A Recipe for State-and-Effect Triangles

CALCO'15, Nijmegen

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Outline

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Where we are, sofar

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Semantics of non-deterministic computation I

There are three equivalent ways of describing non-deterministic programs:

$$\begin{array}{c}
X \xrightarrow{\text{program}} \mathcal{P}(Y) \\
\hline\hline
\mathcal{P}(X) \xrightarrow{\text{state transformer}} \mathcal{P}(Y), \quad \bigvee\text{-preserving} \\
\hline\hline
\mathcal{P}(Y) \xrightarrow[\text{predicate transformer}]{} \mathcal{P}(X), \quad \bigwedge\text{-preserving}
\end{array}$$

- Note the reversal of direction in the last case: predicate transformers take post-conditions to pre-conditions
- The bijective correspondence between programs and predicate transformers is sometimes called healthiness

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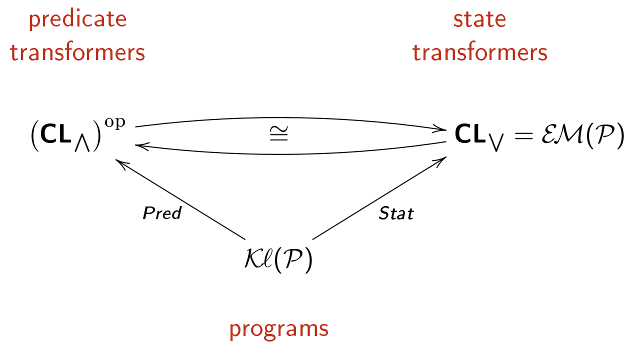
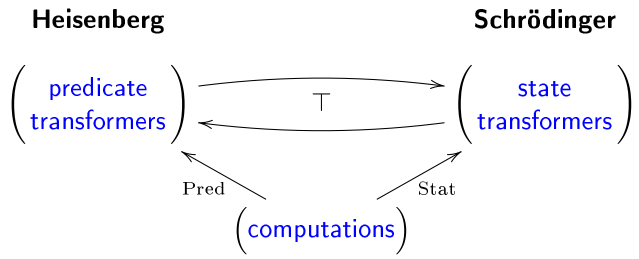
More categorically . . .

There is a "triangle" diagram:



General picture: "state-and-effect triangles"



- This "triangle" view works in many situations, including quantum computation
- It corresponds to the different approaches of Heisenberg (matrix mechanics) and Schrödinger (wave equation, for pure state changes)

This looks too nice . . .

Are these triangles a coincidence, or is there a general construction?

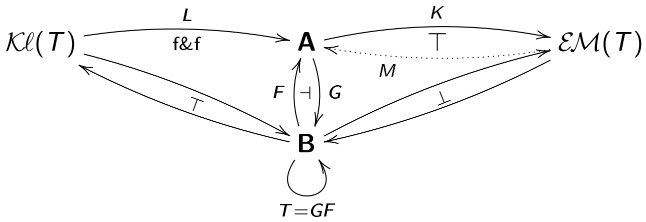
Main point of the paper

- indeed, there is a general construction, covering many monadic examples
- it starts from just an adjunction
- it produces many familiar examples in duality theory
- it gives new descriptions of old monads
- it is based on a standard categorical result, due to John Beck

Where we are, sofar

The main Theorem

Start with an adjunction $F \dashv G$, consider the induced monad $T = GF$, with its Kleisli $\mathcal{K}\ell(T)$ and Eilenberg-Moore categories $\mathcal{EM}(T)$, and draw a diagram:

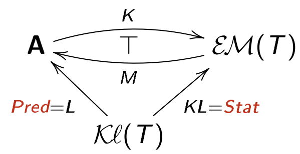


The left adjoint M exists if **A** has coequalisers (of reflexive pairs)

(The monadicity theorem tells when K is an equivalence, but we don't need that)

The triangle recipe

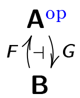
Again, start from adjunction $F \dashv G$, with monad $T = GF$, and turn the top line of the previous diagram into a triangle:



- The adjoint M requires coequalisers in **A**
- The "predicate" and "state" functors are both full and faitful

What we will do next . . .

