Introduction & overview

Towards axiomatisation of quantum logic

Assumption I: basic categorical structure

We have a category $B$ with

- a final object $1$, and finite coproducts $(0, +)$
- the following diagrams are pullbacks:

$$
\begin{array}{ccc}
A + X & \rightarrow & A + Y \\
\downarrow{id + f} & & \downarrow{id + id} \\
B + X & \rightarrow & B + Y
\end{array}
$$

- the following maps are jointly monic:

$$
(A + A) \rightarrow [id, a] A + A
$$

(Actually we need this for $n$-ary coproduct on the left)

Assumption IV

$\text{DistLat}^\tau$, for the Giry monad $G$

$\text{Hom}(X, \kappa 2) = \kappa 2(X, \kappa 2)$ with variations

- completely positive maps, $W^\tau$-algebras, subunital maps
- the crucial, but trivial, logical steps are:
  - not to use Hilbert spaces, but $C^\tau$-algebras
  - to work in the opposite category
  - to use unital positive (UP) maps instead of $*$-homomorphisms

Assumption III

Classical, probabilistic & quantum logic

The aim is to extract the essential properties (and differences) of classical, probabilistic and quantum logic

The idea is to find out what a “quantum topos” could be

The logic will be based on effect modules

- with additional test operators, based on measurement
- crucially, measurement of predicates can have a side effect

There is no finished framework yet, but four successive assumptions for a base category of computations

- a sketch will be given here
- largely unpublished work

Main examples

- Sets, the category of sets and functions
- $K(D)$, the Kleisli category of the distribution monad $D$
  - additionally $K(G)$, for the Giry monad $G$
- $(\text{Cstar}(\text{UP}))^\tau$, with variations
  - completely positive maps, $W^\tau$-algebras, subunital maps
  - the crucial, but trivial, logical steps are:
    - not to use Hilbert spaces, but $C^\tau$-algebras
    - to work in the opposite category
    - to use unital positive (UP) maps instead of $*$-homomorphisms

Conclusions

Other categories, like $\text{Ring}^\tau$ or $\text{DistLat}^\tau$ satisfy some of the assumptions too, and provide additional insight.
**Assumption I: predicate examples**

- In **Sets**, maps $X \to 1 + 1 = 2$ correspond to subsets of $X$.
- In the Kleisli category $\mathcal{K}(\mathcal{D})$, for a set $X$,
  
  Kleisli map $X \to 2$:
  
  function $X \to \mathcal{D}(2) = [0, 1]$.
  
  fuzzy predicate in $[0, 1]^X$.
- The complex numbers $\mathbb{C}$ are initial in $\text{Cstar}_\Gamma$, so final in $(\text{Cstar}_\Gamma)^{\text{op}}$. Hence, $1 + 1 = \mathbb{C} \oplus \mathbb{C} = \mathbb{C}_2$, so:
  
  $A \to 2$ in $(\text{Cstar}_\Gamma)^{\text{op}}$.
  
  $\mathbb{C}_2 \to A$ in $\text{Cstar}_\Gamma$.
  
  effect in $[0, 1]_A \subseteq A$.

**Proposition**

1. Each $\text{Pred}(X)$ is an effect module over the scalars $\text{Pred}(1)$.
2. This yields a functor (or "indexed category") $B \to \text{EMod}^{\text{op}}$.
3. This functor preserves $1, 0, +$.

$\text{Pred}(0) = \{0\}$, $\text{Pred}(X + Y) \cong \text{Pred}(X) \times \text{Pred}(Y)$.

And: the scalars $M = \text{Pred}(1)$ are initial in $\text{EMod}_M$.

**Assumption I: partial sums $\otimes$ of predicates**

**Definition** For predicates $p, q : X \to 1 + 1$ define orthogonality $p \perp q$ as: there is a "bound" map $b : X \to (1 + 1) + 1$ with:

$$
\begin{array}{c}
p \circ X \\
1 + 1 \\
\oplus_{i=0,1} (1 + 1) + 1 \\
\oplus_{i=0,1} \oplus_{i=0,1} + 1
\end{array}
\begin{array}{c}
q \\
(1 + 1) + 1 \\
\oplus_{i=0,1} (1 + 1) + 1 \\
\oplus_{i=0,1} \oplus_{i=0,1} + 1
\end{array}
$$

In that case put $p \otimes q = (\land + \vdash) \circ b : X \to (1 + 1) + 1 \to 1 + 1$.

