Perspectives on Categorical Quantum Logic

Bart Jacobs

Institute for Computing and Information Sciences – Digital Security
Radboud University Nijmegen

Prakash Fest, Oxford University, May 23, 2014

Outline

Classical, probabilistic & quantum logic

- The aim is to extract the essential properties (and differences) of classical, probabilistic and quantum logic
- The idea is to find out what a "quantum topos" could be
- The logic will be based on effect modules
  - with additional test operators, based on measurement
  - crucially, measurement of predicates can have a side effect
- There is no finished framework yet, but four successive assumptions for a base category of computations
  - a sketch will be given here
  - largely unpublished work

Main examples

- **Sets**, the category of sets and functions
- $\mathcal{K}\ell(\mathcal{D})$, the Kleisli category of the distribution monad $\mathcal{D}$
  - additionally $\mathcal{K}\ell(\mathcal{G})$, for the Giry monad $\mathcal{G}$
- $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$, with variations
  - completely positive maps, W^*-algebras, subunital maps
  - the crucial, but trivial mental steps are:
    - not to use Hilbert spaces, but C^*-algebras
    - to work in the opposite category
    - to use unital positive (UP) maps instead of *-homomorphisms

Aside

Other categories, like $\mathbf{Ring}^{\mathrm{op}}$ or $\mathbf{DistLat}^{\mathrm{op}}$ satisfy some of the assumptions too, and provide additional insight.

Assumption I: basic categorical structure

We have a category B with

- a final object 1, and finite coproducts $(0, +)$
- the following diagrams are pullbacks:

$$\begin{array}{ccc} A+X & \xrightarrow{\mathrm{id}+f} & A+Y \\ {\scriptstyle g+\mathrm{id}}\downarrow & & \downarrow{\scriptstyle g+\mathrm{id}} \\ B+X & \xrightarrow[\mathrm{id}+f]{} & B+Y \end{array} \qquad \begin{array}{ccc} A & = & A \\ {\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\ A+X & \xrightarrow[\mathrm{id}+f]{} & A+Y \end{array}$$

- the following maps are jointly monic:

$$(A+A)+A \underset{[[\kappa_2,\kappa_1],\kappa_2]}{\overset{[\mathrm{id},\kappa_2]}{\rightrightarrows}} A+A$$

(Actually we need this for n-ary coproduct on the left)

Assumption I: definitions

- An n-test is a map $X \to n \cdot 1 = 1 + \cdots + 1$
  - We write $\mathrm{Pred}_n(X) = \mathrm{Hom}(X, n \cdot 1)$
- a predicate is a 2-test, ie. a map $X \to 1 + 1 = 2$
  - notation: $\mathrm{Pred}(X) = \mathrm{Pred}_2(X) = \mathrm{Hom}(X, 2)$
- We get some logical structure for free:

$$1 = (1 \xrightarrow{\kappa_1} 1+1) \quad 0 = (1 \xrightarrow{\kappa_2} 1+1) \quad p^\perp = (X \xrightarrow{p} 1+1 \xrightarrow[\cong]{[\kappa_2,\kappa_1]} 1+1)$$

  Then $p^{\perp\perp} = p$, $0^\perp = 1$, $1^\perp = 0$.
- Predicates $1 \to 1+1$ on 1 will be called scalars
  - they carry a monoid structure $p \cdot q = [p, \kappa_2] \circ q$

Assumption I: predicate examples

- In **Sets**, maps $X \to 1+1 = 2$ correspond to subsets of X
- In the Kleili category $\mathcal{K}\ell(\mathcal{D})$, for a set X,

$$\frac{\dfrac{\text{Kleisli map } X \longrightarrow 2}{\text{function } X \longrightarrow \mathcal{D}(2) = [0,1]}}{\text{fuzzy predicate in } [0,1]^X}$$

- The complex numbers $\mathbb{C}$ are initial in $\mathbf{Cstar}_{\mathrm{UP}}$, so final in $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$. Hence, $1+1 = \mathbb{C} \oplus \mathbb{C} = \mathbb{C}^2$, so:

$$\frac{\dfrac{A \longrightarrow 2 \quad \text{in } (\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}}{\mathbb{C}^2 \longrightarrow A \quad \text{in } \mathbf{Cstar}_{\mathrm{UP}}}}{\text{effect in } [0,1]_A \subseteq A}$$