- Instantiate this "triangle recipe" in many cases
- The adjunction that we start from typically involves an "opposite"



- Hence **A** must have equalisers (to get an adjunction)

Where we are, sofar

Sets and Sets

$$\begin{array}{c} \mathbf{Sets}^{\mathrm{op}} \\ \mathcal{P}{=}\mathrm{Hom}(-,2) \uparrow \dashv \downarrow \mathcal{P}{=}\mathrm{Hom}(-,2) \\ \mathbf{Sets} \\ \circlearrowleft \\ \mathcal{N}{=}\mathcal{P}\mathcal{P} \end{array} \qquad \frac{\dfrac{\mathcal{P}(X) \xrightarrow{\mathbf{Sets}^{\mathrm{op}}} Y}{Y \xrightarrow{\mathbf{Sets}} \mathcal{P}(X)}}{X \xrightarrow[\mathbf{Sets}]{} \mathcal{P}(Y)}$$

- $\mathcal{N}$ is the neighbourhood monad, used in modal logic

$$\begin{array}{ccc} \mathbf{Sets}^{\mathrm{op}} & \overset{\simeq}{\rightleftarrows} & \mathcal{EM}(\mathcal{N}) = \mathbf{CABA} \\ {\scriptstyle Pred} \nwarrow & & \nearrow {\scriptstyle Stat} \\ & \mathcal{K}\ell(\mathcal{N}) & \end{array}$$

Sets and Posets

$$\begin{array}{c} \mathbf{PoSets}^{\mathrm{op}} \\ \mathcal{P}{=}\mathrm{Hom}(-,2) \uparrow \dashv \downarrow \mathrm{Up}{=}\mathrm{Hom}(-,2) \\ \mathbf{Sets} \\ \circlearrowleft \\ \mathcal{M}{=}\mathrm{Up}\mathcal{P} \end{array} \qquad \frac{Y \xrightarrow{\mathbf{PoSets}} \mathcal{P}(X)}{X \xrightarrow[\mathbf{Sets}]{} \mathrm{Up}(Y)}$$

$$\begin{array}{ccc} \mathbf{PoSets}^{\mathrm{op}} & \overset{\simeq}{\rightleftarrows} & \mathcal{EM}(\mathcal{M}) = \mathbf{CDL} \\ {\scriptstyle Pred} \nwarrow & & \nearrow {\scriptstyle Stat} \\ & \mathcal{K}\ell(\mathcal{M}) & \end{array}$$

Full and faithfulnes of the predicate functor gives a correspondence:

$$\frac{X \longrightarrow \mathcal{M}(Y)}{\mathcal{P}(Y) \xrightarrow[\text{monotone}]{} \mathcal{P}(X)}$$

see: Hansen, Kupke, Leal '14

Sets and Meet semilattices (MSLs)

$$\begin{array}{c} \mathbf{MSL}^{\mathrm{op}} \\ \mathcal{P}{=}\mathrm{Hom}(-,2) \uparrow \dashv \downarrow \mathrm{Hom}(-,2) \\ \mathbf{Sets} \\ \circlearrowleft \\ \mathcal{F}{=}\mathbf{MSL}(\mathcal{P}(-),2) \end{array} \qquad \frac{Y \xrightarrow{\mathbf{MSL}} \mathcal{P}(X)}{X \xrightarrow[\mathbf{Sets}]{} \mathbf{MSL}(Y,2)}$$

- $\mathcal{F}$ is the filter monad

$$\begin{array}{ccc} \mathbf{MSL}^{\mathrm{op}} & \overset{\simeq}{\rightleftarrows} & \mathcal{EM}(\mathcal{F}) = \mathbf{CCL} \\ {\scriptstyle Pred} \nwarrow & & \nearrow {\scriptstyle Stat} \\ & \mathcal{K}\ell(\mathcal{F}) & \end{array}$$

Sets and Boolean algebras

$$\begin{array}{c} \mathbf{BA}^{\mathrm{op}} \\ \mathcal{P}{=}\mathrm{Hom}(-,2) \uparrow \dashv \downarrow \mathrm{Hom}(-,2) \\ \mathbf{Sets} \\ \circlearrowleft \\ \mathcal{U}{=}\mathbf{BA}(\mathcal{P}(-),2) \end{array} \qquad \frac{Y \xrightarrow{\mathbf{BA}} \mathcal{P}(X)}{X \xrightarrow[\mathbf{Sets}]{} \mathbf{BA}(Y,2)}$$

