ with:

- for each $v \in V$ there is an $n \in \mathbb{N}$ with $-n \cdot 1 \leq v \leq n \cdot 1$.

An Archimedean unit is a strong unit with:

- $[vr > 0, v \leq r \cdot 1]$ implies $v \leq 0$

An order unit space is an ordered vector space over $\mathbb{R}$ with an Archimedean unit.

The category $\text{OUS} \hookrightarrow \text{OVect}_\leq$ has as maps that are

- linear, positive/monotone, and unital (preserve 1)
- we call them simple Positive-Unital (UP)

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**Sets**

$\text{OUS} \overset{\sim}{\longrightarrow} \text{CCH}$

Conv $\overset{\sim}{\longrightarrow} \text{AC} \overset{\sim}{\longrightarrow} \text{CCH}$

We shall see that all these functors have (right) adjoints, given by “states”

**Recall** that $\text{Conv}$ is the category of convex sets and affine maps

- in a convex set $X$, finite convex sums $\sum_{i} t_i x_i$ exist, of elements $x_i \in X$ and $t_i \in [0,1]$ with $\sum_{i} t_i = 1$
- affine functions preserve such convex sums
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\[ \ell^\infty(X) = \{ \phi : X \to \mathbb{R} \mid \exists b. \forall x. |\phi(x)| \leq b \} \]

- For a set \( X \), write \( \ell^\infty(X) \) for the bounded maps \( X \to \mathbb{R} \).
- Vector space operations on \( \ell^\infty(X) \) are inherited pointwise from \( \mathbb{R} \).
- The order is also pointwise: \( \phi \leq \phi' \) iff \( \phi(x) \leq \phi'(x) \) for all \( x \).
- The unit 1: \( X \to \mathbb{R} \in \ell^\infty(X) \) is the "constant 1" map.
  - This unit is strong because maps in \( \ell^\infty(X) \) are bounded.
  - It's Archimedean since \( \mathbb{R} \) is Archimedean.
- For a function \( f : X \to Y \) we get \((-) \circ f : \ell^\infty(Y) \to \ell^\infty(X) \).

\[ A^\infty(X) = \{ \phi : X \to \mathbb{R} \mid \phi \text{ is bounded and affine} \} \]

- For an affine set \( X \in \text{Conv} \) we take

- OUS-structure is like in \( \ell^\infty(X) \).
  - affine maps are closed under (pointwise) vector operations.

\[ \mathbf{C} : \mathbf{CH} \to \text{OUS}^{\text{op}} \text{ and } \mathbf{AC} : \mathbf{CCH} \to \text{OUS}^{\text{op}} \]

- For a compact Hausdorff space \( X \in \mathbf{CH} \) take

\[ C(X) = \{ \phi : X \to \mathbb{R} \mid \phi \text{ is continuous} \} \]

Such maps \( \phi \) are automatically bounded.
- OUS-structure is obtained pointwise, like before.

- Similarly, for a convex compact Hausdorff space \( X \),

\[ AC(X) = \{ \phi : X \to \mathbb{R} \mid \phi \text{ is affine continuous} \} \]

**Aside**: The \( \ell^\infty \) and \( C \) constructions yield \( \mathcal{C} \)-algebras, but \( A^\infty \) and \( AC \) do not, in the convex case.
- affine maps are not closed under multiplication

\[ \mathbf{OUSS} \ni \text{AEMod} \]

where \( \text{AEMod} \to \text{EMod} \) is the full subcategory of Archimedean effect modules.

By imposing (suitable) completeness requirements it restricts to:

\[ \mathbf{BOUS} \ni \text{BEMod} \]

as discussed in the introduction (with 'B' for 'Banach').

\[ \text{Pre- and post-composition } (-) \circ g \text{ and } h \circ (-) \text{ are affine.} \]

**Proof**

The key point is that \( f = \sum r_i f_i : V \to W \) is unital again:

\[ f(1) = \sum r_i f_i(1) = \sum r_i 1 = (\sum r_i) \cdot 1 = 1 \cdot 1 = 1. \]

**Lemma**

For \( V, W \in \text{OUS} \) the homset \( \text{Hom}(V, W) \) is convex.