Assumption I: categorical structure of predicates

Proposition

1. Each $\mathrm{Pred}(X)$ is an effect module over the scalars $\mathrm{Pred}(1)$
2. This yields a functor (or "indexed category")

$$\mathbf{B} \xrightarrow{\ \mathrm{Pred}\ } \mathbf{EMod}^{\mathrm{op}}$$

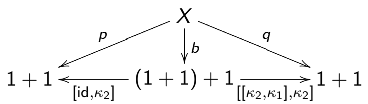
3. This functor preserves $1, 0, +$

$$\mathrm{Pred}(0) = \{0\} \qquad \mathrm{Pred}(X+Y) \cong \mathrm{Pred}(X) \times \mathrm{Pred}(Y)$$

And: the scalars $M = \mathrm{Pred}(1)$ are initial in $\mathbf{EMod}_M$.

Assumption I: partial sums $\otimes$ of predicates

Definition For predicates $p, q \colon X \to 1+1$ define orthogonality $p \perp q$ as: there is a "bound" map $b \colon X \to (1+1)+1$ with:



In that case put $p \otimes q = (\nabla + \mathrm{id}) \circ b \colon X \to (1+1)+1 \to 1+1$.

Lemma There is a bijective correspondence:

$$\frac{\text{predicates } p_1, \ldots, p_n \colon X \longrightarrow 1+1 \text{ with } p_1 \otimes \cdots \otimes p_n = 1}{n\text{-tests } p \colon X \longrightarrow n \cdot 1}$$

Assumption I: states

Definition A state on object X is a map $\omega \colon 1 \to X$.
Write $\mathrm{Stat}(X) = \mathrm{Hom}(1, X)$.
For a predicate $p \colon X \to 1+1$ define the validity probability

$$\omega \models p \quad \stackrel{\text{def}}{=} \quad p \circ \omega \colon 1 \to 1+1$$

Lemma $\mathrm{Stat}(X)$ is a convex sets, closed under convex sums with scalars adding to 1.

Assumption I: states and validity examples

- In **Sets**, states are elements (and predicates subsets), and:

$$x \models p \quad = \quad p(x) \ \in\ \{0,1\}$$

- In $\mathcal{K}\ell(\mathcal{D})$, states are distributions $\varphi \in \mathcal{D}(X)$, and:

$$\varphi \models p \quad = \quad \sum_x \varphi(x) \cdot p(x) \ \in\ [0,1]$$

- In $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$, states are positive unital maps $A \to \mathbb{C}$, and:

$$\omega \models p \quad = \quad \omega(p) \ \in\ [0,1]$$

Assumption I: states, programs, predicates

We read maps in **B** in the following manner

$$\begin{cases} \text{states} & \omega \colon 1 \longrightarrow X \\ \text{programs} & f \colon X \longrightarrow Y \\ \text{predicates} & q \colon Y \to 1+1 \end{cases}$$

Each $f \colon X \to Y$ yields two "transformer" maps:

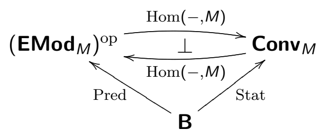
$$\begin{cases} \text{state transformer} & f_* = f \circ (-) \colon \mathrm{Stat}(X) \longrightarrow \mathrm{Stat}(Y) \\ \text{predicate transformer} & f^* = (-) \circ f = \mathrm{wp}(f) \colon \mathrm{Pred}(Y) \longrightarrow \mathrm{Pred}(X) \end{cases}$$

There is the "Galois" equation for the validity probability:

$$(f_*(\omega) \models q) \;=\; (\omega \models f^*(q)) \;=\; (1 \xrightarrow{\omega} X \xrightarrow{f} Y \xrightarrow{q} 1+1).$$

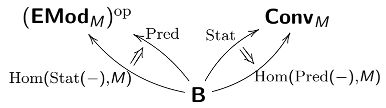
Assumption I: summary

There is a state-and-effect triangle:



where $M = \mathrm{Pred}(1)$

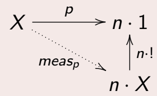
Validity $\models$ yields two natural transformations:



Assumption II: measurement

In addition, to assumption I,

For each n-test $p\colon X \to n \cdot 1$ there is a measurement map $meas_p\colon X \to n \cdot X$ in $\mathbf{B}$ satisfying:



satisfying some "coherence" conditions.