**Lemma** There is a bijective correspondence:

predicates $p_1, \ldots, p_n : X \to 1 + 1$ with $p_1 \otimes \cdots \otimes p_n \equiv 1$

$n$-tests $p : X \to n - 1$

**Assumption I: states**

**Definition** A **state** on object $X$ is a map $\omega : 1 \to X$.

Write $\text{Stat}(X) = \text{Hom}(1, X)$.

For a predicate $p : X \to 1 + 1$ define the validity probability

$$
\omega \models p \overset{\text{def}}{=} p \circ \omega : 1 \to 1 + 1
$$

**Lemma** $\text{Stat}(X)$ is a convex sets, closed under convex sums with scalars adding to 1.

**Assumption I: states and validity examples**

- In **Sets**, states are elements (and predicates subsets), and:
  
  $x \models p = p(x) \in \{0, 1\}$
  
- In $\mathcal{K}(\mathcal{D})$, states are distributions $\varphi \in \mathcal{D}(X)$, and:
  
  $\varphi \models p = \sum_x \varphi(x) \cdot p(x) \in [0, 1]$.
  
- In $(\text{Cstar}_\Gamma)^{\text{op}}$, states are positive unital maps $A \to \mathbb{C}$, and:
  
  $\omega \models p = \omega(p) \in [0, 1]$.

We read maps in $B$ in the following manner:

- states $\omega : 1 \to X$
- programs $f : X \to Y$
- predicates $q : Y \to 1 + 1$

Each $f : X \to Y$ yields two "transformer" maps:

- state transformer $\ell_f = f \circ (\vdash) : \text{Stat}(X) \to \text{Stat}(Y)$
- predicate transformer $\ell' = (\land) \circ f = \text{wp}(f) : \text{Pred}(Y) \to \text{Pred}(X)$

There is the "Galois" equation for the validity probability:

$$
(\ell_f(\omega)) \models q = (\omega \models \ell'(q)) = (1 \land Y) \land 1 + 1).
$$
Assumption I: summary

There is a state-and-effect triangle:

\[(EMod,M)_{PP}^P \xrightarrow{\text{Conv}_M} \text{Hom}(-,M) \xrightarrow{\text{Pred}} B\]

where \(M = \text{Pred}(1)\)

Validity \(\models\) yields two natural transformations:

\[(EMod,M)_{PP}^P \xrightarrow{\text{Hom}(\text{Stat}(-),M)} \text{Pred} \xrightarrow{\text{Stat}} \text{Hom}(\text{Pred}(-),M)\]

Assumption II: test predicates

- An \(n\)-test in \(\text{Sets}\) consists of disjoint subsets \(P_i \subseteq X\) that cover \(X\), and gives \(\text{meas}_p: X \rightarrow n \cdot X\) by:
  \[\text{meas}_p(x) = \kappa_i(x) \quad \text{iff} \quad x \in P_i\]

- An \(n\)-test in \(\mathcal{L}(D)\) consists of \(n\) predicates \(p_i: X \rightarrow \{0,1\}\) that sum to 1, so we get map \(\text{meas}_p: X \rightarrow \mathcal{D}(n \cdot X)\) by:
  \[\text{meas}_p(x) = p_1(x) \kappa_1(x) + \cdots + p_n(x) \kappa_n(x)\]

- An \(n\)-test in a \(C^*\)-algebra \(A\) consists of effects \(e \in [0,1]_A\) summing to 1, and gives \(\text{meas}_p: A \rightarrow n \cdot A\) in \(\text{Cstar}(A)\), so \(\text{meas}_p: A^n \rightarrow A\) in \(\text{Cstar}(A^n)\), with:
  \[\text{meas}_p(x_1,\ldots,x_n) = \sqrt{\nu_1} \cdot x_1 \cdot \sqrt{\nu_2} + \cdots + \sqrt{\nu_n} \cdot x_n \cdot \sqrt{\nu_n}\]

Tests/predicates are side-effect-free in \(\text{Sets}\), in \(\mathcal{L}(D)\), and in commutative \(C^*\)-algebras.