**Corollary**

Taking states yields a functor:

\[ \text{OUS}^{\text{op}} \ni \text{Stat} \ni \text{Conv} \]
Alternative basic opens are, for $\ell_\infty$, the upper-adjoint of $\ell_\infty$, with (Furber & Jacobs, Calco'13):

$$D \cong C_{\text{Cstar}} \cong \mathcal{K} \cong \mathcal{CCH}$$

These adjunctions allow some abstract categorical reasoning.

Proposition $A^\sim \circ D \cong \ell^\sim$, where $D$ is the (discrete probability) distribution monad on $\text{Sets}$

Proof We have $\text{Conv} = \mathcal{E}M(D)$ and so the result follows by composition and uniqueness of adjoints in:

$$\text{OUS} \cong \text{NVect} \cong \text{Sets}$$

Order unit spaces are normed

Definition Each $V \in \text{OUS}$ carries a norm defined by:

$$\|v\|_\text{OUS} = \left\{ r \in \mathbb{R} \geq 0 \mid -r \cdot 1 \leq v \leq r \cdot 1 \right\}.$$

Example The induced norm on $\ell^\sim(X) \in \text{OUS}$ is the supremum (aka. uniform) norm:

$$\|\phi\|_\infty = \sup \{ |\phi(x)| \mid x \in X \}$$

It exists since $\phi$ is bounded. Similarly for $A^\sim(X)$, $C(X)$, $\text{AC}(X)$ (The spaces are actually complete).

Lemma Each map $f$ in $\text{OUS}$ satisfies $\|f\|_\text{op} = 1$. We get a functor:

$$\text{OUS} \cong \text{NVect}$$

In particular, $\text{Stat}(V) \subseteq (V^\#)^{<1}$

Weak-* topology on states

Stat($V$) $\subseteq V^\#$ inherits the weak-* topology, with subbasic opens $\text{dS(v)}^*(S) = \{ \omega \in \text{Stat}(V) \mid \omega(v) \in S \}$, for $v \in V$, $S \subseteq \mathbb{R}$.

Proposition

1. This weak-* topology on $\text{Stat}(V)$ is compact Hausdorff.
2. Alternative basic opens are, for $v \in V$ and $s \in \mathbb{Q}$,

   $$\text{dS}(v) = \{ \omega \in \text{Stat}(V) \mid \omega(v) > s \}$$

3. For each $f : V \to W$ in $\text{OUS}$, the map

   $$(\cdot)^* \circ f : \text{Stat}(W) \to \text{Stat}(V)$$

   is continuous.

Proof $\text{Stat}(V) \subseteq (V^\#)^{<1}$ is a closed subset of a compact Hausdorff space. The rest is easy.

Adjunctions $\text{CH} \cong \text{OUS}^{\text{op}}$ and $\text{CCH} \cong \text{OUS}^{\text{op}}$

Summarising adjunction diagram, with induced monads

Proposition The states functor is also right adjoint in:

$$\text{OUS}^{\text{op}} \cong \text{CH} \cong \text{CCH}$$

As a result, $C \circ U \cong \ell^\sim$ and $\text{AC} \circ E \cong \ell^\sim$ in:

Here $\mathcal{U}$ = ultrafilter, and $E$ = expectation.

Here $\mathcal{R}$ is the Radon monad, with (Furber & Jacobs, Calco'13):

$$\mathcal{K}(\mathcal{R}) \cong (\text{CCstar}^{\text{op}})^{\text{op}}$$

We continue with the lower-right adjunction $\text{OUS}^{\text{op}} \cong \text{CCH}$. 

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Order unit spaces, and the easy half of Kadison duality

Appendix on C*-algebras and effect modules

Radboud University Nijmegen
The easy half of Kadison duality

Theorem The adjunction $OUS^\text{op} \dashv CCH$ restricts to $OUS^\text{op} \dashv CCH_{\text{sep}}$, with separation of points.

The adjunction’s unit, for $X \in CCH_{\text{sep}}$, is the evaluation map

$$X \xrightarrow{ev} \text{Stat}(AC(X))$$

given by $ev(x)(\phi) = \phi(x)$

This unit ev is an isomorphism.