The side-effect of p is the composite:

$$X \xrightarrow{meas_p} n \cdot X \xrightarrow{\nabla} X$$

If this map is the identity, we call p side-effect free.

Assumption II: examples

- An n-test in **Sets** consists of disjoint subsets $P_i \subseteq X$ that cover X, and gives $meas_p\colon X \to n \cdot X$ by:
$$meas_p(x) = \kappa_i x \quad \text{iff} \quad x \in P_i.$$
- An n-test in $\mathcal{K}\ell(\mathcal{D})$ consists of n predicates $p_i\colon X \to [0,1]$ that sum to 1, so we get map $meas_p\colon X \to \mathcal{D}(n \cdot X)$ by:
$$meas_p(x) = p_1(x)\kappa_1 x + \cdots + p_n(x)\kappa_n x$$
- An n-test in a C^*-algebra A consist of effects $e_i \in [0,1]_A$ summing to 1, and gives $meas_{\vec{e}}\colon A \to n \cdot A$ in $(\mathbf{Cstar}_{\mathrm{UP}})^{\mathrm{op}}$, so $meas_p\colon A^n \to A$ in $\mathbf{Cstar}_{\mathrm{UP}}$, with:
$$meas_{\vec{e}}(x_1, \ldots, x_n) = \sqrt{e_1} \cdot x_1 \cdot \sqrt{e_1} + \cdots + \sqrt{e_n} \cdot x_n \cdot \sqrt{e_n}$$

Tests/predicates are side-effect-free in **Sets**, in $\mathcal{K}\ell(\mathcal{D})$, and in *commutative* C^*-algebras.

Assumption II: test maps

- With measurement we define a test map
$$p?[f_1, \ldots, f_n]\colon X \to Y \quad \text{for} \quad \begin{cases} n\text{-test } p\colon X \to n \cdot 1 \\ \text{maps } f_i\colon X \to Y \end{cases}$$
- Explicitly,
$$p?[f_1, \ldots, f_n] = \Big(X \xrightarrow{meas_p} X + \cdots + X \xrightarrow{[f_1,\ldots,f_n]} Y\Big)$$
- In Dijkstra's guarded command style, it can be writte as:

```
begin test
  | p1 → f1
     ⋮
  | pn → fn
end test
```

where predicates $p_1, \ldots, p_n$ with $p_1 \ovee \cdots \ovee p_n = 1$ correspond to n-test p (which may have side-effects)

Assumption II: basic results about side-effect-freeness

Theorem For an effect $e \in [0,1]_A$ in a C^*-algebra A,

$$e \text{ is side-effect-free} \iff e \text{ is in the center } \mathcal{Z}(A)$$

Theorem A C^*-algebra A is commutative iff all its effects are side-effect-free.

Assumption II: test predicates

Definition For predicates $p, q\colon X \to 1+1$, form new predicates:

1. "test p andthen q"
$$\langle p? \rangle(q) = \Big(X \xrightarrow{meas_p} X + X \xrightarrow{[q,0]} 1+1\Big)$$
2. "test p then q"
$$[p?](q) = \Big(X \xrightarrow{meas_p} X + X \xrightarrow{[q,1]} 1+1\Big)$$

Call the model commutative if $\langle p? \rangle(q) = \langle q? \rangle(p)$ for each p, q.
Also, call p a projection of $\langle p? \rangle(p) = p$.

