FACULTY OF SCIENCE

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Structuring Computations

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Contents

No explicit message;
some type/object-related
topics that I like;
and you too, hopefully!

I. Sneak preview

Purely functional programs

Writing X for the type of inputs, Y for outputs . . .

. . . a functional program from X to Y is simply a function

$$X \longrightarrow Y$$

Imperative, state-based programs

Writing S for the type of states ...

... an ***imperative*** program is:

$$X \times S \longrightarrow Y \times S$$

Or, equivalently,

$$X \longrightarrow (Y \times S)^S$$

Involving the ***State Monad*** $Y \longmapsto (Y \times S)^S$

Reactive, stream-based programs

A ***reactive*** program is:

$$X^{\mathbb{N}} \longrightarrow Y^{\mathbb{N}}$$

Or, equivalently,

$$X^{\mathbb{N}} \times \mathbb{N} \longrightarrow Y$$

Involving the ***Stream Comonad*** $X \longmapsto X^{\mathbb{N}} \times \mathbb{N}$

Quantum program

A possible ***quantum*** program is:

$$X \times X \longrightarrow \text{CmplxNr}^{(Y \times Y)}$$

It is a "superoperator" on "density matrices" (or quantum states)—after Vizotto, Altenkirch, Sabry

It forms an example of an ***Arrow***: computations with unit and composition.

Overview

- **Functional:** $X \longrightarrow Y$
- **Imperative:** $X \longrightarrow T(Y)$, with T monad (including Java programs)
- **Reactive:** $G(X) \longrightarrow Y$, with G comonad
- **Quantum:** $A(X, Y)$, with A "arrow"

II. Comonads

Comonads for computations

- Monads are well-established in functional programming & language semantics
- But little attention for the dual notion of comonad . . .
- . . . until Uustalu & Vene recently used them for structuring reactive/dataflow programming—building on Brookes & Geva
- **Slogan:** monads structure output, comonads structure input

Comonad structure

- ***Categorically:*** endofunctor $G: \mathbb{C} \to \mathbb{C}$ with two natural transformations $\varepsilon: G \Rightarrow \mathrm{Id}$ and $\delta: G \Rightarrow G^2$ satisfying standard equations
- ***Computationally:*** Type operator G with
  - *coreturn* : $GX \longrightarrow X$
  - *cobind* : $(GX \to Y) \longrightarrow (GX \to GY)$

  satisfying suitable equations
- ***Logically:*** structure for weakening and contraction (like bang ! in linear logic)

Comonad example

- Mapping $X \longmapsto X^{\mathbb{N}} \times \mathbb{N}$
- Input streams with past / current / future:

$$x_0, x_1, \ldots, x_{n-1}, \boxed{x_n}, x_{n+1}, x_{n+2}, \ldots$$

- Counit / coreturn: $X^{\mathbb{N}} \times \mathbb{N} \longrightarrow X$

$$(\alpha, n) \longmapsto \alpha(n)$$

- Delta: $X^{\mathbb{N}} \times \mathbb{N} \longrightarrow (X^{\mathbb{N}} \times \mathbb{N})^{\mathbb{N}} \times \mathbb{N}$

$$(\alpha, n) \longmapsto (\lambda m{:}\,\mathbb{N}.\, (\alpha, m), n)$$

coKleisli category of computations

- coKleisli maps $X^{\mathbb{N}} \times \mathbb{N} \longrightarrow Y$ form a category
- Identity via coreturn; composition via delta/cobind
- Gives output in Y for completely given input stream of X's
- Basis for dataflow calculus by Uustalu & Vene
  (like in Lustre, Lucid)

Discrete time signals

Three basic comonads:

$$X^{\star} \times X \xleftarrow[\text{no future}]{\text{causality}} X^{\mathbb{N}} \times \mathbb{N} \xrightarrow[\text{no past}]{\text{anti-causality}} X^{\mathbb{N}}$$

$$(\langle \alpha(0), \ldots, \alpha(n-1) \rangle, \alpha(n)) \longleftarrow\!\shortmid (\alpha, n) \longmapsto \lambda m.\, \alpha(n+m)$$

with "comonad homomorphisms" between them

Continuous time signals

Analogues fundamental diagram of comonads:

$$\coprod_{t \in [0,\infty)} X^{[0,t)} \times X \longleftarrow X^{[0,\infty)} \times [0,\infty) \longrightarrow X^{[0,\infty)}$$

where:

$$\coprod_{t \in [0,\infty)} X^{[0,t)} \times X \;\cong\; \coprod_{t \in [0,\infty)} X^{[0,t]} \;\cong\; X^{[0,1]} \times [0,\infty)$$