Proof It is easy to see that ev is continuous and affine. It is injective by the separation property: if $ev(x)(\phi) = \phi(x) = ev(y)(\phi)$ for all $\phi \in \text{Stat}(X)$, then $x = y$.

For surjectivity we prove that the image $ev(X) \subseteq \text{Stat}(AC(X))$ is weak-* dense.

Recall that we still need to prove $ev(X) \subseteq \text{Stat}(AC(X))$ is dense.

Assume a non-empty subbasis open of $\text{Stat}(AC(X))$, namely:

$$\square \omega(\phi) = \{\omega \mid \omega(\phi) > s\}$$

for some $\phi \in AC(X)$ and $s \in \mathbb{Q}$. We must show $\square \omega(\phi) \cap ev(X) \neq \emptyset$.

So let $\omega(\phi) > s$, using that $\square \omega(\phi) \neq \emptyset$. By the Lemma, there is an $x \in X$ with $\omega(\phi) = \phi(x) = ev(x)(\phi)$. But then $ev(x) \in \square \omega(\phi)$. □

Proof, continued, denseness

Lemma For each $\phi \in AC(X)$ and for each state $\omega \in \text{Stat}(AC(X))$ there is an $x \in X$ with $\omega(\phi) = \phi(x)$.

Proof The image $\phi(X) \subseteq X$ is compact, and thus closed and bounded, and also convex. Hence it is a closed interval:

$$\phi(X) = [s, \tilde{s}] \subseteq X.$$ Take the middle: $m = \frac{1}{2}(s + \tilde{s}) \in [s, \tilde{s}]$ and radius $r = \frac{1}{2}(\tilde{s} - s)$, so that $\phi(X) = \{t \in X \mid |m - t| \leq r\}$.

The function $\phi - m \cdot 1 \in AC(X)$ has $r$ as upperbound:

$$\|\phi - m \cdot 1\|_B = \max_{x \in X} |\phi(x) - m| \leq r$$

because $\phi(x) \in \phi(X)$. For $\omega \in \text{Stat}(AC(X))$, since $\|\cdot\|_{\text{op}} = 1$,

$$|\omega(\phi) - m| = |\omega(\phi - m \cdot 1)| \leq \|\phi - m \cdot 1\|_B \leq r$$

But then $\omega(\phi) \in \phi(X)$, and thus $\omega(\phi) = \phi(x)$, for some $x \in X$. □

Where do we stand?

- Our aim to prove an equivalence $OUS^\text{op} \cong CCH_{\text{sep}}$
- We have adjoint functors in two directions:

$$OUS^\text{op} \xrightarrow{\text{Hom}(\cdot, X)} \text{CCH} \xrightarrow{\text{ev}} AC$$

- The unit of this adjunction is an isomorphism: for $X \in CCH_{\text{sep}}$,

$$X \xrightarrow{ev} \text{Stat}(AC(X))$$

- The counit is the (restricted) double dual map, for $V \in OUS$,

$$V \xrightarrow{\text{id}} AC(\text{Stat}(V))$$

Next, we will prove two things:

- this id is injective — in fact an isometry
- it is also surjective — via the Krein-Šmulian Theorem

Basic results about order unit spaces

Lemma Let $V$ be an order unit space with, norm $|\cdot|$ and $v \in V$

1. one can write $v = v_+ - v_-$, for positive vectors $v_+, v_- \in V$,
2. $|v| = |v_+| - |v_-| \leq 1$;
3. if $f(1) = 0$ for a positive linear map $f : V \rightarrow W$, then $f = 0$;
4. each positive linear functional $\rho : V \rightarrow R$ is bounded, with $\|\rho\|_{\text{op}} = \rho(1)$;
5. for each non-zero positive linear map $\rho : V \rightarrow R$ there is a unique $r > 0$ such that $\rho r$ is a state, namely $r = \frac{1}{\|\rho\|_{\text{op}}}$;
6. if $-w \leq v \leq w$ then $|v| \leq |w|$.

Proposition Let $V$ be an order unit space.