Assumption II: test predicate examples

- In **Sets** we get ordinary conjunction and implication:
  $$\langle p?\rangle(q) = p \cap q \qquad [p?](q) = \neg p \cup q$$
- In $\mathcal{K}\ell(\mathcal{D})$ we get product and Reichenbach implication:
  $$\langle p?\rangle(q)(x) = p(x)\cdot q(x) \qquad [p?](q)(x) = p(x)\cdot q(x) + (1 - p(x))$$
- In C^*-algebras we get Gudder's sequential effect algebra formula:
  $$\langle e?\rangle(d) = \sqrt{e}\cdot d \cdot \sqrt{e} \qquad [e?](d) = \sqrt{e}\cdot d\cdot\sqrt{e} + (1 - e)$$

Assumption II: basic results about test predicates

Lemma

- $\langle 1?\rangle(p) = p = \langle p?\rangle(1)$ and $\langle 0?\rangle(p) = 0 = \langle p?\rangle(0)$
- $\langle p?\rangle(q_1 \owedge q_2) = \langle p?\rangle(q_1) \owedge \langle p?\rangle(q_2)$
- $\langle p?\rangle(s \bullet q) = s \bullet \langle p?\rangle(q)$
- $[p?](q) = \langle p?\rangle(q^\perp)^\perp = \langle p?\rangle(q) \owedge p^\perp$

Lemma (Test map formula). Write $\mathrm{wp}(f) = f^*$ in:
$$\mathrm{wp}(p?[f_1, f_2])(q) = \langle p?\rangle(\mathrm{wp}(f_1)(q)) \owedge \langle p^\perp?\rangle(\mathrm{wp}(f_2)(q)).$$

Note the similarity with the standard rule for if-then-else:
$$\mathrm{wp}(\text{if } p \text{ then } f_1 \text{ else } f_2)(q) = (p \wedge \mathrm{wp}(f_1)(q)) \vee (\neg p \wedge \mathrm{wp}(f_2)(q))$$

Assumption II: pure maps

Definition Call a map $f\colon X \to Y$ pure if it commutes with all measurement maps, as in:
$$\begin{array}{ccc} X & \xrightarrow{\mathit{meas}_{f^*(q)}} & n\cdot X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle n\cdot f} \\ Y & \xrightarrow[\mathit{meas}_q]{} & n \cdot Y \end{array}$$

Such pure f satisfies: $f^*(\langle p?\rangle(q)) = \langle f^*(p)?\rangle(f^*(q))$

Lemma

1. In **Sets** all maps are pure.
2. In $\mathcal{K}\ell(\mathcal{D})$ the maps in the image of $\mathbf{Sets} \to \mathcal{K}\ell(\mathcal{D})$ are pure.
3. In $\mathbf{Cstar}_{\mathrm{UP}}$ all *-homomorphisms (MIU-maps) are pure.

Assumption III: tensor structure

In addition to assumptions I & II,

- the category **B** is symmetric monoidal (has tensors $\otimes$),
- with the final object 1 as tensor unit (giving projections)
- $\otimes$ distributes over coproduct $(+, 0)$
- the monoidal isomorphisms are pure
- and with "coherent measurement maps", as in:
  $$\begin{array}{ccc} X \otimes A & \xrightarrow{\mathit{meas}_p \otimes \mathrm{id}} & (n\cdot X)\otimes A \\ & \searrow{\scriptstyle \mathit{meas}_{\pi_1^*(p)}} & \downarrow\cong \\ & & n\cdot(X \otimes A) \end{array}$$

Assumption III: some consequences

Proposition

1. The object $2 = 1 + 1$ is a commutative monoid — using Eckmann-Hilton style argument
2. Predicates can be paired, via:
   $$p_1 \odot p_2 = \left(X_1 \otimes X_2 \xrightarrow{p_1 \otimes p_2} 2 \otimes 2 \xrightarrow{\;\cdot\;} 2\right)$$
   States can also be paired, via:
   $$\omega_1 \odot \omega_2 = \left(1 \xrightarrow{\cong} 1 \otimes 1 \xrightarrow{\omega_1 \otimes \omega_2} X_1 \otimes X_2\right)$$
3. Then: $(\omega_1 \odot \omega_2 \models p_1 \odot p_2) = (\omega_1 \models p_1)\cdot(\omega_2 \models p_2)$.

More formally, pairing $\odot$ is a bi-homomorphism, both on predicates and on states, and makes the functors $\mathrm{Pred}, \mathrm{Stat}$ (co)monoidal.