Assumption II: test maps

- With measurement we define a test map
  \[p?[f_1,\ldots,f_n]: X \rightarrow Y\]
  for \(n\)-test \(p: X \rightarrow n \cdot 1\) maps \(f_i: X \rightarrow Y\)

- Explicitly,
  \[p?[f_1,\ldots,f_n] = \left(X \xrightarrow{\text{meas}} X + \cdots + X \xrightarrow{[f_1,\ldots,f_n]} Y\right)\]

- In Dijkstra’s guarded command style, it can be written as:
  \[
  \begin{align*}
  \text{begin test} & \quad \text{where predicates } p_1,\ldots,p_n \\
  | & \quad p_1 \rightarrow f_1 \\
  | & \quad \ldots \\
  | & \quad p_n \rightarrow f_n \\
  \text{end test} & \quad \text{(which may have side-effects)}
  \end{align*}
  \]

Definition

For predicates \(p, q: X \rightarrow 1 + 1\), form new predicates:

- “test \(p.q\) and then \(p\)”
  \[\langle p.q \rangle(p) = \left(X \xrightarrow{\text{meas}} X + X \xrightarrow{[p.q]} 1 + 1\right)\]

- “test \(p\) then \(q\)”
  \[\langle p.q \rangle(p) = \left(X \xrightarrow{\text{meas}} X + X \xrightarrow{[p.q]} 1 + 1\right)\]

Call the model commutative if \(\langle p?q\rangle(p) = \langle q?p\rangle(p)\) for each \(p, q\).

Assumption II: basic results about side-effect-freeness

Theorem

For an effect \(e \in [0,1]_{\text{A}}\) in a \(C^*\)-algebra \(A\),
\[e\] is side-effect-free \(\iff\) \(e\) is in the center \(Z(A)\)

Theorem

A \(C^*\)-algebra \(A\) is commutative iff all its effects are side-effect-free.
• In **Sets** we get ordinary conjunction and implication:
\[
\langle q \wedge q \rangle = q, \quad \langle p \vee q \rangle = \neg p \vee q
\]
• In \(\text{Kl}(D)\) we get product and Reichenbach implication:
\[
\langle q \wedge (x) = p \wedge q \rangle \quad \langle q \circ (x) = p \circ q \rangle
\]
• In \(\mathcal{C}^*\)-algebras we get Gudder’s sequential effect algebra formula:
\[
\langle e \rangle (d) = \sqrt{e} \cdot d \cdot \sqrt{e} \quad \langle e \rangle (d) = \sqrt{e} \cdot d \cdot \sqrt{e} + (1 - e)
\]

**Lemma**
- \(\langle q \rangle = p = \langle p \rangle (1) \) and \(\langle q \rangle = 0 = \langle p \rangle (0)
- \(\langle q \circ (x) = p \circ q \rangle \)
- \(\langle p \circ q \rangle = \langle p \rangle \circ q \)
- \(\langle p \circ q \rangle = \langle p \rangle \circ q \)

**Lemma** (Test map formula). Write \(wp(f) = f^*\) in:
\[
wp(p \circ f)(q) = \langle p \rangle (wp(f)(q)) \circ (p^* - wp(f)(q))
\]

Note the similarity with the standard rule for if-then-else:
\[
wp(p \text{ if then } f \text{ else } g) = (p \land wp(f)(q)) \lor (\neg p \land wp(g)(q))
\]

**Assumption II: pure maps**
- Call a map \(f : X \to Y\) pure if it commutes with all measurement maps, as in:

\[
\begin{array}{ccc}
X & \xrightarrow{\text{meas}_X} & n \cdot X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{\text{meas}_Y} & n \cdot Y \\
\end{array}
\]

Such pure \(f\) satisfies: \(f^*(\langle p \rangle (q)) = \langle f^* p \rangle (f^* q)\)

**Lemma**
1. In **Sets** all maps are pure.
2. In \(\text{Kl}(D)\) the maps in the image of \(\text{Sets} \to \text{Kl}(D)\) are pure.
3. In \(\mathcal{C}^*\)-algebras all *-homomorphisms (MIU-maps) are pure.