- $\mathcal{U}$ is the ultrafilter monad

$$\begin{array}{ccc} \mathbf{BA}^{\mathrm{op}} & \overset{\top}{\rightleftarrows} & \mathcal{EM}(\mathcal{U}) = \mathbf{CompHaus} \\ {\scriptstyle Pred} \nwarrow & & \nearrow {\scriptstyle Stat} \\ & \mathcal{K}\ell(\mathcal{U}) & \end{array}$$

Sets and complete Boolean algebras

The adjunction

$$\begin{array}{c}\mathbf{BA}^{\mathrm{op}}\\ \mathrm{Hom}(-,2)\uparrow\dashv\downarrow\mathrm{Hom}(-,2)\\ \mathbf{Sets}\end{array} \qquad \text{restricts to} \qquad \begin{array}{c}\mathbf{CBA}^{\mathrm{op}}\\ \mathrm{Hom}(-,2)\uparrow\dashv\downarrow\mathrm{Hom}(-,2)\\ \mathbf{Sets}\\ \circlearrowleft\\ \mathbf{CBA}(\mathcal{P}(-),2)\cong\mathrm{id}\end{array}$$

But here we hit a wall, since the induced monad is the identity:

Lemma

For each set X the unit map $\eta\colon X \to \mathbf{CBA}(\mathcal{P}(X), 2)$, given by $\eta(x)(U) = 1$ iff $x \in U$, is an isomorphism.

Where we are, sofar

Boolean vs. probabilistic

- So far we have used functors $\mathrm{Hom}(-, 2)$, for the set $2 = \{0, 1\}$ of Booleans
  - that is, we have been "homming into 2"
- Next, we will be "homming into $[0, 1]$"
  - where $[0, 1] \subseteq \mathbb{R}$ is the unit interval of probabilities
  - we will deal with "quantitative logic"
  - the relevant algebraic structures are effect modules

Sets and effect modules

$$\begin{array}{c}\mathbf{EMod}^{\mathrm{op}}\\ \mathrm{Hom}(-,[0,1])\uparrow\dashv\downarrow\mathrm{Hom}(-,[0,1])\\ \mathbf{Sets}\\ \circlearrowleft\\ \mathcal{E}=\mathbf{EMod}([0,1]^{(-)},[0,1])\end{array} \qquad \begin{array}{c} Y \xrightarrow{\ \mathbf{EMod}\ } [0,1]^X \\ \hline\hline X \xrightarrow[\ \mathbf{Sets}\ ]{} \mathbf{EMod}(Y,[0,1])\end{array}$$

- $\mathcal{E}$ is the expectation monad

$$\begin{array}{ccc}\mathbf{EMod}^{\mathrm{op}} & \overset{\longrightarrow}{\underset{\longleftarrow}{\top}} & \mathcal{EM}(\mathcal{E}) = \mathbf{ConvCompHaus}\\ & {}_{Pred}\nwarrow \quad \nearrow_{Stat} & \\ & \mathcal{K}\ell(\mathcal{E}) & \end{array}$$

Basic facts about the expectation monad $\mathcal{E}$

- It is an "extension" of the finite distribution monad $\mathcal{D}$
  - $\mathcal{E}(X) \cong \mathcal{D}(X)$, for finite sets X
- There is a full and faitful functor to commutative C^*-algebras with positive unital maps:

$$\mathcal{K}\ell(\mathcal{E}) \longrightarrow (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$$

- There is Kadison duality between $\mathcal{EM}(\mathcal{E})$ and "Banach" effect modules (metrically complete)

So what are effect modules?

Intuitively:
- "probabilistic vector spaces", with scalars from $[0, 1]$
- algebraic logic for probabilistic/quantum predicates

Mathematically:

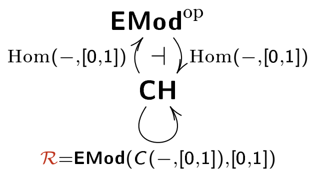
(1) a partial commutative monoid, with partial sum $\ovee$ and 0

(2) an orthocomplement, with $x \ovee x^{\perp} = 1$, where $1 = 0^{\perp}$

(3) a scalar multiplication $s \cdot x$, for $s \in [0, 1]$

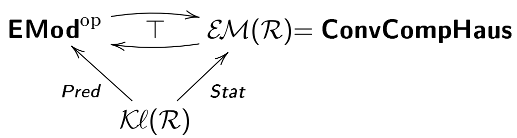
Main examples:
- $[0, 1]$, and more generally, fuzzy predicates $[0, 1]^X$ on a set X
- continuous or measurable functions $X \to [0, 1]$
- effects on a Hilbert space $\mathcal{H}$: $E : \mathcal{H} \to \mathcal{H}$ with $0 \leq E \leq \mathrm{id}$
- predicates in a C^*-algebra A: $a \in A$ with $0 \leq a \leq 1$.