Assumption III: projections

- Since the monoidal unit 1 is final, we get a tensor with projections:
  $$X \xleftarrow{\cong} X \otimes 1 \xleftarrow{\mathrm{id}\otimes !} X \otimes Y \xrightarrow{! \otimes \mathrm{id}} 1 \otimes Y \xrightarrow{\cong} Y$$
- Note: there are no diagonals, because of no-cloning
- There are predicate and state transformers for projections, viz. weakening and restriction (aka. **marginal** or **partial trace**)

$$\mathrm{Pred}(X) \xrightarrow{(\pi_1)^*} \mathrm{Pred}(X \otimes Y) \qquad \mathrm{Stat}(X \otimes Y) \xrightarrow{(\pi_1)_*} \mathrm{Stat}(X)$$

Assumption III: dependence and entanglement

In **Sets** there is an isomorphism:

$$\mathrm{Stat}(X \otimes Y) \underset{\odot}{\overset{\langle (\pi_1)_*, (\pi_2)_* \rangle}{\rightleftarrows}} \mathrm{Stat}(X) \times \mathrm{Stat}(Y)$$

But in general, this is only a retraction,

- in $\mathcal{K}\ell(\mathcal{D})$ because a joint distribution may involve dependencies: it need not be the pairing of its marginals
- in C^*-algebras because states may be entangled

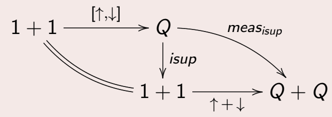
Assumption IV: qubits

In addition to assumptions I–III,

There is a special object $Q \in \mathbf{B}$ with two states and a predicate:

$$1 \xrightarrow{\uparrow} Q \qquad 1 \xrightarrow{\downarrow} Q \qquad Q \xrightarrow{isup} 1+1$$

such that the following diagram commutes,



Superdense coding example

In the superdense coding protocol Alice sends two classical bits to Bob by transferring her part of a shared, entangled quantum state.

In a category with a quantum object Q this is a map $sdc\colon 4 \to 4$ consisting of three consecutive steps:

$$sdc = \left(4 \xrightarrow{init} 4 \otimes Q \otimes Q \xrightarrow{test_A \otimes \mathrm{id}} Q \otimes Q \xrightarrow{test_B} 4\right)$$

One proves that this map is the identity

Superdense coding, as pseudo code

$$
\begin{array}{rl}
sdc(z_1,z_2,z_3,z_4) = & \text{let } v_1, v_2 = \uparrow \text{ in} \\
& \quad \text{let } b_1 = \mathrm{CNOT}(H \otimes \mathrm{id})(v_1 \otimes v_2) \text{ in} \\
& \text{let } t_A = \text{ begin test } (z_1,z_2,z_3,z_4) \\
& \qquad\qquad \mid 1 \longrightarrow b_1 \\
& \qquad\qquad \mid 2 \longrightarrow (X \otimes \mathrm{id})(b_1) \\
& \qquad\qquad \mid 3 \longrightarrow (Z \otimes \mathrm{id})(b_1) \\
& \qquad\qquad \mid 4 \longrightarrow (XZ \otimes \mathrm{id})(b_1) \\
& \qquad\quad \text{end test} \quad \text{in} \\
& \text{let } t_B = \text{ begin test } t_A \\
& \qquad\qquad \mid |b_1\rangle\langle b_1| \longrightarrow 1 \\
& \qquad\qquad \mid |b_2\rangle\langle b_2| \longrightarrow 2 \\
& \qquad\qquad \mid |b_3\rangle\langle b_3| \longrightarrow 3 \\
& \qquad\qquad \mid |b_4\rangle\langle b_4| \longrightarrow 4 \\
& \qquad\quad \text{end test} \quad \text{in} \\
& t_B
\end{array}
$$

Final remarks

- Effect algebras/modules arise naturally
  - not only in examples: fuzzy predicates, effects in C^*-algebras
  - but also from basic categorical structure
- States-and-effect triangles capture basics of program semantics
  - duality between state- and predicate-transformations
- Axiomatisation of (categorical) qantum logic proposed via four assumptions
  - further examples & constructions are needed
- A corresponding calculus of types, terms and formulas will be presented by Robin Adams, at QPL'14 in Kyoto