The CPS transform from Griffin. Let us first repeat what Herman already showed on the blackboard. The definition of the CPS transform in Griffin is:

$$\begin{aligned} \overline{x} &= \lambda k. kx \\ \overline{\lambda x. M} &= \lambda k. k \left(\lambda x. \overline{M} \right) \\ \overline{MN} &= \lambda k. \overline{M} \left(\lambda m. \overline{N} \left(\lambda n. mnk \right) \right) \end{aligned}$$

If we define a type transformation by:

$$\begin{array}{rcl} \alpha^* & = & \alpha \\ (\sigma \rightarrow \tau)^* & = & \sigma^* \rightarrow (\tau^* \rightarrow o) \rightarrow o \end{array}$$

Then we have the following implication:

$$\begin{array}{rrrr} \Gamma & \vdash & M \ : \ \sigma \\ & & \Downarrow \\ \Gamma^* & \vdash & \overline{M} \ : \ (\sigma^* \to o) \to o \end{array}$$

We check that this holds for each rule:

• The first rule is:

$$\overline{x} = \lambda k. \, kx$$

In this the left hand side is typed as:

$$\begin{array}{rcl} x & : & \sigma \\ \\ \overline{x} & : & (\sigma^* \to o) \to o \end{array}$$

and the right hand side is typed as:

$$k : \sigma^* \to o$$

$$x : \sigma^*$$

$$kx : o$$

$$\lambda k. kx : (\sigma^* \to o) \to o$$

• The second rule is:

$$\overline{\lambda x.\,M} = \lambda k.\,k\,(\lambda x.\,\overline{M})$$

In this the left hand side is typed as:

$$\begin{array}{rcl} x & : & \sigma \\ M & : & \tau \\ \lambda x. M & : & \sigma \to \tau \\ \overline{\lambda x. M} & : & ((\sigma \to \tau)^* \to o) \to o \end{array}$$

and the right hand side is typed as:

$$k : (\sigma \to \tau)^* \to o$$

$$x : \sigma^*$$

$$\overline{M} : (\tau^* \to o) \to o$$

$$\lambda x. \overline{M} : \sigma^* \to (\tau^* \to o) \to o = (\sigma \to \tau)^*$$

$$k (\lambda x. \overline{M}) : o$$

$$\lambda k. k (\lambda x. \overline{M}) : ((\sigma \to \tau)^* \to o) \to o$$

• The third rule is:

$$\overline{MN} = \lambda k. \, \overline{M} \left(\lambda m. \, \overline{N} \left(\lambda n. \, mnk \right) \right)$$

In this the left hand side is typed as:

$$\begin{array}{rcl} M & : & \sigma \to \tau \\ N & : & \sigma \\ MN & : & \tau \\ \overline{MN} & : & (\tau^* \to o) \to o \end{array}$$

and the right hand side is typed as:

$$k : \tau^* \to o$$

$$m : (\sigma \to \tau)^* = \sigma^* \to (\tau^* \to o) \to o$$

$$n : \sigma^*$$

$$mn : (\tau^* \to o) \to o$$

$$mnk : o$$

$$\lambda n. mnk : \sigma^* \to o$$

$$\overline{N} : (\sigma^* \to o) \to o$$

$$\overline{N} (\lambda n. mnk) : o$$

$$\lambda m. \overline{N} (\lambda n. mnk) : (\sigma \to \tau)^* \to o$$

$$\overline{M} : ((\sigma \to \tau)^* \to o) \to o$$

$$\overline{M} (\lambda m. \overline{N} (\lambda n. mnk)) : o$$

$$\lambda k. \overline{M} (\lambda m. \overline{N} (\lambda n. mnk)) : (\tau^* \to o) \to o$$

This actually corresponds to an inductive proof of the implication $M: \sigma \Rightarrow \overline{M}: (\sigma^* \to o) \to o$ (where the induction is on the type derivation of $M: \sigma$), but I did not want to write it that way to prevent clutter. Here I just wanted to show what the types of the various parts of the rules are.

In my understanding of this, the types that look like $\ldots \rightarrow o$ correspond to continuations, while types that look like $(\ldots)^*$ correspond to non-continuations.

Note that the type o in all this is arbitrary. There are two different ways that one can look at this. Either we are doing Curry-style type theory, and the

types will be correct for any choice of o. Or we are doing Church-style type theory, like in Femke's course notes, which means that really there everywhere should be types given with the bound variables in the lambda terms. In that case the o is an extra argument of the CPS transform.

