Measurable Spaces and their Effect Logic

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LICS 2013

Introduction & overview
Computational logic

Effect modules and integration
Effect modules
Integration

Harvest time
Conclusions

Outline

Prerequisites

• Basic knowledge of category theory
• Some familiarity with program semantics via monads
  Eg. via the powerset monad $\mathcal{P}$ for non-determinism
  Here also the distribution monad $D$ and the Giry monad $\mathcal{G}$, for
discrete and continuous probabilistic computation
• Also, basic knowledge of measure theory & integration

Main point

Where to position “effect modules”

Both logic & probability, via indexed categories

Effect Algebras & Effect Modules
allow partial $\vee$

Quantum logic
Orthomodular lattice

Intuitionistic logic
Heyting algebra

Boolean logic/algebra

A bird’s eye view on non-deterministic computation

• Semantics of non-deterministic program is given by:
  • relations $R \subseteq X \times Y$, or, more categorically:
    functions $X \to \mathcal{P}(Y)$, ie. maps in the Kleisli category $\mathcal{K}(\mathcal{P})$
  • Full & faithful functor “from Kleisli to Eilenberg-Moore”
    here: $\mathcal{K}(\mathcal{P}) \to \mathcal{E}(\mathcal{M}(\mathcal{P})) = \mathcal{E}\mathcal{M}(\mathcal{P})$
    where $\mathcal{E}\mathcal{M}(\mathcal{P})$ is complete lattices with join-preserving maps
  • According to Dijkstra, each program $s: X \to \mathcal{P}(Y)$ gives
    weakest precondition operation $wp(s): \mathcal{P}(Y) \to \mathcal{P}(X)$
    Explicitly, $wp(s)(Q) = \{ x \mid x(s) \subseteq Q \}$
    $wp(s)$ preserves meets, so is map in $\mathcal{E}\mathcal{M}(\mathcal{P})$
• Summary of non-deterministic “logic and computation”
  $X \xrightarrow{P(Y)} \mathcal{P}(X)$
  $\frac{P(Y)}{\wp(s)} \xrightarrow{\mathcal{E}(\mathcal{M}(\mathcal{P}))}$
  $\xrightarrow{wp(s)} \mathcal{P}(X)$
  $\xrightarrow{\mathcal{K}(\mathcal{P})}$

Effect modules and integration

Algebraic logic for probabilism/quantum uses effect modules

These structure have been introduced in theoretical physics in the
mid-1990s (Foulis, Gudder, . . . ), for quantum probability
Here we elaborate the special case of (continuous) probabilistic
computation
Measurable Spaces and their Effect Logic

Each Boolean algebra is an effect algebra: it is a distributive orthomodular lattice. Projections / closed subspaces on a Hilbert space form an effect algebra, where $\mathcal{M}(\mathcal{D})$ is a distributive orthomodular lattice, in which $x \perp 1$ and $x \perp x = 0$ must hold. Additionally, $x \leq y \Rightarrow x = 0$ must hold.

Effect algebras generalise the unit interval $[0, 1]$ with its (partial) addition $+$ and “negation” $x \mapsto 1 - x$.

A Partial Commutative Monoid (PCM) consists of a set $M$ with zero $0 \in M$ and partial operation $\odot: M \times M \to M$, which is suitably commutative and associative. One writes $x \perp y$ if $x \odot y$ is defined.

An effect algebra is a PCM in which each element $x$ has a unique ‘orthosuplement’ $x^\perp$ with $x \odot x^\perp = 1 (\neq 0^\perp)$. Additionally, $x \perp (1 \Rightarrow x = 0$ must hold.

Effect algebras with countable sums

- Each effect algebra is a partial order, via $x \leq y$ iff $y = x \odot z$, for some $z$.
- One speaks of a $\sigma$-effect algebra if countable joins (wrt. $\leq$) exist.
- Examples are $[0, 1]$, but also measurable subsets $\Sigma X \subseteq \mathcal{P}(X)$, wrt. a measurable space $(X, \Sigma X)$.