1. If $S \subseteq V$ is a linear subspace containing $1$, then each state $\omega : S \rightarrow R$ extends to a state $\omega' : V \rightarrow R$ with $\omega'(1) = \omega$
2. Each $\rho \in \gamma(V)$ can be written as $\rho = \rho_+ - \rho_-$, where $\rho_+ , \rho_- \in \gamma(V)$ are positive and $\|\rho_+\|_{\text{op}} + \|\rho_-\|_{\text{op}} = \|\rho\|_{\text{op}}$
3. $\rho \geq 0$ iff $\omega(v) \geq 0$ for all $\omega \in \text{Stat}(V)$
4. If $v \neq v'$, then there is an $\omega \in \text{Stat}(V)$ with $\omega(v) \neq \omega(v')$
5. $\|v\| = \sqrt{|\omega(v)|}$ for $\omega \in \text{Stat}(V)$
The counit of $OUS^{op} \cong CCH_{sep}$ is an isometry.

The “double dual” counit map

$$V \longrightarrow AC(\text{Stat}(V))$$

- is an isometry: $\|\dd(V)\|_{op} = \|V\|$ — and thus injective
- preserves and reflects the order.

Proof of the second point

$$\dd(V) \leq \dd(V') \iff \dd(V)(\omega) \leq \dd(V')(\omega), \text{ for all } \omega$$

since the order is pointwise

$$\iff \omega(V) \leq \omega(V'), \text{ for all } \omega$$

$$\iff V \leq V' \text{ by the previous result} \square$$

Proof, step 2, given $\dd: \text{Stat}(V) \to \mathbb{R}$

Recall that each $\rho \in V^#$ can be written as difference $\rho = \rho_1 - \rho_2$ of positive maps $\rho_1, \rho_2 \geq 0$.

If $\rho_2 \neq 0$, then we can write $\rho_2 = \rho_1\omega_1$ for $\omega_1 \in \text{Stat}(V)$, where

$$\rho_1 = \rho_1(1) \text{ and } \omega_1 = \frac{1}{\rho_1(1)}\rho_2.$$ 

Then define:

$$\dd_\rho(v) = \rho_1\phi(\omega_1) - \rho_2\phi(\omega_2) = m(1)(\phi\left(\frac{1}{\rho_1(1)}\rho_1\right) - \rho_2(1)(\phi\left(\frac{1}{\rho_1(1)}\rho_2\right)),$$

One now proves that this $\dd: V^# \to \mathbb{R}$

- is well-defined, i.e. independent of choice of $\rho_1, \rho_2$
- is linear — and also bounded
- extends $\phi: \text{Stat}(V) \to \mathbb{R}$. 

These verifications use that $\phi$ is affine.

Theorem (Kadison) The counit $\dd: V \to AC(\text{Stat}(V))$ is an isomorphism iff $V$ is complete (Banach).

Proof The (only if) part is easy, since $AC(\text{Stat}(V))$ is complete.

The (if) part requires much more work.

Strategy: extend $\phi \in AC(\text{Stat}(V))$ to a weakly continuous $\dd: V^# \to \mathbb{R}$ in:

$$\Phi\downarrow \longrightarrow \dd(V)^#$$

affine continuous $\phi$ weakly continuous $\dd$

Then use the isomorphism $V \cong V^{op}$.

Finally, Kadison duality

There is an equivalence of categories

$$BOUS^{op} \cong CCH_{sep}$$

Main occurrence: relating observables and states in $C^*$-algebras:

$$BOUS^{op} \cong CCH_{sep}$$

self-adjoints which gives the same picture as $\Phi \in \text{Stat}(V) \times \mathbb{R}$ in the non-deterministic case.

$$\mathbb{C}L_{\Phi} \cong \mathbb{C}L_V$$

Such triangles form a general pattern for predicate and state transformer semantics.
A cousin-duality: Yosida

Recall: A Riesz space is an ordered vector space with binary joins \( V \) — and thus also \( \wedge \).