Compact Hausdorff spaces and effect modules



$$\frac{Y \xrightarrow{\ \mathbf{EMod}\ } [0,1]^X}{X \xrightarrow[\ \mathbf{CH}\ ]{} \mathbf{EMod}(Y, [0, 1])}$$

- $\mathcal{R}$ is the Radon monad



Fundamental result, with Robert Furber (CALCO'13)

Theorem

There is an equivalence of categories:

$$\mathcal{K}\ell(\mathcal{R}) \xrightarrow{\ \simeq\ } (\mathbf{CCstar}_{\mathrm{PU}})^{op}$$

Two more variations

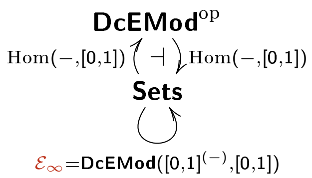
Effect modules are automatically posets; hence we can impose further order completeness conditions, as in the subcategories:

$$\mathbf{DcEMod} \hookrightarrow \omega\text{-}\mathbf{EMod} \hookrightarrow \mathbf{EMod}$$

- ω-**EMod** contains effect modules with joins of ascending ω-chains
- **DcEMod** contains effect modules with joins of all directed subsets

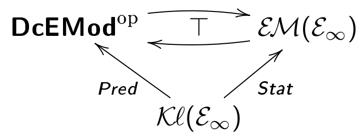
(Maps preserve the relevant structure)

Sets and directed complete effect modules



$$\frac{Y \xrightarrow{\ \mathbf{DcEMod}\ } [0,1]^X}{X \xrightarrow[\ \mathbf{Sets}\ ]{} \mathbf{DcEMod}(Y,[0,1])}$$

- $\mathcal{E}_\infty$ looks like a new monad ...



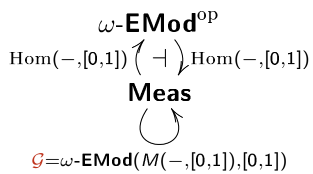
Nothing new, really

Theorem

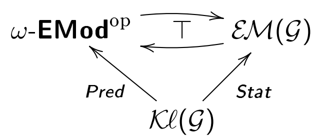
*$\mathcal{E}_\infty \cong \mathcal{D}_\infty$, where $\mathcal{D}_\infty \colon$ **Sets** $\to$ **Sets** is the "infinite distribution" monad:*

$$\begin{aligned}\mathcal{D}_\infty(X) &= \{\phi\colon X \to [0,1] \mid \textstyle\sum_x \phi(x) = 1\}\\ &= \{\phi\colon X \to [0,1] \mid supp(\phi) \text{ is countable, and } \textstyle\sum_x \phi(x) = 1\}.\end{aligned}$$

Measurable spaces and ω-complete effect modules



$$\frac{Y \xrightarrow{\ \omega\text{-}\mathbf{EMod}\ } [0,1]^X}{X \xrightarrow[\ \mathbf{Meas}\ ]{} \omega\text{-}\mathbf{EMod}(Y,[0,1])}$$



The name $\mathcal{G}$ is a give-away

Theorem

The monad $\mathcal{G}$ on measurable spaces defined by:

$$\mathcal{G}(X) = \omega\text{-}\mathbf{EMod}\Big(\mathbf{Meas}(X, [0,1]),\ [0,1]\Big)$$

is (isomorphic to) the Giry monad given by:

$$X \longmapsto \{\phi\colon \Sigma_X \to [0,1] \mid \phi \text{ is a probability distribution}\}$$

For the big picture, including the role of effect modules in Lebesgue integration, see the MFPS'15 paper with Bram Westerbaan.

Where we are, sofar

Main points

- Many "state-and-effect" triangles arise via a basic recipe
- The recipe is obtained by "morphing" Beck's theorem
- "Healthiness" is built-in via bijective correspondences:

$$\frac{\frac{\text{programs}}{\text{state transformers}}}{\text{predicate transformers}}$$

- Many dual adjunctions, equivalences, and monads arise in this manner.