IPA Course on Formal Methods

An introduction to theorem proving using PVS

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Today

What is a theorem prover and why would you use one?

A first taste of using a theorem prover

Rough plan for the day:

- **Introduction to PVS: specification language + prover**
- **Exercise 1**
- **Lunch**
- **A typical computer science application:**
  state machines in PVS
- **Exercise 2**
- **Demo: use of *automated* theorem proving for program verifcation (ESC/Java2)**

What is a theorem prover?

A theorem prover is a tool for logical reasoning, like a calculator is a tool for arithmetic.

Theorem provers are *not* as mature or widely used (yet?) as calculators; it takes considerable expertise to use one.

What is a theorem prover?

Theorem provers such as PVS, Isabelle/HOL, Coq are capable of expressing *any piece of mathematics or computer science*. This involves

1. **modelling/specification/definition of constructs involved**
2. **proving results about them, interactively and/or automatically.**

‘theorem prover’ aka ‘proof assistant’ is a bit of a misnomer, as it ignores the first part, which is already interesting in itself.

The first theorem prover was AUTOMATH, by de Bruijn & co here at TU/e in 1970’s

There are also less expressive theorem provers (eg. first-order theorem provers or SAT solvers) which provide better automation of proofs.

Why use a theorem prover?

It can give the highest level of confidence in correctness, but at very high cost: lots of effort by experts

So primarily of interest for applications where cost of failure is highest:

- **safety-critical systems (eg. Ariane 5)**
- **security-critical systems (eg. Chipknip software)**
- **mass-produced products (eg. Pentium bug)**

Example applications

- **hardware**
- **algorithms, esp. distributed or real-time algorithms**
- **(security) protocols**
- **programming language theory**
  formalising programming languages: their type systems, semantics, or program verification logics
- **mathematics**
  eg Four Color Theorem in Coq (Georges Gonthier, 2004)

Theorem proving vs model checking

+ **Theorem provers are more expressive**
+ **Model checkers can run into limitations due to the state explosion problem; theorem provers don't, and can cope with infinite state spaces.**

  Model checker can verify dining philisophers for 4 philosophers, theorem proving can do it for arbitrary number.
– **Theorem provers are more labour-intensive**
– **Model checkers provide better feedback.**
  - **Failed modelcheck attempt concrete counterexample trace.**
  - **Failed proof attempt may be due to missing lemma (invariant), or wrong proof strategy.**
- **Model checking can be used as part of theorem proving; indeed, PVS includes a model checker**

The PVS specification language

PVS specification language

The PVS specification language consists of

- **a typed lambda calculus,**
  simlar to functional programming languages à la Haskell or ML, but more expressive

  Eg. `reverse : [List -> List]`

- **a typed higher-order logic on top of this**

  Eg. `(FORALL (x:List): rev(rev(x)) = x )`

Many theorem provers, notably Isabelle and Coq, are based on similar typed languages, if slightly less baroque.

Types

- Base types `bool, int, real`
- Function types `[bool,int -> int]`
- Enumeration types `{red, white, blue}`
- Tuple types `[A,B]`
- Record types `[# x:int, y:int #]`
- Algebraic datatypes (ADTs) `Stack, List, Tree`
- Subset types `{ i:int | i >= 0 }`

Subset types are peculiar to PVS, and do not exist in for instance Isabelle or Coq.

Expressions

- basic expressions
  ```
  TRUE, FALSE: bool
  0, 1, -23, 23+5, 24*5 : int
  ```
- function abstraction and application
  ```
  (LAMBDA(i,j:nat):i+j) : [nat,nat->nat]
  f(i,j)
  ```
- tuples and projection
  ```
  (1,true) : [int,bool]
  tup`2 ,proj_2(tup)
  ```
- records and projection
  ```
  (# x:=1, y:=4 #) : [# x:int, y:int #]
  point`x
  ```

More expressions

- **let-expressions**

```
LET name = e1 IN e2
```

- **conditionals**

```
IF c THEN e1 ELSE e2 ENDIF
COND c1 -> e1, ..., cn -> E2 ENDCOND
COND c1 -> e1, ..., ELSE -> E2 ENDCOND
```