Effect algebras with countable sums

- For discrete probability:

  \[
  \text{EMod} \circ \text{Conv} = \mathcal{E}\mathcal{M}(\mathcal{D})
  \]

- For quantum, the D’Hondt-Panangaden program logic is captured by:

  \[
  \text{EMod} \circ \text{Conv} = \mathcal{E}\mathcal{M}(\mathcal{G})
  \]

Homomorphisms of effect algebras

A homomorphism of effect algebras $f: X \to Y$ satisfies:

- $f(1) = 1$
- if $x \perp x'$ then both $f(x) \perp f(x')$ and $f(x \odot x') = f(x) \odot f(x')$.

This yields a category $\text{EA}$ of effect algebras.

The subcategory $\sigma\text{-EA} \hookrightarrow \text{EA}$ contains $\sigma$-algebras with maps also preserving countable joins.

- There is a functor $\text{Meas} \circ \sigma\text{-EA}$, via $(A, \Sigma X) \mapsto \Sigma X$.
- A probability measure is a map $\Sigma X \to [0, 1]$ in $\sigma\text{-EA}$.

Conclusions

We need to spend quite some time on the various structures involved.
Effect modules are effect algebras with a scalar multiplication, with scalars not from \( \mathbb{R} \) or \( \mathbb{C} \), but from \([0,1]\).

**Definition**

A \((\sigma,\alpha)\)-effect module \(M\) is a \((\sigma,\alpha)\)-effect algebra with an action \([0,1] \times M \to M\) that is a "bihomomorphism".

A map of effect modules is a map of effect algebras that commutes with scalar multiplication.

We get a category \(\text{EMod}\), with subcategory \( \sigma\text{-EMod} \to \text{EMod}\). Here we always assume "\(\sigma\)" and drop it for convenience.

### Probability measures and predicates

For a measurable space \((X, \Sigma_X)\) we define:

\[
\mathcal{G}(X) = \{ \phi : \Sigma_X \to [0,1] \mid \phi \text{ is a probability measure} \}
\]

\[
\text{Pred}(X) = \text{Meas}(X, [0,1]) = \{ p : X \to [0,1] \mid p \text{ is a measurable function} \}.
\]

**Functionality**

- \(\mathcal{G}(X)\) is a measurable space again, giving a functor \(\mathcal{G} : \text{Meas} \to \text{Meas}\)
- \(\text{Pred}(X)\) is an effect module, via operations inherited pointwise from \([0,1]\), giving \(\text{Pred} : \text{Meas} \to \text{EMod}^{\text{op}}\)

### Integration in logic and computation

For \(f : X \to \mathcal{G}(Y)\) in \(\text{Meas}\) we define:

- **Kleisli extension** \(f^\sharp : \mathcal{G}(X) \to \mathcal{G}(Y)\) in \(\text{Meas}\)
  \[
  f^\sharp(\phi) = \lambda N \in \Sigma_Y. \int (\lambda x. f(x)(N)) \, d\phi
  \]

- **Substitution** \(f^* : \text{Pred}(Y) \to \text{Pred}(X)\) in \(\text{EMod}\)
  \[
  f^*(q) = \lambda x \in X. \int q \, df(x)
  \]

**Now**: Fundamental Galois-style relation between the two

\[
\int f^*(q) \, d\phi = \int q \, df^\sharp(\phi)
\]
Integration yields two isomorphisms:

\[ G(X) \cong \text{EMod}(\text{Pred}(X), [0, 1]) \]
\[ \phi \mapsto \lambda \mu. \int \mu \, d\phi \]

And similarly:

\[ \text{Pred}(X) \cong \text{EM}(G)(G(X), [0, 1]) \]
\[ p \mapsto \lambda \phi. \int p \, d\phi \]

Note: we are "homming into \([0, 1]\)". This suggests a duality.

Main points

Using effect modules we have a clean redescription of
- Lebesgue integration and the Giry monad
- Probabilistic predicates and substitution
- Kozen's duality, via state-and-effect triangles:

Also, we've put continuous probabilistic computation in this general framework, that also applies to quantum computation.