III. Arrows

Arrow overview

- Introduced in Haskell by Hughes in 2000, as common interface extending monads (parser as main example)
- Binary type operation $A(-,+)$ with three operations: arr, $\ggg$, first.
- Folklore claim: Arrows are Freyd categories (Power & Robinson'99)
- Recently substantiated by first describing arrows as ***monoids*** in a category of bifunctors $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to$ **Sets**

Arrow in Haskell

Introduced as type class:

```
class Arrow A where
  arr :: (X → Y) → A X Y
  (>>>) :: A X Y → A Y Z → A X Z
  first :: A X Y → A (X, Z) (Y, Z)
```

Which should satisfy 8 equations, such as:

$$\begin{aligned} (a \ggg b) \ggg c &= a \ggg (b \ggg c) \\ a \ggg \mathsf{arr}(1) &= a \\ \mathsf{first}(\mathsf{arr}(f)) &= \mathsf{arr}(f \times 1), \quad \textit{etc} \end{aligned}$$

Arrow examples

- $(X, Y) \longmapsto (X \to T(Y))$, for T monad
  $(X, Y) \longmapsto (G(X) \to Y)$, for G comonad
- $(X, Y) \longmapsto (X \times X \to \mathsf{CmplxNr}^{(Y \times Y)})$ for quantum computation
- $(X, Y) \longmapsto (X^{\mathbb{N}} \to \mathcal{P}(Y^{\mathbb{N}}))$ for "non-deterministic dataflow"
- $$(X, Y) \longmapsto \; (2 \times S^{\star}) \times ((S^{\star} \times X) \to (1 + (S^{\star} \times Y)))$$
  for Swierstra-Duponcheel parser that motivated Hughes

Arrows, categorically

- A is functorial: for $f \colon X' \to X$ and $g \colon Y \to Y'$,

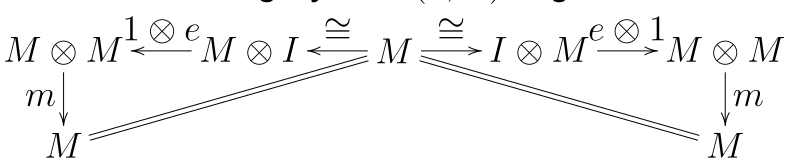
$$A(X, Y) \xrightarrow{\;A(f,g)\;} A(X', Y')$$

$$a \longmapsto \mathsf{arr}(f) \ggg a \ggg \mathsf{arr}(g)$$

- arr: $(+)^{(-)} \to A(-,+)$ is natural transformation (natro, for short)
- $\ggg$ is natro $A \otimes A \to A$, for tensor product of distributors / profunctors
- first corresponds to "internal strength"

Excurs: monoid in a category

- Standardly, a monoid is a set M with associative $m\colon M \times M \to M$ and two-sided unit $e\colon 1 \to M$
- Can be formulated in category with finite products $(1, \times)$: equations become diagrams
- No projections/diagonals needed: also in monoidal category with $(I, \otimes)$. Eg.

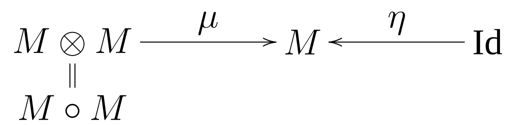


Excurs: monads are monoids

- The functor category $\mathbb{C}^{\mathbb{C}}$ is monoidal:

$$F \otimes G \;=\; F \circ G \qquad\qquad I \;=\; \mathrm{Id}$$

- A monoid in $\mathbb{C}^{\mathbb{C}}$ is a functor $M\colon \mathbb{C} \to \mathbb{C}$ with natros:



  satisfying the monoid equations
- A monoid in $\mathbb{C}^{\mathbb{C}}$ is precisely a monad!

Arrows are also monoids

- Arrows are monoids in category of bifunctors $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to$ **Sets**
- Tensor $\otimes$ more complicated, with exponentiation/hom as unit
- Allows for precise comparison with Freyd categories
  (bijective correspondence)
- Details in Heunen & Jacobs, MFPS'06.

