On Block Structures in Quantum Computation

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Effect algebras, definition

Effect algebras generalise the unit interval $[0,1]$ with (partial) addition.

A Partial Commutative Monoid (PCM) consists of a set $M$ with zero $0 \in M$ and partial operation $\odot : M \times M \to M$, which is suitably commutative and associative.

One writes $x \perp y$ if $x \odot y$ is defined.

An effect algebra is a PCM in which each element $x$ has a unique 'orthosuplement' $x'= 1 - x$.

Additionally, $x \perp 1 \Rightarrow x = 0$ must hold.

For $x \in [0,1]$ the orthosuplement is $x' = 1 - x$.

Homomorphisms of effect algebras

A homomorphism of effect algebras $f : X \to Y$ satisfies:

- $f(1) = 1$
- if $x \perp x'$ then both $f(x) \perp f(x')$ and $f(x \odot x') = f(x) \odot f(x')$.

This yields a category $\mathbf{EA}$ of effect algebras.

A state of an effect algebra $X$ is a homomorphism $X \to [0,1]$.

A state of a measurable space is the same as a (finitely additive) measure.
Effect modules are effect algebras with a scalar multiplication, with scalars not from $\mathbb{R}$ or $\mathbb{C}$, but from $[0, 1]$.

**DEFINITION**
An effect module $M$ is an effect algebra with an action $[0, 1] \times M \to M$ that is a “bihomomorphism.”

A map of effect modules is a map of effect algebras that commutes with scalar multiplication. This yields a category $\mathtt{EMod}$.

Effect modules, main examples

1. Effects $\mathcal{E}(H)$ on a Hilbert space: operators $A : H \to H$ satisfying $0 \leq A \leq 1$, with scalar multiplication $(r, A) \mapsto rA$.
2. Fuzzy predicates $[0, 1]^X$ on a set $X$, with scalar multiplication $r \cdot p \overset{\text{def}}{=} \lambda x \in X \cdot r \cdot p(x)$.
3. Measurable predicates $\mathcal{H}om([0, 1])$, for a measurable space $X$, with the same scalar multiplication.
4. Effects in a C*-algebra $A$: positive elements below the unit: $[0, 1]_A = \{ a \in A \mid 0 \leq a \leq 1 \}$.

This one covers the previous three illustrations.

Effect modules, as indexed category

All the previous examples lead to functors / indexed categories of the form $B \to \mathtt{EMod}^{op}$, used for predicate logic

1. Fuzzy predicates $[0, 1]^{(-)} : \mathcal{K}(D) \to \mathtt{EMod}^{op}$, on the Kleisli category of the distribution monad $D$.
2. Measurable predicates $\mathcal{K}((\mathcal{G})) \to \mathtt{EMod}^{op}$ for the Giry monad $\mathcal{G}$ (see my LIC'13 paper).
3. Hilbert space effects $\mathtt{Hilb}_{\text{num}} \to \mathtt{EMod}^{op}$ using only isometries (dagger monos) as morphisms.
4. C*-algebra effects, $[0, 1]^{(-)} : \mathtt{Cstar}_{\text{num}} \to \mathtt{EMod}^{op}$ using positive unital maps as morphisms.

The predicate logic of effects

- These four indexed categories $B \to \mathtt{EMod}^{op}$ capture the essence of (probabilistic & quantum) predicate logic.
- There is more (dynamical) logical structure, but it is not needed here.
- We will need certain characteristic maps later.

Corresponding idea in quantum programming

- Quantum programs are standardly modelled as completely positive maps (acting on density matrices, or C*-algebras).
- According to Stinespring’s theorem, each such completely positive $S : DM(H) \to DM(H)$ is of the form:
  \[ S(\rho) = \text{tr}_K \left( \rho \otimes \xi \right) U^U \]
- where:
  1. $U$ is a unitary operator on a state space $H \otimes K$ enlarging $H$ with an “ancilla” space $K$.
  2. $\xi$ is an “initial” pure state $|\psi\rangle$ for some vector $|\psi\rangle \in K$.
  3. $\text{tr}_K$ is the partial trace operation, acting as “return”.

- Typical block structure:
  \{int \ v = 0; ...; return\}
- Temporary extension of the state space: opened by initialisation of variables, and closed by a return statement.
First steps towards general approach

**DEFINITION**

A block structure on a category $A$ consists of:

- endofunctors $B_n : A \to A$, for $n > 0$, with natural isos:
  $$B_1(X) \cong X \quad B_m(B_n(X)) \cong B_{m \cdot n}(X),$$

- two collections of natural transformations $i_n, o_n : I \to B_n$ with $o_n \circ i_n = id$, as in:
  $$\text{Example: blocks in } K\ell(D)$$

- Take $B_n(X) = n \cdot X$, with count $\text{out} = \epsilon = \nabla : B_n(X) \to X$

- An obvious \textquote{in} map uses uniform distribution
  $$X \xrightarrow{\text{in}} n \cdot X \quad \text{where} \quad x \mapsto \frac{1}{n} x_1 \cdot \ldots \cdot \frac{1}{n} x_n.