If we want to get the value out at the end by applying the CPS transformed term to the identity continuation (as also happens in Theorem 3.2 of Sabry & Felleisen), i.e., by evaluating

 $\overline{M}\left(\lambda x.\,x\right)$

then that will fix the o. It then will be the * of the type of M, which is natural considering we will be running this identity continuation after evaluating M.

The CPS transform from Sabry & Felleisen. We now repeat this whole thing for the definition in Sabry & Felleisen. Everything is exactly the same, except that the extra argument for the continuation (that additional argument, as Fabio explained it to us) is added at the front instead of at the back. This has the disadvantage that the definition of the CPS transform looks less regular, but the advantage that the definition of the type transformation looks more like contraposition in logic.

The definition of the CPS transform in Sabry & Felleisen is:

$$\begin{aligned} \mathcal{F}\llbracket V \rrbracket &= \lambda k. \, k \, \Psi\llbracket V \rrbracket \\ \mathcal{F}\llbracket M N \rrbracket &= \lambda k. \, \mathcal{F}\llbracket M \rrbracket \left(\lambda m. \, \mathcal{F}\llbracket N \rrbracket \left(\lambda n. \, mkn \right) \right) \\ \Psi\llbracket x \rrbracket &= x \\ \Psi\llbracket \lambda x. \, M \rrbracket &= \lambda k. \, \lambda x. \, \mathcal{F}\llbracket M \rrbracket k \end{aligned}$$

If we define a type transformation by:

$$\begin{array}{rcl} \alpha^* & = & \alpha \\ (\sigma \rightarrow \tau)^* & = & (\tau^* \rightarrow o) \rightarrow (\sigma^* \rightarrow o) \end{array}$$

Then we have the following implications:

$$\begin{array}{rcl} \Gamma & \vdash & M \, : \, \sigma \\ & & \downarrow \\ \Gamma^* & \vdash & \mathcal{F}\llbracket M \rrbracket \, : \, \left(\left(\sigma^* \to o \right) \to o \right) \\ & & \Gamma & \vdash & V \, : \, \sigma \\ & & \downarrow \\ \Gamma^* & \vdash & \Psi \llbracket V \rrbracket \, : \, \sigma^* \end{array}$$

We check that this holds for each rule:

• The first rule is:

$$\mathcal{F}\llbracket V \rrbracket = \lambda k. \, k \, \Psi \llbracket V \rrbracket$$

In this the left hand side is typed as:

$$V : \sigma$$
$$\mathcal{F}\llbracket V \rrbracket : (\sigma^* \to o) \to o$$

and the right hand side is typed as:

$$\begin{array}{rcl} k & : & \sigma^* \to o \\ \Psi[\![V]\!] & : & \sigma^* \\ k \, \Psi[\![V]\!] & : & o \\ \lambda k. \, k \, \Psi[\![V]\!] & : & (\sigma^* \to o) \to o \end{array}$$

• The second rule is:

$$\mathcal{F}\llbracket MN \rrbracket = \lambda k. \mathcal{F}\llbracket M \rrbracket (\lambda m. \mathcal{F}\llbracket N \rrbracket (\lambda n. mkn))$$

In this the left hand side is typed as:

$$\begin{array}{rcl} M & : & \sigma \to \tau \\ N & : & \sigma \\ MN & : & \tau \\ \mathcal{F}\llbracket MN \rrbracket & : & (\tau^* \to o) \to o \end{array}$$

and the right hand side is typed as:

$$\begin{aligned} k &: \tau^* \to o \\ m &: (\sigma \to \tau)^* = (\tau^* \to o) \to \sigma^* \to o \\ n &: \sigma^* \\ mk &: \sigma^* \to o \\ mkn &: o \\ \lambda n. mkn &: \sigma^* \to o \\ \mathcal{F}[N]] &: (\sigma^* \to o) \to o \\ \mathcal{F}[N]] &: (\sigma^* \to o) \to o \\ \mathcal{F}[N]] (\lambda n. mkn) &: o \\ \lambda m. \mathcal{F}[N] (\lambda n. mkn) &: (\sigma \to \tau)^* \to o \\ \mathcal{F}[M]] &: ((\sigma \to \tau)^* \to o) \to o \\ \mathcal{F}[M]] (\lambda m. \mathcal{F}[N]] (\lambda n. mkn)) &: o \\ \lambda k. \mathcal{F}[M]] (\lambda m. \mathcal{F}[N]] (\lambda n. mkn)) &: (\tau^* \to o) \to o \end{aligned}$$