Arrows, intuitively

- Most fundamental mathematical structure in computing?
- Monoid $(A, ;, \mathrm{skip})$ of programs/actions $A \in$ **Sets** with sequential composition
- Adding input and output makes $A(-, +)$ binary operator
- Hence carrier A becomes bifunctor $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to$ **Sets**
- Keeping the monoid structure leads to Hughes' **Arrow**

IV. Monads

Monad overview

- Introduced by Moggi (1991), popularised in functional programming by Wadler
- for structuring outputs / computational effects
- Standard examples:
  - lift / maybe $1 + (-)$
  - exception $E + (-)$
  - list $(-)^{\star}$
  - state $(- \times S)^S$
  - non-determinism $\mathcal{P}$ (powerset)
  - probability $\mathcal{D}$ (distribution)

Java monad

- Definition [Jacobs & Poll'03]:

$$J(X) \;=\; (1 + S \times X + S \times E)^S$$

- Combination of state, lift, exception monad
- Actual "abnormal" termination in Java more complicated: exceptions, return, break, continue
- Exception mechanism (plus logic) axiomatised as equaliser by [Schröder & Mossakowski]

Kleisli composition for Java monad

- Kleisli composition for J is "argument evaluation, before use" (and not sequential composition ;)
- For $a \colon X \to J(Y)$, and $p \colon Y \to J(Z)$,

$$p \bullet a \;=\; \lambda x \colon X.\, \lambda s \colon S.$$

CASES $a\, x\, s$ OF

$*$	$\longmapsto$	$*$	// non-termination
(s', y)	$\longmapsto$	$p\, y\, s'$	// normal termination
(s', e)	$\longmapsto$	(s', e)	// except. termination

V. Java program verification (at Nijmegen)

Developments

- **Original focus:** theorem proving for small Java programs (for smart cards)
- **Outcome:**
  - No scaling beyond couple of pages
  - Practical experience, formalisations & deeper theory
- **Shift of focus:**
  - Extension to security properties (esp. confidentiality)
  - Static checking primary, theorem proving secondary

JML: Java Modeling Language

JML [Leavens et al.] adds specifications as special comments in Java code, mainly for:

- Class invariants and constraints
- Method specifications:

```
/*@ behavior
  @ requires    <precondition>
  @ assignable  <items that may be modified>
  @ diverges    <precondition for non-termination>
  @ ensures     <postcond for normal termination>
  @ signals     <postcond for exceptional
  @              termination>
  @*/
void method() { ...  }
```


JML: example

JML method specifications may clarify the behaviour of Java methods:

```
/*@ normal_behavior
  @ requires     x >= 0;
  @ assignable   \nothing;
  @ ensures      \result * \result <= x &&
  @              x < (\result+1) * (\result+1);
  @*/
int f(int x) {
  int count = 0, sum = 1;
  while (sum <= x) {
    count++;
    sum += 2 * count + 1;
  }
  return count;
}
```

LOOP project

- LOOP tool: compiles Java+JML to PVS
- Based on formalised semantics of Java+JML in PVS
- Including Hoare logic (see later) & WP-reasoner (all with provably sound rules)
- Used for several non-trivial case studies, but now in "sleep mode"
- Static checking is simply more effective; theorem proving best for difficult left-overs.

VI. Static Checking for Java

ESC/Java and ESC/Java2

Extended static checker: original ESC/Java by Leino et. al at Compaq, but no longer supported.

- *tries* to *prove* correctness of specifications, at compile-time, fully automatically
- *not sound*, *not complete*, but finds lots of potential bugs quickly
- Original ESC/Java only supports a (not fully compatible) subset of full JML
- New ESC/Java2 is open source, compatible and handles more (eg. **`assignable`** clauses).