- **record and function updates**

```
point WITH [‘x:=24]
f WITH [(0):=1]
```

Declarations and definitions

- **declarations**

```
 i : int
A : TYPE
```

- **definitions**

```
twentyeight : int = 25+3
Point: TYPE = (# x:int, y:int #)
p:Point = (# x:=1, y:=4 #)
square: [int->int] =(LAMBDA (n:nat): n*n)
pred(n:int) : int = n-1
```

Logic

A typed higher-order logic, with

- conjunction, disjunction, negation, implication

  `AND OR NOT IMPLIES IFF`

  Alternative syntax: `&, =>` for `AND, IMPLIES`

- (in)equality

  `= /=`

- (typed) universal/extensional quantification

  `FORALL EXISTS`

Eg. `(FORALL (i,j:int): i>0 AND j>0 => i*j/=0)`

Theories

Specifications are built from theories with definitions, declarations and named axioms and lemmas. Eg.

```
MyFirstTheory: THEORY
 BEGIN
   square(n:nat): nat  = n*n
   square_nondecreasing: LEMMA
      FORALL (n:nat) : square(n) >= n
   sqrt : [nat-> nat]
   axiom_sqrt: AXIOM
      FORALL (n:nat) :
        square(sqrt(n)) <= n AND n < square(sqrt(n)+1)
 END MyFirstTheory
```

You could also *define* `sqrt` and turn the axiom into a lemma. (This would be better. Why?)

Theories

Trick to avoid lots of explicit type information

```
SquareTheory : THEORY
BEGIN
  n: VAR nat    % ie. n will range over nat
  square(n) : nat = n *n
  square_nondecreasing : LEMMA
     FORALL (n:nat) : square(n) >= n
  ...
```

Theories can be parameterized, eg

```
stack[A:Type] : THEORY
  ...
```

Recursion

All recursive functions *must be shown to terminate* by supplying a measure function.

```
fac(n:nat) : RECURSIVE nat =
    IF n=0 THEN 1 ELSE n*f(n-1) ENDIF
  MEASURE n
```

`fac` is only well-typed if

- **measure decreases, ie. `n/=0 => n-1<n`**
- **measure remains non-negative, ie. `n/=0 => n-1>=0`**

These are the so-called *type checking conditions (TCCs)*

Here PVS differs from typical functional programming languages!

TCCs

Expressions are only well-typed after all type checking conditions (TCCs) have been proven.

- **type checking is undecidable, in principle**
+ **but usually PVS prover discharges most TCCs fully automatically, in practice**

Warning: unsolved TCCs may leave inconsistencies in your theories.

Subset types

Subset types, eg

```
nat : TYPE = { i:int | i >= 0 }
subrange(n,m:int) : TYPE
            = { i:int | n <=i & i <= m }
```

are useful for partial operations, e.g. division

```
/ : [int, { n:int | n /= 0} -> int]
```

and also give rise to type checking conditions (TCCs)

Eg

```
average = sum / numbers :  int
```

is only well-typed if `numbers/=0`

The PVS prover

The PVS prover

Once we have defined – and type-checked! – a theory, we can prove any lemmas and theorems it contains.

Lemmas can be done in any order; PVS keeps track of what has been proved.

Proving is done interactively, by the user giving commands, tactics, to the PVS prover.

A tactic

- **either solves a proof obligation, or**
- **gives rise to one of more new, hopefully simpler, proof obligations.**

Sequents

PVS proof obligations are *sequents* of the form

```
[-1]  P
[-2]  Q
[-3]  R
----------
{1}   S
{2}   T
```

Intuitive meaning: `(P AND Q AND R)` => `(S OR T)`

- negatively numbered *ancedents/assumptions* above line,
- positively numbered *consequents/goals* below line

PVS maintains a *proof tree* of such sequents.

Tactics

The user interacts with the prover by tactics (which are actually LISP expressions).

There are *many* tactics, and you can define additional ones yourself.

Below we give an overview of the more common ones.

A full list is included in the ‘PVS Prover Guide’.