**Assumption III: tensor structure**
- The category \(B\) is symmetric monoidal (has tensors \(\otimes\)),
- with the final object 1 as tensor unit (giving projections)
- \(\otimes\) distributes over coproduct (+, 0)
- the monoidal isomorphisms are pure
- and with “coherent measurement maps”, as in:

\[
\begin{array}{c}
X \otimes A \xrightarrow{\text{meas}_X \otimes \text{id}_A} (n \cdot X) \otimes A \\
\downarrow \text{id}_X & & \downarrow \text{id}_X \\
\end{array}
\]

**Proposition**
- The object \(2 = 1 + 1\) is a commutative monoid — using Eckmann-Hilton style argument
- Predicates can be paired, via:
\[
p_1 \otimes p_2 = (X_1 \otimes X_2 \xrightarrow{\text{proj}} 2 \otimes 2 \xrightarrow{\text{proj}} 2)
\]
- States can also be paired, via:
\[
\omega_1 \otimes \omega_2 = \left(1, 1 \xrightarrow{\text{proj}} X_1 \otimes X_2\right)
\]
- Then: \(\{x \otimes \omega_1 : x \vdash p_1 \otimes \omega_2 \} = \{x \mid x \vdash p_1 \} \cdot \{x \mid \omega_2 \} = \{x \mid p_2 \}
\]

More formally, pairing \(\otimes\) is a bi-homomorphism, both on predicates and on states, and makes the functors \(\text{Pred}, \text{Stat}\) (co)monoidal.

**Assumption III: projections**
- Since the monoidal unit 1 is final, we get a tensor with projections:
\[
\begin{array}{ccc}
X & \xrightarrow{\text{proj}} & X \otimes 1 \\
\downarrow \text{id}_X & & \downarrow \text{id}_X \\
1 \otimes Y & \xrightarrow{\text{proj}} & Y \\
\end{array}
\]
- Note: there are no diagonals, because of no-cloning
- There are predicate and state transformers for projections, viz. weakening and restriction (aka. marginal or partial trace)
\[
\begin{array}{ccc}
\text{Pred}(X) & \xrightarrow{\text{proj}} & \text{Pred}(X \otimes Y) \\
\downarrow (\text{proj})^* & & \downarrow (\text{proj}) \\
\text{Stat}(X \otimes Y) & \xrightarrow{\text{proj}} & \text{Stat}(X) \\
\end{array}
\]
Assumption III: dependence and entanglement

In \textbf{Sets} there is an isomorphism:

$$\text{Stat}(X \otimes Y) \cong \text{Stat}(X) \times \text{Stat}(Y)$$

But in general, this is only a retraction,
- in \mathcal{K}(\mathcal{D}) because a joint distribution may involve dependencies: it need not be the pairing of its marginals
- in \mathcal{C}^\ast\text{-algebras} because states may be entangled

Superdense coding example

In the superdense coding protocol Alice sends two classical bits to Bob by transferring her part of a shared, entangled quantum state.

In a category with a quantum object $Q$ this is a map $\text{sdc}: 4 \to 4$ consisting of three consecutive steps:

$$\text{sdc} = (4 \overset{\text{init}}{\rightarrow} 4 \otimes Q \otimes Q \overset{\text{testa} \otimes 1}{\rightarrow} Q \otimes Q \overset{\text{testb}}{\rightarrow} 4)$$

One proves that this map is the identity

Assumption IV: qubits

In addition to assumptions I–III,
There is a special object $Q \in \mathcal{B}$ with two states and a predicate:

$1 \overset{\uparrow}{\rightarrow} Q \overset{1}{\rightarrow} Q \overset{\text{meas}}{\rightarrow} 1 + 1$

such that the following diagram commutes,

$1 + 1 \overset{1 + 1}{\rightarrow} Q \overset{\text{meas}}{\rightarrow} 1 + 1 \overset{1 + 1}{\rightarrow} Q \overset{1}{\rightarrow} Q$

Final remarks

- \textbf{Effect algebras/modules} arise naturally
  - not only in examples: fuzzy predicates, effects in \mathcal{C}^\ast\text{-algebras}
  - but also from basic categorical structure

- \textbf{States-and-effect triangles} capture basics of program semantics
  - duality between state- and predicate-transformations

- Axiomatisation of (categorical) quantum logic proposed via four assumptions
  - further examples \& constructions are needed

- A corresponding \textbf{calculus} of types, terms and formulas will be presented by Robin Adams, at QPL’14 in Kyoto