Measurable Spaces and their Effect Logic

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Outline

Prerequisites

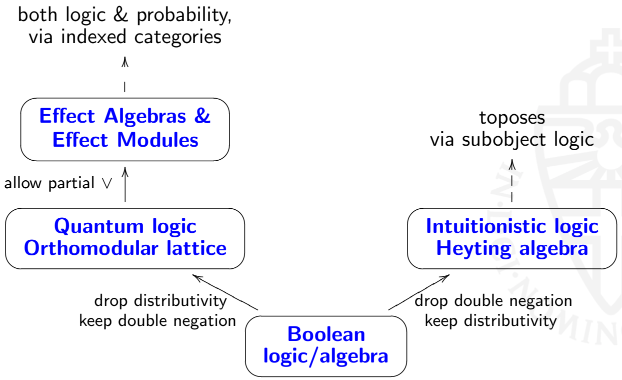
- Basic knowledge of category theory
- Some familiarity with program semantics via monads
  - Eg. via the powerset monad $\mathcal{P}$ for non-determinism
  - Here also the distribution monad $\mathcal{D}$ and the Giry monad $\mathcal{G}$, for discrete and continuous probabilistic computation
- Also, basic knowledge of measure theory & integration

Main point

Algebraic logic for probabilism/quantum uses effect modules

- These structure have been introduced in theoretical physics in the mid-1990s (Foulis, Gudder, ...), for quantum probability
- Here we elaborate the special case of (continuous) probabilistic computation

Where to position "effect modules"



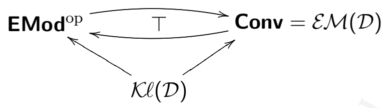
A bird's eye view on non-deterministic computation

- Semantics of non-deterministic program is given by:
  - relations $R \subseteq X \times Y$, or, more categorically:
  - functions $X \to \mathcal{P}(Y)$, ie. maps in the Kleisli category $\mathcal{K}\ell(\mathcal{P})$
- Full & faithful functor "from Kleisli to Eilenberg-Moore"
  - here: $\mathcal{K}\ell(\mathcal{P}) \to \mathcal{EM}(\mathcal{P}) = \mathbf{CL}_\bigvee$
  - where $\mathbf{CL}_\bigvee$ is complete lattices with join-preserving maps
- According to Dijkstra, each program $s\colon X \to \mathcal{P}(Y)$ gives weakest precondition operation $\mathrm{wp}(s)\colon \mathcal{P}(Y) \to \mathcal{P}(X)$
  - Explicitly, $\mathrm{wp}(s)(Q) = \{x \mid s(x) \subseteq Q\}$
  - $\mathrm{wp}(s)$ preserves meets, so is map in $\mathbf{CL}_\bigwedge$
- Summary of non-deterministic "logic and computation"

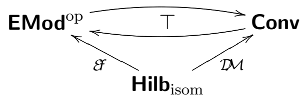
$$\begin{array}{c} X \xrightarrow{\;s\;} \mathcal{P}(Y) \\ \hline\hline \mathcal{P}(X) \xrightarrow{\;\bigvee\;} \mathcal{P}(Y) \\ \hline\hline \mathcal{P}(Y) \xrightarrow[\mathrm{wp}(s)]{\;\bigwedge\;} \mathcal{P}(X) \end{array} \qquad \begin{array}{ccc} (\mathbf{CL}_\bigwedge)^{\mathrm{op}} & \cong & \mathbf{CL}_\bigvee = \mathcal{EM}(\mathcal{P}) \\ & \nwarrow \quad \nearrow & \\ & \mathcal{K}\ell(\mathcal{P}) & \end{array}$$

Similar triangles

- For *discrete* probability:

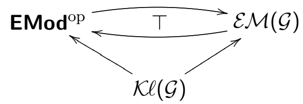


- For quantum, the D'Hondt-Panangaden program logic is captured by:



Technical contribution

There is also such a "states-and-effects" triangle for continuous probability, via the Giry monad $\mathcal{G}$:



This gives categorical reformulation of Kozen's duality.