Basic tactics

- `(undo)` undo the last step in the proof
- `(quit)` quit the current proof
- `(postpone)` go to the next proof obligation
- `(help)`, `(help postpone)` get help

Propositional logic

The proof obligation

```
------------------
{1} P => Q
```

after `(flatten 1)` **becomes**

```
[-1] P
------------------
{1}  Q
```

You can omit the argument -1 and let PVS guess this.
Useful shorthard : `TAB f`

Propositional logic

```
[-1]  P1 AND P2
-----------------
...
```

after `(flatten -1)` becomes

```
[-1]  P1
[-2]  P2
-----------------
...
```

Propositional logic

Similarly,

```
...
-----------------
{1} Q1 OR Q2
```

after `(flatten 1)` **becomes**

```
...
----------------
{1}   Q1
{2}   Q2
```

Propositional logic

```
[1]  P1 OR P2
----------------
...
```

after `(split 1)` results in two proof obligations

```
[1]  P1              [1] P2
----------           ---------
...                  ....
```

This also works for antecedents of the form
`IF c THEN e1 ELSE e2 ENDIF`,
resulting in distinction of the cases `c` and `NOT c`.

Propositional logic (split)

Similarly,

```
...
-------------------
{1} P1 AND P2
```

after `(split 1)` **results in two proof obligations**

```
......          ......
----------      ---------
[1] P1          [1] P2
```

Note: many tactics can often be used on dual constructs – eg `AND` and `OR` – on different sides of the line.

Tactics for propositional logic

- `(flatten [fnum])`
  flatten antedents `(P1 AND P2)`
  and consequents `(Q1 OR Q2)` and `(Q1 => Q2)`
- `(split [fnum])`
  split based on consequent `(P1 AND P2)`
  or antecedent `(Q1 OR Q2)` or `(IF ...)`
- `(case "formula")`
  case distinction on formula, eg `(case "x>0")`
- `(lift-if [fnum])`
  replace `f(IF b THEN e1 ELSE e2 ENDIF)`
  by `IF b THEN f(e1) ELSE f(e2) IF`,
  typically as precursor to splitting
- `(prop)` automatic strategy for propositional logic

The argument [fnum] is optional; if you omit it PVS chooses one.

Predicate logic

```
...
--------------------------
{1} FORALL (x:A) P(x)
```

after `(skolem! 1)` becomes

```
...
-------------------
{1} P(x!1)
```

`(skolem 1 "name")` uses `name` instead of `x!1`

`(skosimp)` does `(skolem!)` and `(flatten)`

`(skosimp*)` does this repeatedly

Predicate logic

```
[1] EXIST (x) P(x)
---------------------
...
```

after `(skolem! -1)` becomes

```
[-1] P(x!1)
-----------------
...
```

Here `x!1` is the so-called witness

`(skolem 1 "name")` calls the witness `name`

Predicate logic

Proof obligations

```
[fnum] FORALL (x) P(x)          ...
----------------------          ----------------------
...                            {fnum} EXISTS (x) P(x)
```

after `(inst [fnum] "expr")` **become**

```
[fnum] P(expr)                  ...
------------------              -------------
...                            {fnum} P(expr)
```

`(inst? [fnum])` **lets PVS guess** `expr`**; only works in simple cases!!**

`(inst-cp ...)` **leaves copy of the quantification**

Tactics for predicate logic

- `(skolem! [fnum])`
  introduces skolem constants for consequent `(FORALL(x) P)` or antecendent `(EXIST(x) P)`
- `(skolem [fnum] "name1" ... "namen")`
  let's you choose name of these constants.
- `(inst [fnum] "expr1" ... "exprn")`
  instantiates antecedent `(FORALL(x) P)` or provides witness for consequent `(EXIST(x) P)`
- `(inst? [fnum])`
  lets PVS guess the expression; only works in simple cases!