Write \( \text{BARsz} \to \text{BOUS} \) for the category of "Banach-Archimedean Riesz spaces"

**Theorem (Yosida/Stone)** There is an equivalence of categories:

\[
\text{BARsz} \cong \text{CH} \quad (\text{Stat} = \text{Hom}(\mathbb{R}) )
\]

The following diagram commutes

\[
\begin{array}{ccc}
\text{CH} & \to & \text{BOUS} \\
\uparrow & & \uparrow \\
\text{BARsz} & \cong & \text{CH}
\end{array}
\]

As a result, there is a left adjoint to the forgetful functor in:

\[
\text{BOUS} \leftarrow \text{BARsz}
\]

Some characteristic differences between MIU and UP

**MIU-maps example**

For \( n, m \in \mathbb{N} \), there is a bijective correspondence:

\[
\text{MIU-maps } \mathbb{C}^n \to \mathbb{C}^m \\
\text{functions } m \to n
\]

Essentially, this is the finite-dimensional version of Gelfand duality:

\[
\text{FinSets} \cong (\text{FidCstar}_{\text{MIU}})^{\text{op}}
\]

Maps between \( C^* \)-algebras

- Traditionally, almost always, \( C^* \)-algebras are used with \(*\)-homomorphisms as maps
- However, there are good reasons to consider them with the (weaker) \( \text{unit} \) \( \text{positive} \) (UP) maps
  - probabilistic nature of UP-maps, in the commutative case
  - relation to order unit spaces and effect algebras
- The goal here is to explain some of the differences — and to promote UP-maps a bit
- (I don’t wish to distinguish positive and completely positive maps at this stage)

A linear map \( f: A \to B \) between \( C^* \)-algebras is called:

- **unital** \((\text{U})\), if \( f(1) = 1 \)
  (variation: subunital means \( 0 \leq f(1) \leq 1 \))
- **positive** \((\text{P})\), if \( a \geq 0 \Rightarrow f(a) \geq 0 \)
  (where \( a \geq 0 \) means \( a = x^*x \), for some \( x \))
- **multiplicative** \((\text{M})\), if \( f(a \cdot b) = f(a) \cdot f(b) \)
- **involutive** \((\text{I})\), if \( f(a^*) = f(a)^* \)

**FACTS** \( \text{UP} \Rightarrow \text{I} \) and \( \text{MIU} \Rightarrow \text{UP} \)

We use categories \( \text{Cstar}_{\text{MIU}} \) and \( \text{Cstar}_{\text{UP}} \leftrightarrow \text{Cstar}_{\text{MIU}} \)

MIU-maps are usually called \( \sim\)-homomorphisms; they are the "standard" maps in \( C^* \)-algebra theory

- eg. Gelfand duality: \( \text{CH} \cong (\text{Cstar}_{\text{MIU}})^{\text{op}} \)
UP-maps example, continued

Proof of the correspondence.

- Each \( f : m \to n \) obviously gives \((\circ) \circ f : C^n \to C^m\). It preserves the (pointwise) structure.

- Assume \( \varphi : C^n \to C^m \) is a MIU map. Write the standard base vectors as \( |i⟩ = (0, \ldots, 0, 1, 0, \ldots, 0) \in C^n \).
  Since \( |j⟩:|i⟩ = |i⟩ \), we get \( \varphi(|j⟩) = \varphi(|i⟩) = \varphi(|i⟩) \), so that \( \varphi(|i⟩) = (r_1, \ldots, r_m) \in C^m \) consists of \( r_j \in \{0, 1\} \).
  Since \( \sum_i |i⟩ = 1 \in C^n \), we get \( \sum_i \varphi(|i⟩) = \varphi(1) = 1 \in C^m \), so that \( \sum_j r_j = 1 \), for each \( j \leq m \).
  But then: for each \( j \leq m \) there is precisely one \( i \leq n \) with \( r_j = \varphi_i(1) \). This yields a function \( m \to n \).

For \( n, m \in \mathbb{N} \), there is a bijective correspondence:

\[
\begin{align*}
\text{UP-maps} & \quad C^n \longrightarrow C^m \\
\text{functions} & \quad m \longrightarrow D(n)
\end{align*}
\]

where \( D \) is the distribution monad.

This gives “probabilistic” Gelfand duality, in the finite case:

\[
\mathcal{K}(D) \simeq (\text{FdCCstar}_{\text{UP}})^{op}
\]

where \( \mathcal{K}(D) \to \mathcal{K}(D) \) is the full subcategory with numbers \( n \in \mathbb{N} \) as objects.

Thus, \( \text{FdCCstar}_{\text{UP}} \) is the Lawvere theory of the distribution monad.

UP-maps example, continued

Proof of the correspondence.