We need to spend quite some time on the various structures involved.

Effect algebras, definition

Effect algebras generalise the unit interval $[0,1]$ with its (partial!) addition $+$ and "negation" $x \mapsto 1-x$.

A Partial Commutative Monoid (PCM) consists of a set M with zero $0 \in M$ and partial operation $\ovee \colon M \times M \to M$, which is suitably commutative and associative.

One writes $x \perp y$ if $x \ovee y$ is defined.

An effect algebra is a PCM in which each element x has a unique 'orthosuplement' $x^{\perp}$ with $x \ovee x^{\perp} = 1\ (= 0^{\perp})$

Additionally, $x \perp 1 \Rightarrow x = 0$ must hold.

Effect algebras, main examples

1. Projections / closed subspaces on a Hilbert space form an effect algebra; $P^{\perp}$ is orthocomplement:
$$\langle x \mid y \rangle = 0 \qquad \text{for all } x \in P, y \in P^{\perp}$$
2. Orthomodular lattices are effect algebras, with $\ovee$ as join $x \vee y$ only for elements with $x \perp y$, i.e. $x \leq y^{\perp}$
3. Each Boolean algebra is an effect algebra: it is a distributive orthomodular lattice, in which $x \perp y$ iff $x \wedge y = 0$.
   In particular, the Boolean algebra of measurable subsets of a measure space forms an effect algebra, where $U \ovee V$ is defined if $U \cap V = \emptyset$, and is then equal to $U \cup V$.

Effect algebras with countable sums

- Each effect algebra is a partial order, via $x \leq y$ iff $y = x \ovee z$, for some z.
- One speaks of a σ-effect algebra if countable joins (wrt. $\leq$) exist.
- Examples are $[0,1]$, but also measurable subsets $\Sigma_X \subseteq \mathcal{P}(X)$, wrt. a measurable space (X, Σ_X).

Homomorphisms of effect algebras

DEFINITION

A homomorphism of effect algebras $f \colon X \to Y$ satisfies:

- $f(1) = 1$
- if $x \perp x'$ then both $f(x) \perp f(x')$ and $f(x \ovee x') = f(x) \ovee f(x')$.

This yields a category **EA** of effect algebras.

The subcategory σ-**EA** $\hookrightarrow$ **EA** contains σ-algebras with maps also preserving countable joins.

Examples:

- There is a functor **Meas**$^{op} \to \sigma$-**EA**, via $(A, \Sigma_X) \mapsto \Sigma_X$
- A probability measure is a map $\Sigma_X \to [0,1]$ in σ-**EA**.

Effect modules

Effect modules are effect algebras with a scalar multiplication, with scalars not from $\mathbb{R}$ or $\mathbb{C}$, but from $[0, 1]$.

DEFINITION

A (σ-)effect module M is a (σ-)effect algebra with an action $[0, 1] \times M \to M$ that is a "bihomomorphism"

A map of effect modules is a map of effect algebras that commutes with scalar multiplication.

We get a category **EMod**, with subcategory σ-**EMod** $\hookrightarrow$ **EMod**.

Here we always assume "σ" and drop it for convenience.

Effect modules, main examples

Probabilistic examples

- Fuzzy predicates $[0, 1]^X$ on a set X, with scalar multiplication
$$r \cdot p \stackrel{\text{def}}{=} \lambda x \in X.\, r \cdot p(x).$$
- Measurable predicates $\mathrm{Hom}(X, [0, 1])$, for a measurable space X, with the same scalar multiplication.

Quantum examples

- Effects $\mathcal{E}(H)$ on a Hilbert space: operators $A\colon H \to H$ satisfying $0 \le A \le I$, with scalar multiplication $(r, A) \mapsto rA$.
- Effects in a C^*-algebra A: positive elements below the unit:
$$[0, 1]_A = \{a \in A \mid 0 \le a \le 1\}.$$

This one covers the previous three illustrations.