Tactics for equational reasoning

- `(expand "name" [fnum] [n])`
  expand nth occurrence of `name` by its definition in `fnum`; the default is all occurrences
  Shorthand: put cursor on `name` and type `TAB r`
- `(replace fnum [fnums] LR)`
  use antedent `fnum` of the form `l = r`, to replace occurrences of `l` by `r` in `fnums`.
  Shorthand: `TAB r`, which interactively ask for all the options.
- `(replace fnum [fnums] RL)`
  idem, but in other direction
- `(assert)`
  built-in decision procedure for equality

Using lemmas

- `(lemma "name")`
  add lemma `name` as an assumption
- `(rewrite "name" [fnums] RL)`
  like `(replace)`, but using lemma instead of antecedent

Tactics for induction

- `(induct "n")`
  for goal of the form `(FORALL (..,n:nat,..) P)`
- `(induct-and-simplify "n")`
  Idem, but combined with simplification

Exercise 1

Load the file `pvs_intro.pvs` into PVS and start proving!

This involves basic propositional, equational, and equational logic.

Instructions & hints are included in file.

Background for exercise 2

Something with a more computer science flavour: deterministic state machines and refinements

Deterministic state machines

```
machine[S, I, O: TYPE+]: THEORY
BEGIN
  machine: TYPE =
    [# delta : [S, I -> S], output: [S -> O] #]

  m: VAR machine

  trace(m)(s:S, inp:sequence[I])(n:nat): RECURSIVE S =
    IF n = 0 THEN s
      ELSE trace(m)(delta(m)(s, inp(n)), inp)(n - 1)
    ENDIF
  MEASURE n
```

Sequences over type `I` are elements of type `[nat->I]`

Trace refinement

Let `m1:machine[S1,I,O]` and `m2:machine[S2,I,O]`

`rho`: `[S1->S2]` gives a trace-refinement from `m1` to `m2`
iff

```
FORALL (i:I, s1:S1)
  rho(m1‘delta(s1,i)) = m2‘delta(rho(s1),i)
```

ie.

$$\begin{tikzcd}
\texttt{S1} \arrow[r, "\texttt{rho}"] \arrow[d, "\texttt{m1‘delta(_,i)}"'] & \texttt{S2} \arrow[d, "\texttt{m2‘delta(_,i)}"] \\
\texttt{S1} \arrow[r, "\texttt{rho}"] & \texttt{S2}
\end{tikzcd}$$

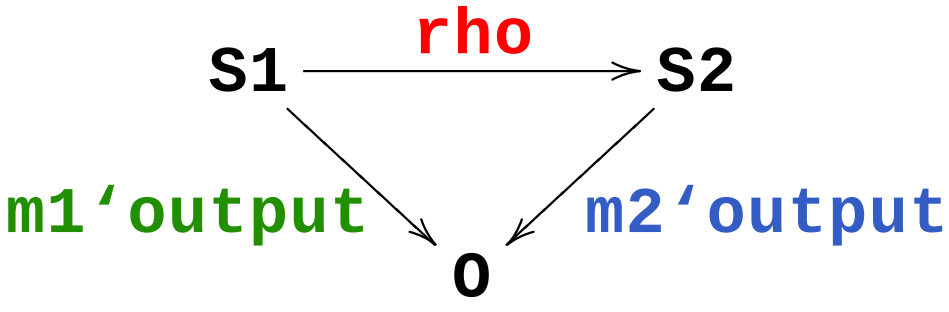
commutes

Output refinement

`rho`: `[S1->S2]` gives an output-refinement from `m1` to `m2` iff

```
FORALL (s1:S1)
  m1‘output(s1) = m2‘output(rho(s1))
```

ie.



commutes

Tripmeter: informal spec

A tripmeter, with input type `real` and output type `boolean`, monitors if input stays between thresholds `HI` and `LO`.

The meter is *tripped – ie. outputs true* – iff

1. `input > HI`
2. `input < LO`
3. `input > HI - D` but meter was tripped in previous state, because of conditions 1 or 3.
4. `input < LO + D` but meter was tripped in previous state, because of conditions 2 or 4.

for certain `0 <= D, LO < HI, LO+D <= HI-D`

A formal model `tm1`

Capture state as two booleans

```
state1: TYPE = [# LT, HT: bool #]
```

where

- **`HT` is true iff meter tripped due to high input (1 & 3)**
- **`LT` is true iff meter tripped due to low input (2 & 4)**

Could we just use one boolean, to record if meter is tripped?