- Each \( f : m \to D(n) \) gives a map \( C^m \to C^n \) by:

\[
\varphi \mapsto \lambda y \leq m. \sum_{j \leq n} f_j(j) \cdot \varphi(i)
\]

- Assume \( \varphi : C^n \to C^m \) is a UP map. The base vector \( |i⟩ \in C^n \) is positive, and so \( \varphi(|i⟩) = (r_1, \ldots, r_m) \in C^m \) consists of positive (real) numbers \( r_j \).
  As before, \( \sum_i \varphi(|i⟩) = \varphi(1) = 1 \in C^m \), so for each \( j \leq m \) we have \( \sum_j r_j = 1 \).
  Thus we get the required map \( m \to D(n) \).

Effect algebras, main examples

1. **Projections / closed subspaces** on a Hilbert space form an effect algebra; \( P^\perp \) is orthocomplement:

\[
(x \downarrow y) = 0 \quad \text{for all } x \in P, y \in P^\perp
\]

2. **Orthomodular lattices** are effect algebras, with \( \oplus \) as join \( x \vee y \) only for elements with \( x \perp y \), i.e. \( x \leq y \perp 1 \).

3. Each **Boolean algebra** is an effect algebra: it is a distributive orthomodular lattice, in which \( x \perp y \iff x \wedge y = 0 \).
   In particular, the Boolean algebra of measurable subsets of a measure space forms an effect algebra, where \( U \cup V \) is defined if \( U \cap V = \emptyset \), and is then equal to \( U \cup V \).
Homomorphisms of effect algebras

**Definition**
A homomorphism of effect algebras \( f : X \rightarrow Y \) satisfies:
- \( f(1) = 1 \)
- If \( x \perp x' \) then both \( f(x) \perp f(x') \) and \( f(x \otimes x') = f(x) \otimes f(x') \).
This yields a category \( \text{EA} \) of effect algebras.

**Examples:**
- There is a functor \( \text{Meas}^{op} \rightarrow \text{EA} \), via \( (A, \Sigma_X) \mapsto \Sigma_X \).
  (This forms an adjunction, via homming into \( 2 = \{0, 1\} \)).
- A probability measure yields a map \( \Sigma_X \rightarrow [0,1] \) in \( \text{EA} \).

Effect modules

**Definition**
An effect module \( M \) is an effect algebra with an action \( [0,1] \times M \rightarrow M \) that is a "bihomomorphism".

A map of effect modules is a map of effect algebras that commutes with scalar multiplication.

We get a category \( \text{EMod} \rightarrow \text{EA} \).

Effect modules, main examples

**Probabilistic examples**
- Fuzzy predicates \( [0,1]^X \) on a set \( X \), with scalar multiplication \( r \cdot p \overset{\text{def}}{=} \lambda x \in X. r \cdot p(x) \).
- Measurable predicates \( \text{Hom}(X, [0,1]) \), for a measurable space \( X \), with the same scalar multiplication.

**Quantum examples**
- Effects \( \mathcal{E}(H) \) on a Hilbert space: operators \( A : H \rightarrow H \) satisfying \( 0 \leq A \leq I \), with scalar multiplication \( (r,A) \mapsto rA \).
- Effects in a \( C^* \)-algebra \( A \): positive elements below the unit: \( [0,1]_A = \{ a \in A \mid 0 \leq a \leq 1 \} \).

This one covers the previous three illustrations.

Observation about predicates, and sneak-preview

There is a general construction that yields effect algebra/module structure on predicates \( X \rightarrow 1 + 1 \) in a category.

The examples below discuss maps \( X \rightarrow 1 + 1 \) as predicates.
- Obviously in \( \text{Sets} \)
- In \( KL(\mathcal{P}) \) one gets fuzzy predicates \( [0,1]^X \)
- In \( (C\text{star}^{op})^{op} \) on gets effects
- In rings \( \text{Ring}^{op} \) one gets idempotents \( e^2 = e \), forming a Boolean algebra — used in the sheaf theory of rings
- In distributive lattices \( \text{DLat}^{op} \) one gets elements with complements, ie. \( x \) with \( x' \) satisfying \( x \wedge x' = 0, x \vee x' = 1 \).