Probability measures and predictes

For a measurable space $X = (X, \Sigma_X)$ we define:

$$\mathcal{G}(X) = \{\phi\colon \Sigma_X \to [0, 1] \mid \phi \text{ is a probability measure}\}$$

$$\begin{aligned} Pred(X) &= \mathbf{Meas}(X, [0, 1]) \\ &= \{p\colon X \to [0, 1] \mid p \text{ is a measurable function}\}. \end{aligned}$$

Functoriality

- $\mathcal{G}(X)$ is a measurable space again, giving a functor $\mathcal{G}\colon \mathbf{Meas} \to \mathbf{Meas}$
- $Pred(X)$ is an effect module, via operations inherited pointwise from $[0, 1]$, giving $Pred\colon \mathbf{Meas} \to \mathbf{EMod}^{\mathrm{op}}$

Lebesgue integration

- Lebesgue integration yields for $\phi \in \mathcal{G}(X)$ and $p \in Pred(X)$,
$$\int p \,\mathrm{d}\phi \in [0, 1]$$
- It is defined, in standard manner:
  - on *indicator functions* as $\int \mathbf{1}_M \,\mathrm{d}\phi = \phi(M)$
  - then extended linearly to step functions $\sum_i s_i \mathbf{1}_{M_i}$
  - and extended to arbitrary predicates via limits

Integration is a map of effect modules

$p \longmapsto \int p \,\mathrm{d}\phi$ is a map $Pred(X) \to [0, 1]$ in **EMod**

Integration in logic and computation

For $f: X \to \mathcal{G}(Y)$ in **Meas** we define:

- **Kleisli extension** $f^{\S}\colon \mathcal{G}(X) \to \mathcal{G}(Y)$ in **Meas**
$$f^{\S}(\phi) = \lambda N \in \Sigma_Y.\, \int \Big(\lambda x.\, f(x)(N)\Big) \,\mathrm{d}\phi$$
- **Substitution** $f^*\colon Pred(Y) \to Pred(X)$ in **EMod**
$$f^*(q) = \lambda x \in X.\, \int q \,\mathrm{d}f(x)$$

Now: Fundamental Galois-style relation between the two

$$\int f^*(q) \,\mathrm{d}\phi = \int q \,\mathrm{d}f^{\$}(\phi)$$

Predicate logic, categorically

- (Giry 1982) $\mathcal{G}$ is a monad
- Taking predicates yields a functor ("indexed category")
$$\mathcal{K}\ell(\mathcal{G}) \xrightarrow{\;Pred\;} \mathbf{EMod}^{\mathrm{op}}$$

On arrows it is given by substitution

Alternatives, for Giry and predicates

Integration yields two isomorphisms:

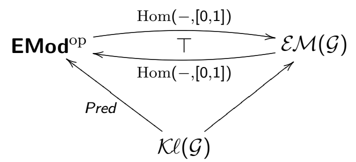
$$\mathcal{G}(X) \;\cong\; \mathbf{EMod}\Big(\mathit{Pred}(X), [0,1]\Big)$$
$$\phi \longmapsto \lambda p.\int p\,\mathrm{d}\phi$$

And similarly:

$$\mathit{Pred}(X) \;\cong\; \mathcal{EM}(\mathcal{G})\Big(\mathcal{G}(X), [0,1]\Big)$$
$$p \longmapsto \lambda\phi.\int p\,\mathrm{d}\phi$$

Note: we are "homming into $[0,1]$". This suggests a duality

The full picture



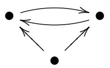
We have recovered Kozen's (1981) version for $\mathcal{G}$ of Dijkstra for $\mathcal{P}$:

$X \longrightarrow \mathcal{G}(Y)$	probabilistic computations as Kleisli maps
$\mathcal{G}(X) \longrightarrow \mathcal{G}(Y)$	computations as algebra maps
$\mathit{Pred}(Y) \longrightarrow \mathit{Pred}(X)$	weakest precondition as effect module maps

Main points

Using effect modules we have a clean redescription of

- Lebesgue integration and the Giry monad
- Probabilistic predicates and substitution
- Kozen's duality, via state-and-effect triangles:



Also, we've put continuous probabilistic computation in this general framework, that also applies to quantum computation