• The third rule is:

$$\Psi[\![x]\!] = x$$

In this the left hand side is typed as:

$$\begin{array}{rrrr} x & : & \sigma \\ \Psi[\![x]\!] & : & \sigma^* \end{array}$$

and the right hand side is typed as:

x : σ^*

• The fourth rule is:

$$\Psi[\![\lambda x. M]\!] = \lambda k. \, \lambda x. \, \mathcal{F}[\![M]\!] k$$

In this the left hand side is typed as:

$$\begin{array}{rcccc} x & : & \sigma \\ M & : & \tau \\ \lambda x.\,M & : & \sigma \to \tau \\ \Psi[\![\lambda x.\,M]\!] & : & (\sigma \to \tau)^* \end{array}$$

and the right hand side is typed as:

$$k : \tau^* \to o$$

$$x : \sigma^*$$

$$\mathcal{F}\llbracket M \rrbracket : (\tau^* \to o) \to o$$

$$\mathcal{F}\llbracket M \rrbracket k : o$$

$$\lambda x. \mathcal{F}\llbracket M \rrbracket k : \sigma^* \to o$$

$$\lambda k. \lambda x. \mathcal{F}\llbracket M \rrbracket k : (\tau^* \to o) \to \sigma^* \to o = (\sigma \to \tau)^*$$

Griffin-compatible CPS transform in the style of Sabry & Felleisen. The CPS transforms from Griffin and from Sabry & Felleisen are exactly the same, only the order of the arguments of the transformed functions is different. We now show that the way that Sabry & Felleisen have two variants of the transform that correspond to the distinction between values and non-values, also can be used for the CPS transform from Griffin.

The definition of the Griffin-compatible CPS transform in the style of Sabry & Felleisen is:

$$\begin{split} \mathcal{F}\llbracket V \rrbracket &= \lambda k. \, k \, \Psi\llbracket V \rrbracket \\ \mathcal{F}\llbracket MN \rrbracket &= \lambda k. \, \mathcal{F}\llbracket M \rrbracket \left(\lambda m. \, \mathcal{F}\llbracket N \rrbracket \left(\lambda n. \, mnk \right) \right) \\ \Psi\llbracket x \rrbracket &= x \\ \Psi\llbracket \lambda x. \, M \rrbracket &= \lambda x. \, \lambda k. \, \mathcal{F}\llbracket M \rrbracket k \end{split}$$

Even nicer (and this is what I would have liked to have seen in Sabry & Felleisen) is to take the version of this in which the last rule has been eta-reduced:

$$\begin{split} \mathcal{F}\llbracket V \rrbracket &= \lambda k. \, k \, \Psi\llbracket V \rrbracket \\ \mathcal{F}\llbracket MN \rrbracket &= \lambda k. \, \mathcal{F}\llbracket M \rrbracket \left(\lambda m. \, \mathcal{F}\llbracket N \rrbracket \left(\lambda n. \, mnk \right) \right) \\ \Psi\llbracket x \rrbracket &= x \\ \Psi\llbracket \lambda x. \, M \rrbracket &= \lambda x. \, \mathcal{F}\llbracket M \rrbracket \end{split}$$

This is exactly the definition of the CPS transform from Griffin, if we expand the definition of Ψ in the definition of \mathcal{F} .

If we define a type transformation by:

$$\begin{array}{rcl} \alpha^* & = & \alpha \\ (\sigma \rightarrow \tau)^* & = & \sigma^* \rightarrow (\tau^* \rightarrow o) \rightarrow o \end{array}$$

Then we have the following implication:

$$\begin{array}{rrrr} \Gamma & \vdash & M \, : \, \sigma \\ & \downarrow & \\ \Gamma^* & \vdash & \mathcal{F}\llbracket M \rrbracket \, : \, \left((\sigma^* \to o) \to o \right) \\ \Gamma^* & \vdash & \Psi\llbracket M \rrbracket \, : \, \sigma^* \end{array}$$

We check that this holds for each rule:

• The first rule is:

$$\mathcal{F}\llbracket V \rrbracket = \lambda k. \, k \, \Psi \llbracket V \rrbracket$$