Can both booleans become true at the same time?

An implementation `tm2`

Implementation `tm2` differs only slightly from the model `tm1`:
capture state using 3-element enumeration type

```
state2: TYPE = {Ht, Lt, Ut}
```

where

- **`Ht` means meter tripped due to high input (1 & 3)**
- **`Lt` means meter tripped due to low input (2 & 4)**
- **`Ut` means meter untripped**

What is the relation with `tm1`?

Another formal model `tm3`

Capture state as a list of reals

```
state3: TYPE = list[real]
```

Intuition: the state records the history of all inputs.
So transition function just adds input to list.

What should the output function be?

What is the relation with `tm2`?

Another formal model `tm3`

Output `TRUE` iff for state `x:list[real]`

```
(EXISTS(i): i < length(x) & nth(x,i) > HI &
    FORALL(j): j<i => HI-D < nth(x,j) & nth(x,j) <= HI)
 OR
  (EXISTS(i): i < length(x) & nth(x,i) < LO &
    FORALL(j): j<i => LO < nth(x,j) & nth(x,j) <= LO+D)
```

Here `nth(x,j)` is the `j`-th element of `x`.
The first element of `x` is the latest input.

What does `(EXISTS(i): i<length(x) & nth(x,i)>HI & (FORALL(j): j<i => HI-D<nth(x,j) & nth(x,j)<=HI))` ***mean?***

What is the relation with `tm2`***?***

Exercise 2

Load files `machine.pvs`and `tripmeter.pvs`into PVS.

Complete definitions and prove lemmas in `tripmeter.pvs`

Automation

PVS philosophy: automate everything that can be automated

- `(prop)`
  decision procedure for propositional logic
- `(assert)`
  decision procedure for equational logic

To find out how a tactic work: append a `$`, eg. `(prop$)` instead of `prop`.
This is a useful way to discover more tactics that may be useful in a particular situation

Some more PVS tactics

There are more powerful tactics than `(prop)` and `(assert)`, eg.

- `(grind)` - the general purpose workhorse

But beware that many automated tactics

- may produce far too many subgoals; then `(undo)`
  Moral: if `(grind)` fails to prove a goal, always `(undo)` as it probably did something silly.
- may take too long, or fail to terminate; then interrupt prover with `CTRL-c CTRL-c`, followed by `(restore)`

More about equality

- `(apply-extensionality [fnum])`
  replace goal `f=g` by `FORALL(x):f(x)=g(x)`

  Also replaces `(#x:=e1,y:=e2#) = (#x:=e3,y:=e4#)` by `e1=e3` and `e2=e4`
- `(decompose-equality [fnum])`
  idem, but also works on assumptions

Hints for complicated proofs

- **Try to understand what the assumptions/goals mean**
  This is often the bottleneck in verifications; ugly PVS syntax can be hard to read
- **Which instantiations of assumptions are useful ?**
- **Which lemmas might be useful?**
- **Carefully expand definitions**
  Too much expansion makes things unreadable, so use `("expand "expr" fnum n)`rather than `("expand "expr")`
- **Which case distinctions are useful ?**
  Many useful case-distinctions can be made by expanding definition, `lift-if`ing and `split`ing
- **exercise: if automated tactics (eg. induct-and-simplify, grind) succeed, try to do the proof "by hand" without using such automated techniques**
  Not just masochism: the improved insight can be crucial once automated tactics fail.

Concluding remarks

- **PVS is a very general tool**
- **Still a *BIG* step from being formal with pencil & paper to being formal in theorem prover**
- **Experience: specification is easy, verification is difficult,**
  . . . relatively speaking
- **But, errors often exposed during specification, *not* verification**
- **Mainly for experts on critical applications and academics**

Our work with PVS

Use of PVS in Digital Security group includes

- **formalisation of single-threaded Java in LOOP project, incl. denotational semantics and Hoare logic for program verification for specification language JML**
- **formalisation of semantics of C++ in ROBIN project**
- **formalisation of scheduling protocol [FMICS'2007] and C++ implementation of ReadersWritersLock [TBA]**