ESC/Java "demo"

```
class Bag {
  int[] a;
  int   n;
  int extractMin() {
   int m = Integer.MAX_VALUE;
   int mindex = 0;
   for (int i = 1; i <= n; i++) {
        if (a[i] < m) { mindex = i; m = a[i]; } }
   n--;
   a[mindex] = a[n];
   return m;
 }
```

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   return m;
 }
```

Warning: possible null deference. Plus other warnings

ESC/Java "demo"

```
class Bag {
  int[] a;  //@ invariant a != null;
  int   n;
  int extractMin() {
   int m = Integer.MAX_VALUE;
   int mindex = 0;
   for (int i = 1; i <= n; i++) {
        if (a[i] < m) { mindex = i; m = a[i]; } }
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   n--;
   a[mindex] = a[n];
   return m;
 }
```

Warning: Array index possibly too large

ESC/Java "demo"

```
class Bag {
  int[] a;  //@ invariant a != null;
  int   n;  //@ invariant 0 <= n && n <= a.length;
  int extractMin() {
   int m = Integer.MAX_VALUE;
   int mindex = 0;
   for (int i = 1; i <= n; i++) {
        if (a[i] < m) { mindex = i; m = a[i]; } }
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        if (a[i] < m) { mindex = i; m = a[i]; } }
   n--;
   a[mindex] = a[n];
   return m;
 }
```

Warning: Possible negative array index

ESC/Java "demo"

```
class Bag {
  int[] a;  //@ invariant a != null;
  int   n;  //@ invariant 0 <= n && n <= a.length;
  //@ requires n > 0;
  int extractMin() {
   int m = Integer.MAX_VALUE;
   int mindex = 0;
   for (int i = 0; i < n; i++) {
        if (a[i] < m) { mindex = i; m = a[i]; } }
   n--;
   a[mindex] = a[n];
   return m;
 }
```

ESC/Java “demo”

```
class Bag {
  int[] a;  //@ invariant a != null;
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   int mindex = 0;
   for (int i = 0; i < n; i++) {
        if (a[i] < m) { mindex = i; m = a[i]; } }
   n--;
   a[mindex] = a[n];
   return m;
 }
```

No more warnings about this code

ESC/Java “demo”

```
class Bag {
  int[] a;  //@ invariant a != null;
  int   n;  //@ invariant 0 <= n && n <= a.length;
  //@ requires n > 0;
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   int mindex = 0;
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        if (a[i] < m) { mindex = i; m = a[i]; } }
   n--;
   a[mindex] = a[n];
   return m;
 }
```


. . . but warnings about calls to **`extractMin()`** that do not ensure precondition : design by contract

VII. Hoare logic for JML

Hoare logic issues for Java & JML

- Complications in Hoare logic for Java:
  - exceptions and other abrupt control flow
  - expressions may have side effects
- Thus:
  - not Hoare *triples* but Hoare *n-tuples*,
  - both for statements & expressions

Hoare Logic assertions

For $\{\,\textit{Pre}\,\}\, m\, \{\,\textit{Post}\,\}$ write

$$\left(\begin{array}{rcl} \text{requires} & = & \textit{Pre} \\ \text{statement} & = & m \\ \text{ensures} & = & \textit{Post} \end{array}\right)$$

For JML one needs:
$$\left(\begin{array}{rcl} \text{diverges} & = & D \\ \text{requires} & = & \textit{Pre} \\ \text{statement} & = & m \\ \text{ensures} & = & \textit{Post} \\ \text{signals} & = & S \end{array}\right)$$

Hoare composition Rule

$$\frac{\left(\begin{array}{rcl} \text{diverges} & = & \lambda x.\, b \\ \text{requires} & = & \textit{Pre} \\ \text{statement} & = & s_1 \\ \text{ensures} & = & Q \\ \text{signals} & = & S \end{array}\right) \quad \left(\begin{array}{rcl} \text{diverges} & = & \lambda x.\, b \\ \text{requires} & = & Q \\ \text{statement} & = & s_2 \\ \text{ensures} & = & \textit{Post} \\ \text{signals} & = & S \end{array}\right)}{\left(\begin{array}{rcl} \text{diverges} & = & \lambda x.\, b \\ \text{requires} & = & \textit{Pre} \\ \text{statement} & = & s_1 \,;\, s_2 \\ \text{ensures} & = & \textit{Post} \\ \text{signals} & = & S \end{array}\right)}$$

Intermediate predicate provided by the user in JML

Use of the Hoare logic

- Actual use seems clumsy, but PVS takes care of the bookkeeping
- This logic forms basis for semantics of JML

VIII. Conclusions

Main points

- There is mathematical uniformity & elegance in the structure of computation
- Main notions: monad / comonad / arrow
- This elegance is not completely lost in concrete languages / systems
- For our Java work: practice preceded theory
- Theorem proving cannot beat static checking in program verification

Thanks for your attention!