In this the left hand side is typed as:

$$V : \sigma$$
$$\mathcal{F}\llbracket V \rrbracket : (\sigma^* \to o) \to o$$

and the right hand side is typed as:

$$k : \sigma^* \to o$$

$$\Psi[\![V]\!] : \sigma^*$$

$$k \Psi[\![V]\!] : o$$

$$\lambda k. k \Psi[\![V]\!] : (\sigma^* \to o) \to o$$

• The second rule is:

$$\mathcal{F}\llbracket MN \rrbracket = \lambda k. \, \mathcal{F}\llbracket M \rrbracket \left(\lambda m. \, \mathcal{F}\llbracket N \rrbracket \left(\lambda n. \, mnk \right) \right)$$

In this the left hand side is typed as:

$$\begin{array}{rcl} M & : & \sigma \to \tau \\ N & : & \sigma \\ MN & : & \tau \\ \mathcal{F}\llbracket MN \rrbracket & : & (\tau^* \to o) \to o \end{array}$$

and the right hand side is typed as:

$$\begin{array}{rcl} k & : & \tau^* \to o \\ m & : & (\sigma \to \tau)^* = \sigma^* \to (\tau^* \to o) \to o \end{array}$$

$$\begin{array}{rcl}n&:&\sigma^{*}\\\\mn&:&(\tau^{*}\rightarrow o)\rightarrow o\\\\mnk&:&o\\\\\lambda n.\,mnk&:&\sigma^{*}\rightarrow o\\\\\mathcal{F}\llbracket N\rrbracket &:&(\sigma^{*}\rightarrow o)\rightarrow o\\\\\mathcal{F}\llbracket N\rrbracket (\lambda n.\,mnk)&:&o\\\\\lambda m.\,\mathcal{F}\llbracket N\rrbracket (\lambda n.\,mnk)&:&(\sigma\rightarrow \tau)^{*}\rightarrow o\\\\\mathcal{F}\llbracket M\rrbracket &:&((\sigma\rightarrow \tau)^{*}\rightarrow o)\rightarrow o\\\\\mathcal{F}\llbracket M\rrbracket &:&((\sigma\rightarrow \tau)^{*}\rightarrow o)\rightarrow o\\\\\mathcal{F}\llbracket M\rrbracket (\lambda m.\,\mathcal{F}\llbracket N\rrbracket (\lambda n.\,mnk))&:&o\\\\\lambda k.\,\mathcal{F}\llbracket M\rrbracket (\lambda m.\,\mathcal{F}\llbracket N\rrbracket (\lambda n.\,mnk))&:&(\tau^{*}\rightarrow o)\rightarrow o\end{array}$$

• The third rule is:

$$\Psi[\![x]\!] = x$$

In this the left hand side is typed as:

$$\begin{array}{rrrr} x & : & \sigma \\ \Psi[\![x]\!] & : & \sigma^* \end{array}$$

and the right hand side is typed as:

$$x$$
 : σ^*

• The fourth rule is:

$$\Psi\llbracket\lambda x.\,M\rrbracket = \lambda x.\,\lambda k.\,\mathcal{F}\llbracketM\rrbracket k$$

In this the left hand side is typed as:

$$\begin{aligned} x &: \sigma \\ M &: \tau \\ \lambda x. M &: \sigma \to \tau \\ \Psi \llbracket \lambda x. M \rrbracket &: (\sigma \to \tau)^* \end{aligned}$$

and the right hand side is typed as:

$$\begin{aligned} k &: \quad \tau^* \to o \\ x &: \quad \sigma^* \\ \mathcal{F}\llbracket M \rrbracket &: \quad (\tau^* \to o) \to o \\ \mathcal{F}\llbracket M \rrbracket k &: \quad o \\ \lambda k. \, \mathcal{F}\llbracket M \rrbracket k &: \quad (\tau^* \to o) \to o \\ \lambda x. \, \lambda k. \, \mathcal{F}\llbracket M \rrbracket k &: \quad \sigma^* \to (\tau^* \to o) \to o = (\sigma \to \tau)^* \end{aligned}$$

Or, if we consider the eta-reduced version of the fourth rule:

$$\Psi[\![\lambda x. M]\!] = \lambda x. \mathcal{F}[\![M]\!]$$

the right hand side is typed as:

$$\begin{array}{rcl} x & : & \sigma^* \\ & \mathcal{F}\llbracket M \rrbracket & : & (\tau^* \to o) \to o \\ \lambda x. \, \mathcal{F}\llbracket M \rrbracket & : & \sigma^* \to (\tau^* \to o) \to o = (\sigma \to \tau)^* \end{array}$$