- The equation $\text{out} \circ \text{in} = id$ involves Kleisli composition

- Also power $(-)^n$ forms a block structure on $K\ell(D)$, see paper

**Example: blocks in $K\ell(P) = \text{Rel}$**

- Recall that the category $K\ell(P) = \text{Rel}$
  - is used as model for non-deterministic computation
  - has $+$ has biproduct (both product and coproduct)

- Now copower comonad $n \cdot -$ and power monad $(-)^n$ coincide

- We get a block structure with (co)diagonals:
  $$X \xrightarrow{\text{in}} n \cdot X \quad \text{out} = \nabla

**Example: blocks in $\text{Hilb}$**

- Also the category $\text{Hilb}$ has biproducts: the direct sum $\oplus$ is both a product and coproduct

- Now we need to use a scaling factor for (co)diagonals, as in:
  $$X \xrightarrow{\text{in}} n \cdot X \quad \text{out} = \nabla

- We have $\text{out} \circ \text{in} = id$, with $\text{out} = i_n^\dagger$, making in a dagger mono.

**DEFINITION**

A characteristic maps

- Assume a predicate logic (model) $\text{Pred} : A \to \text{EMod}^{\text{op}}$, with:
  - for a map $f$ in $A$ there is substitution $\text{Pred}(f) = f^{-1}$
  - there is a block structure $B_n : A \to A$

- An $n$-test on $X \in A$ is an $n$-tuple $p = (p_1, \ldots, p_n)$ of predicates $p_i \in \text{Pred}(X)$ with $p_1 \odot \cdots \odot p_n = 1$.

**CONCLUSION**

- for each $X \in A$ and $n > 0$ there is a “universal” $n$-test given by $\Omega_n = B_n(X)$, stable under substitution
- for each $n$-test $p = (p_1, \ldots, p_n)$ on $X$, there is a characteristic map $\text{char}_p : X \to B_n(X)$ in $A$ with $\text{char}_p^\dagger(\Omega_n) = p$.

The $\text{char}_p$ maps open a block, following the test $p$
Non-deterministic computation: $\mathcal{K}(P) = \text{Rel}$

- Formal point: we have a logic $\text{Pred}: \mathcal{K}(P) \to \text{EA}^n$, forming effect modules over $\{0,1\}$ instead of $[0,1]$.
- On objects, we have classical predicates $\text{Pred}(X) = P(X)$. A test in it consists of disjoint subsets $U_i \in P(X)$ with $U_i \cup \cdots \cup U_n = X$.
- We have $\Omega_i = \{x \in X | n \cdot x = B_n(X)$
- For test $U_i \in P(X)$ can be defined by disjointness:

$X \xrightarrow{\text{char}} B_n(X) \quad \text{by} \quad x \xrightarrow{\{\kappa_i x\} \quad \text{if} \quad x \in U_i} \quad \text{if} \quad x \in U_i$.

- We have $\text{char}_U(X) \to B_n(X)$, where $B_n = n \cdot (-)$ is the copower comonad on $\mathcal{K}(P)$
- An obvious question is: when is $\text{char}_U$ an Eilenberg-Moore coalgebra?
- There is a clear answer, via a bijective correspondence:

$\text{Boolean n-tests } U = (U_1, \ldots, U_n) \ \text{in } P(X)$

Eilenberg-Moore coalgebras $X \xrightarrow{\text{B}_n(X)} \mathcal{K}(P)$

- Given a coalgebra $c: X \to n \cdot X$, we get an $n$-test with predicates $U_i = \{x | n \cdot x \in c(x)\}$.

Probabilistic computation example: $\mathcal{K}(D)$

- Logic of fuzzy predicates $[0,1]^n$: $\mathcal{K}(D) \to \text{EMod}^\text{op}$
- an $n$-test $p_i \in [0,1]^X$ satisfies $p_i(x) + \cdots + p_n(x) = 1$.
- Generic predicate $\Omega_i \in [0,1]^n$, with $\Omega_i(n_jx) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
- For $n$-test $p$ on $X$, there is in $\mathcal{K}(D)$ via the convex sum:

$\text{char}_p(x) = p_1(x)\kappa_1 x + \cdots + p_n(x)\kappa_n x$.

It opens a block of options, in a probabilistic manner.

- Again we have the coalgebra question. It works here only for a subset of $n$-tests, the projections, with $p_i^2 = p_i$.
- Then: $p_i(x) \in [0,1] \subseteq [0,1]$, so projections correspond to subsets.
- Thus we have correspondences:

$\text{Boolean n-tests } U = (U_1, \ldots, U_n) \ \text{in } P(X)$

$n$-tests of projections $p = (p_1, \ldots, p_n) \in [0,1]^n$ with $p_i^2 = p_i$

Eilenberg-Moore coalgebras $X \xrightarrow{\text{B}_n(X)} \mathcal{K}(D)$.

Continuous probabilistic computation

Example: computation in Hilbert spaces

- Effect logic: $\mathcal{G}: \text{Hilb}_{\text{num}} \to \text{EMod}^\text{op}$
- $B_n(H) = n \cdot H = H \oplus \cdots \oplus H$ endofunctor on $\text{Hilb}_{\text{num}}$
- with $n = \frac{1}{\sqrt{2}}\Delta$ as it is a dagger mono
- but not with out $= \frac{1}{\sqrt{2}}\nabla$, as it is a dagger epi

Hence this does not give a block structure on $\text{Hilb}_{\text{num}}$.
- But we do have characteristic maps: for an $n$-test $E_i \in \mathcal{G}(H)$ have a dagger mono:

$H \xrightarrow{\text{char}_{E_i} = (\sqrt{1}, \ldots, \sqrt{1})} H \oplus \cdots \oplus H = B_n(H)$
Hilbert spaces, continued

\[ E:E_i = E_i, \text{ so that } E_i \text{ is a projection} \]

\[ E:E_i = 0, \text{ if } i \neq j \]

\[ \text{Coecke} \& \text{Pavlović (2008)} \text{ prove a bijective correspondence:} \]

\[ \text{von Neumann } n\text{-tests } E = (E_1, \ldots, E_n) \text{ in } \mathcal{B}(H) \]

\[ \text{is self-adjoint: } c_i^\dagger = c_i \]

\[ \text{C}^\ast\text{-algebras} \]

- Call an \( n \)-test \( E_i \in \mathcal{B}(H) \) a von Neumann test if:
  - \( E:E_i = E_i \), so that \( E_i \) is a projection
  - \( E:E_i = 0 \), if \( i \neq j \)

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\[ \text{is self-adjoint: } c_i^\dagger = c_i \]

\[ \text{Final remarks} \]

- These categories are most naturally used in reverse form:
  - \( C^\ast\text{-algebras} \) maps are positive and unital
  - \( C^\ast\text{-algebras} \) maps are completely positive

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  - \( C^\ast\text{-algebras} \) maps are completely positive

- There is an effect logic \([0,1]_A: (C^\ast\text{-algebras})^\text{op} \rightarrow \text{EMod}^\text{op} \), and similarly for the ‘CPU’ case.

- There are two logical block structures, via powers and via matrices

\[ \text{Power block structure} \]

- There is a product \( \oplus \) of \( C^\ast\text{-algebras} \)
  - forming a coproduct in \( (C^\ast\text{-algebras})^\text{op} \)
  - yielding a comonad on \( (C^\ast\text{-algebras})^\text{op} \)

- In this opposite category \( (C^\ast\text{-algebras})^\text{op} \) we get

\[ \begin{align*}
  A \xrightarrow{\text{in}_a} A^n \\
  & \quad \text{via } \begin{cases} 
    \text{in}_a(\underbrace{a_1, \ldots, a_n}_{A}) = \frac{a_1 + \cdots + a_n}{n} \\
    \text{out}_a(A) = \Delta(a) = (a, \ldots, a)
  \end{cases}
\end{align*} \]

- For an \( n \)-test of effects \( e_i \in [0,1]_A \) we get in \( (C^\ast\text{-algebras})^\text{op} \)

\[ \begin{align*}
  A \xrightarrow{\text{char}_a} A^n \\
  & \quad \text{where } \text{char}_a(\underbrace{a_1, \ldots, a_n}_{A}) = \sum_i \sqrt{e_i} a_i \sqrt{e_i}
\end{align*} \]

\[ \text{Matrix block structure} \]

- Taking \( n \times n \) matrices forms a functor:

\[ \begin{align*}
  C^\ast\text{-algebras} \xrightarrow{\text{Mat}_n} C^\ast\text{-algebras} \text{\quad via } \begin{cases} 
    \text{in}_a(\underbrace{a_1, \ldots, a_n}_{A}) = \frac{1}{n} \sum_{i=1}^n M_i \\
    \text{out}_a(a) = a_{1,0} = (a \ 0 \ 0 \ 0) \ 0 \ 0 \ 0 \ a)
  \end{cases}
\end{align*} \]

\[ \text{Matrix logical block structure} \]

- the universal \( n \)-test consists of matrices

\[ \Omega_i = |i\rangle\langle i| \in \text{Mat}_n(A) \]

- for an \( n \)-test \( e_i \in [0,1]_A \) a characteristic map

\[ \text{char}_a: A \rightarrow \text{Mat}_n(A) \text{ in } (C^\ast\text{-algebras})^\text{op} \] is given by:

\[ \text{char}_a(M) = (\sqrt{e_1} \ldots \sqrt{e_n}) M (\sqrt{e_1} \ldots \sqrt{e_n})^\dagger \]

\[ \text{Final remarks} \]

- Investigation of block structures, as an abstract programming language construct
  - with "open" and "close" maps
  - opening also via characteristic / measurement maps
  - logic of effect modules is needed

- This structure is present in non-deterministic, probabilistic and quantum computation

- On \( C^\ast\text{-algebras} \): both \text{copower} and \text{matrix} block structures
  - Copower is comonad (has \text{copy}), matrix is \text{not} a comonad
  - precise relationship & usage requires further investigation.